

# **abstract**

MINI-COURSE IN GEOMETRIC PDE

Topic:

Speaker:

Affiliation:

Date:

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In 2005 Ma, Trudinger and Wang introduced a fourth-order differential condition which comes close to be necessary and sufficient for the smoothness of solutions to optimal transport problems with a given cost function. If the cost function is the squared distance on a Riemannian manifold, this condition can be rephrased as the positivity of a fourth-order nonlocal curvature-type tensor.

To apply the Ma-Trudinger-Wang results one needs to keep the optimal transport map away from the cut locus. This leads to the study of the cut locus, which is still one of the most poorly understood objects in Riemannian geometry.

In this course I shall analyze some properties of this Ma-Trudinger-Wang (MTW) tensor, in relation with optimal transport theory and with the geometry of the cut locus. This will be based on a series of works by Figalli, Loeper, Rifford and myself.

Part I: Regularity and non-regularity

In this lecture I shall start with a negative result showing that regularity does not hold without the MTW condition [1].

Then a positive result according to which, under a certain non-focalization condition, strict MTW is sufficient to get regularity. The main new insight is that (somewhat unexpectedly) the MTW condition provides a strong control of the geometric shape of the cut locus. [2]

Part II: Stability and non-stability

Because of nonlocality, the stability of the MTW condition under limits is not obvious, even

under  $C^k$  convergence for large  $k$ . However, a synthetic formulation allows to prove the stability under the extremely weak notion of Gromov-Hausdorff convergence. Two schemes of proof will be discussed. [4]

The stability under perturbation is easy when there is no focalization [2], but difficult otherwise, e.g. on the sphere. In dimension 2, this problem can be solved satisfactorily. [5,6] A consequence of independent interest is the Figalli-Rifford theorem about the stability of shape of cut locus near the 2-dimensional sphere. [5] In higher dimension the problem becomes extremely rich and it is hard to formulate even a guess.

[1] Loeper. On the regularity of solutions to optimal transportation problems.

[2] Loeper, Villani. Regularity of optimal transport in curved geometry: the nonfocal case.

[3] Figalli, Villani. An approximation lemma about the cut locus, with applications to optimal transport theory.

[4] Villani. Stability of a fourth-order curvature condition arising in optimal transport theory.

[5] Figalli, Rifford. Continuity of optimal transport maps and convexity of injectivity domains on small deformations of  $S^2$ .

[6] Figalli, Rifford, Villani. Stability of the Ma-Trudinger-Wang conditions.