Iterative construction of Cayley Expander graphs

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Abstract

We construct a sequence of groups G_n, and explicit sets of generators $Y_n \subset G_n$, such that all generating sets have bounded size, and the associated Cayley graphs are all expanders. The group G_1 is the alternating group A_d, the set of even permutations on the elements $\{1, 2, \ldots, d\}$. The group G_n is the group of all even symmetries of the rooted d-regular tree of depth n. Our results hold for any large enough d.

We also describe a finitely-generated infinite group G_∞ with generating set Y_∞, given with a mapping f_n from G_∞ to G_n for every n, which sends Y_∞ to Y_n. In particular, under the assumption described above, G_∞ has property (τ) with respect to the family of subgroups $\ker(f_n)$.

The proof is elementary, using only simple combinatorics and linear algebra. The recursive structure of the groups G_n (iterated wreath products of the alternating group A_d) allows for an inductive proof of expansion, using the group theoretic analogue [4] of the zig-zag graph product of [42]. The basis of the inductive proof is a recent result by Kassabov [22] on expanding generating sets for the group A_d.

Essential use is made of the fact that our groups have the *commutator property*: every element is a commutator. We prove that direct products of such groups are expanding even with highly correlated tuples of generators. Equivalently, highly dependent random walks on several copies of these groups converge to stationarity on all of them essentially as quickly as independent random walks. Moreover, our explicit construction of the generating sets Y_n above uses an efficient algorithm for solving certain equations over these groups, which relies on the work of [37] on the commutator width of perfect groups.

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