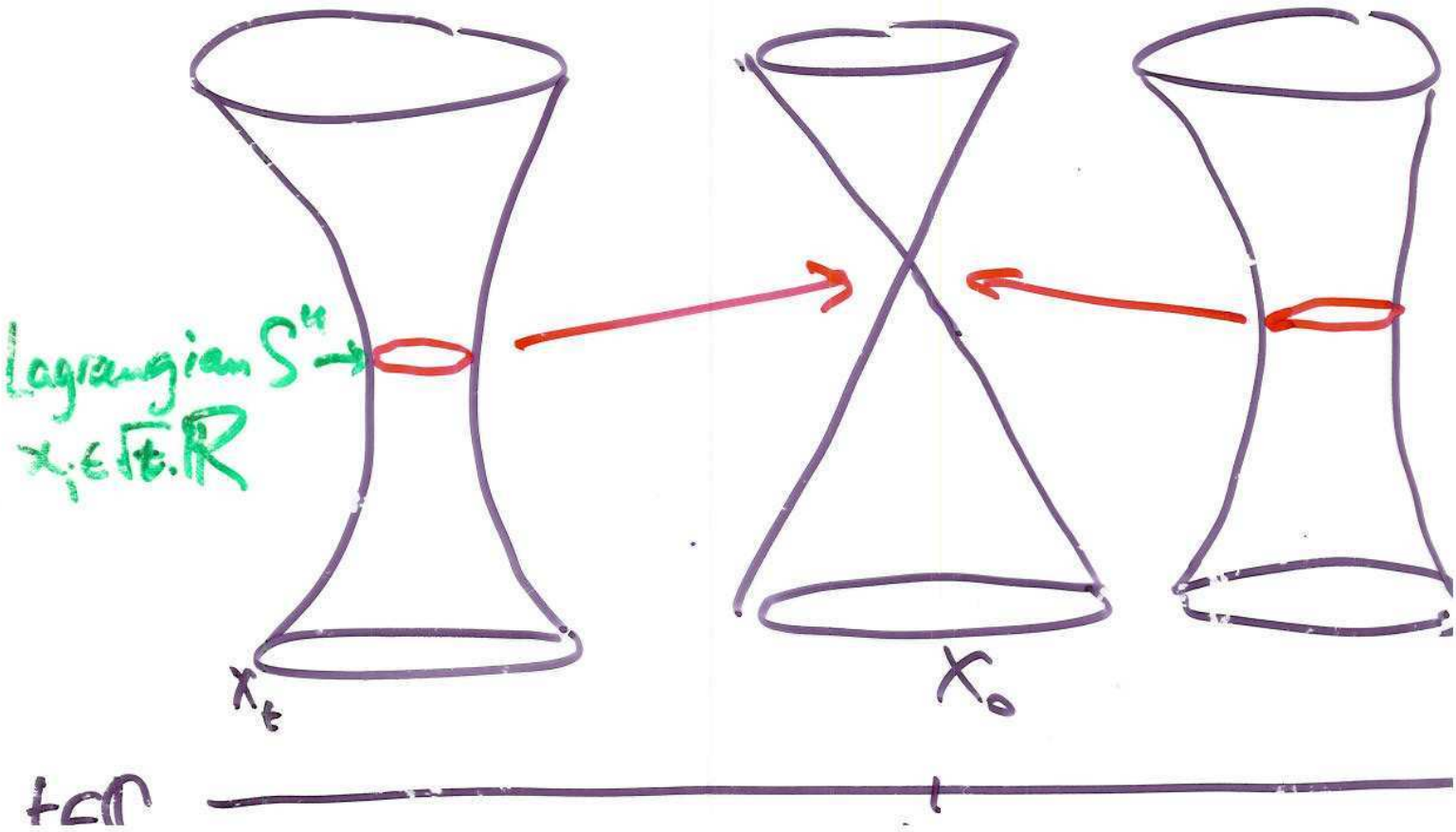
Configuration Spaces, Simultaneous Resolutions, and Braid Group actions on Categories.

Joint work with
P. Seidel, M. Khovanov

Dehn Twists.

$$(X,\omega) = \left\{ \sum_{i=1}^{n+1} x_i^2 = t \right\} \subseteq (\mathbb{C}^{n+2}, \omega_{standard})$$



2.

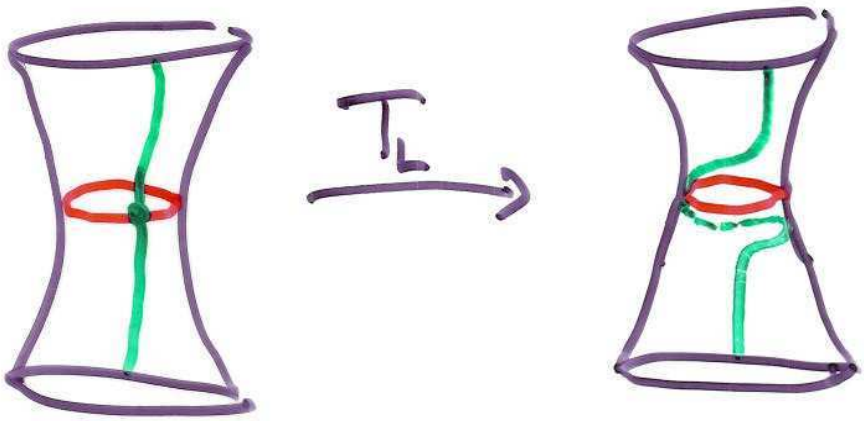
<u>Symplectically</u> this is a fibre bundle

Symplectic connection : horizontal subspace $TX_t^{\perp\omega}$.

