Y. Soibelman - 24/1/07 - Algebra & geometry of CY varieties

Def: a noncommutative CY var. of $\dim = d$ is a <u>compact</u>, <u>homologically smooth</u> triangulated A_∞-category with Serre functor $S \simeq [d]$.

A A_∞-algebra, $\mathcal{C} = A\text{-mod}$, $m_k : A^{\otimes k} \to A$ $k \geq 1$

- <u>compact</u> := $\operatorname{rank} H^*(A, m_1) < \infty$
- <u>hom. smooth</u> := A is a perfect A-A-bimodule.

Lemma: if A is homol. smooth & compact then $\operatorname{rank} HH(A, A) < \infty$.

Conj. 1: a) If X NCCY then the moduli space $\mathcal{M}_X$ is smooth ← focus on this
b) $\exists$ embedding of $\mathcal{M}_X$ into an affine space
c) $\exists$ global moduli stack $\mathcal{M}^{glob}$, which is an orbifold.

cf. Toen's work

- X CY, $\dim X = d$, $X/\mathbb{C}$, P a generator of $\operatorname{Perf}(X)$

$A = \Omega^{0,*}(X, \operatorname{End} P)$ dg-algebra, vol = holom. vol. form

$\alpha \mapsto \int_X \operatorname{tr}(\alpha) \wedge \mathrm{vol}$ gives a nondegenerate pairing on $H^\bullet(A)$

- Let A = compact (weakly unital) A_∞-algebra

$$\rightsquigarrow CC_\bullet(A) := (C_\bullet(A, A)[u^{-1}], b + uB)$$

$\deg u = 2$

Hochschild chains — Hochschild diff. — connes diff.

Let $[\varphi] \in H^d(CC_\bullet(A))$.

(A, m_1) is a subcomplex of the Hochschild complex;

$$(A, m_1) \xrightarrow[Tr_{[\varphi]}]{} k[-d]$$

Def: φ is homologically nondegenerate if $Tr_{[\varphi]}(ab)$ is a nondegenerate pairing.

Prop: such φ gives rise to an isom class of scalar products on a minimal model of A.

Def: φ gives rise to a CY structure of degree d on A.

Conj 2: the 2 definitions of NC CY structures $\begin{cases} - \ S \simeq [d] \\ - \ \mathrm{Tr}_{C(\varphi)} \text{ nondeg. pairing} \end{cases}$ are equivalent for homol. smooth compact A.

Also:

$A^! = \mathrm{Hom}(A, A \otimes A)$ inverse dualizing bimodule

$\left(F \mapsto F \otimes K_X^{-1}[-\dim X] \right)$

Prop: if A is homol. smooth then $A^! \in \mathrm{Perf}(A\text{-mod-}A)$

Conj 2': The existence of a CY struct. of deg. d is equiv^t to $A^! \simeq A[d]$.

(proved by V-Ginzburg for case A = assoc. algebra).

Smoothness of moduli space: involves **noncomm. Hodge-to-de Rham degeneration**

(char $k=0$) **Def:** $CC_\bullet^-(A) = (CC_\bullet(A)[[u]], b + uB)$

negative cyclic complex

Def: Say A satisfies degeneration property if $H^*(CC_\bullet^-(A)/(u^n))$ is flat / $k[u]/(u^n)$ $\forall n \geq 1$

$\Rightarrow H^*(CC_\bullet^-(A))$ is a flat $k[[u]]$-module

ie. a vector bundle over $\mathbb{A}^1_{\text{formal}}[-2]$

Generic fiber $\simeq HP_\bullet(A)$ periodic cyclic homology $\mathbb{Z}/2$-graded

Fiber at $u=0$ $\simeq HH_\bullet(A)$

Conj 3 (degeneration of Hodge-to-DeRham)

If A finite dim. compact homologically smooth, then degeneration property holds.

- Recently announced by Kaledin in case A is entirely in degrees ≥ 0.

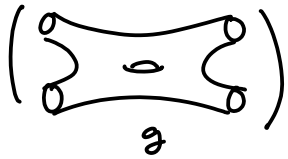
Corollary of Conj-3:

Let A = CY algebra. Then assuming Conj-3, $\mathcal{M}_A$ is smooth.

NB: $\mathcal{M}_A :=$ we can define 1) A_∞-structure (on a fixed underlying vect-space, define m_k)
2) bilinear pairing $(\cdot,\cdot): A \otimes A \to k[-d]$ st. $(m_k(a_0 \dots a_{n-1}), a_n)$ is cyclically invariant

In case 1), $T\mathcal{M}$ = Hochschild cohomology
In case 2), $T\mathcal{M}$ = cyclic cohomology. related in CY case.

[Costello; Kontsevich-S., Kontsevich-Katzarkov-Pantev]

----- Chains 

$$\begin{tikzcd} M_{g,n} \arrow[r] & \overline{M}_{g,n} \\ & \end{tikzcd}$$

Top FT ← $M_{g,n}$; Cohomological FT ← $\overline{M}_{g,n}$