We saw in class the Gilbert-Varashamov and Plotkin bounds for the case of binary alphabet. Please prove the generalizations of these bounds to $q$-ary alphabet:

1. **Gilbert-Varashamov bound for $q$-ary alphabet:** For all $q \geq 2$, and for $0 \leq \delta \leq 1 - \frac{1}{q}$, there exists a family of $q$-ary codes $C$ having rate $R$ and relative distance $\delta$ s.t.

   $$R \geq 1 - H_q(\delta)$$

   **Hint.** Let the $q$-ary entropy function $H_q$ be

   $$H_q(x) = x \log_q(q - 1) - x \log_q x - (1 - x) \log_q(1 - x)$$

   Show that the volume $Vol_q(n, \delta n)$ of a Hamming ball of radius $\delta n$ in the space of length $n$ vectors over $q$-ary alphabet is bounded by

   $$q^{H_q(\delta)n - o(n)} \leq Vol_q(n, \delta n) \leq q^{H_q(\delta)n}$$

2. **Plotkin bound for $q$-ary alphabet:** For any $(n, k, d)\_q$-code $C$,

   (a) If $d = (1 - \frac{1}{q})n$, then $k \leq 1 + \log q + \log n$

   (b) If $d > (1 + \varepsilon)(1 - \frac{1}{q})n$, then $k \leq \log(1 + \varepsilon)$

   (c) For $R = k/n$ and $\delta = d/n$, $R \leq 1 - \frac{q - 1}{q - 1} \delta + o(1)$

   **Hint.** Show that there exists a mapping $f: C \to \mathbb{R}^{nq}$ s.t. $\forall C \in C$, $\|f(C)\|_2 = 1$ and for all $C \neq C' \in C$, $\langle f(C), f(C') \rangle \leq 1 - \frac{q - 1}{q - 1} \frac{\Delta(C, C')}{n}$. 

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