All fibres $(X_{t\neq 0}, \omega) \cong (T^*S^n, \omega_{standard})$

Monodromy $\in \mathrm{Aut}(X_t, \omega)$

<u>Dehn twist</u> (Arnol'd, Seidel)
about L.



$$[T_L]_* : [A] \longmapsto [A] + ([L].[A])\,[L]$$

Picard-Lefschetz

3.

In $\mathrm{Fuk}(X_t)$,

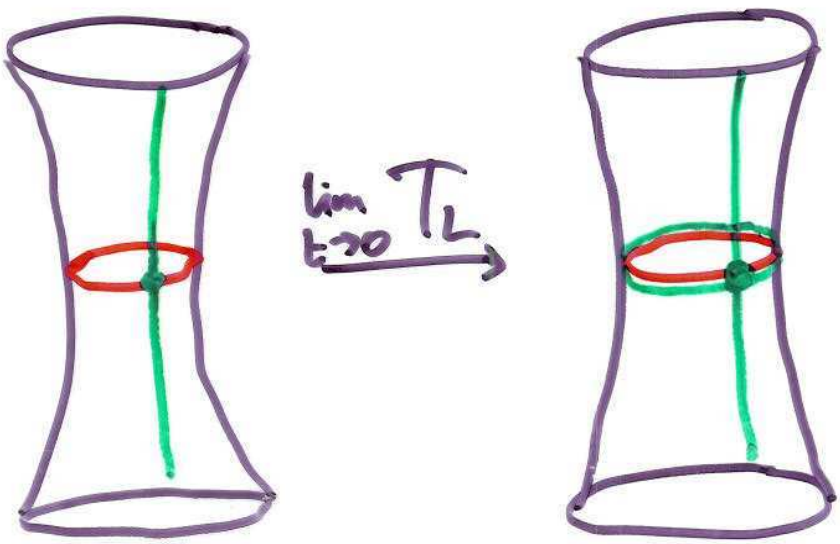
$$\begin{tikzcd} HF^*(L,A)\otimes L \arrow[r, "ev"] & A \arrow[r] & T_L A \arrow[ll, bend left, "{[1]}"] \end{tikzcd}$$

$$ev = id \in HF^*(HF^*(L,A)\otimes L, A) \cong HF^*(L,A)^* \otimes HF^*(L,A)$$

Tighten loop about $t=0$

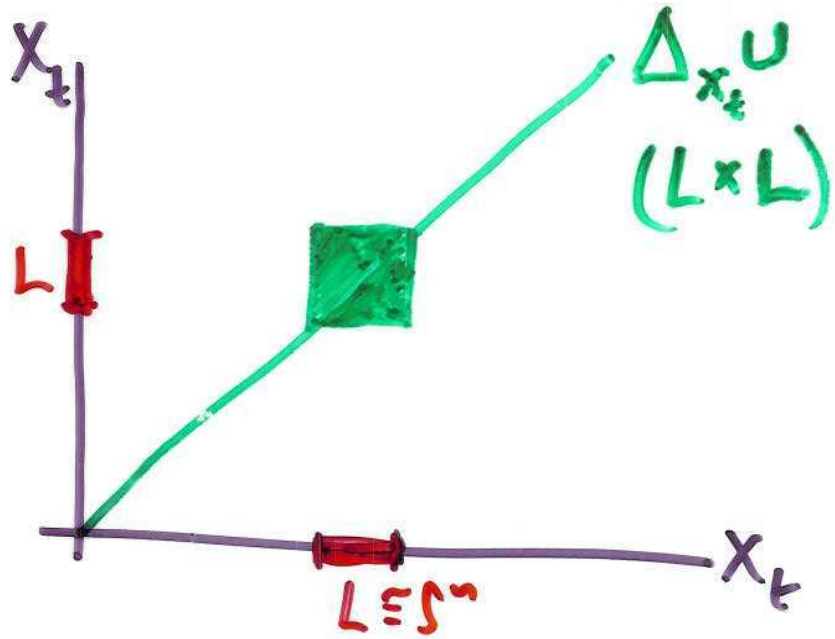
($\Rightarrow$ different T_L due to curvature of symplectic connection $\in \mathrm{Ham}(X_t, \omega)$)

$\lim_{t\to 0}$



4.

$\lim_{t\to 0} \mathrm{Graph}(T_L)$:



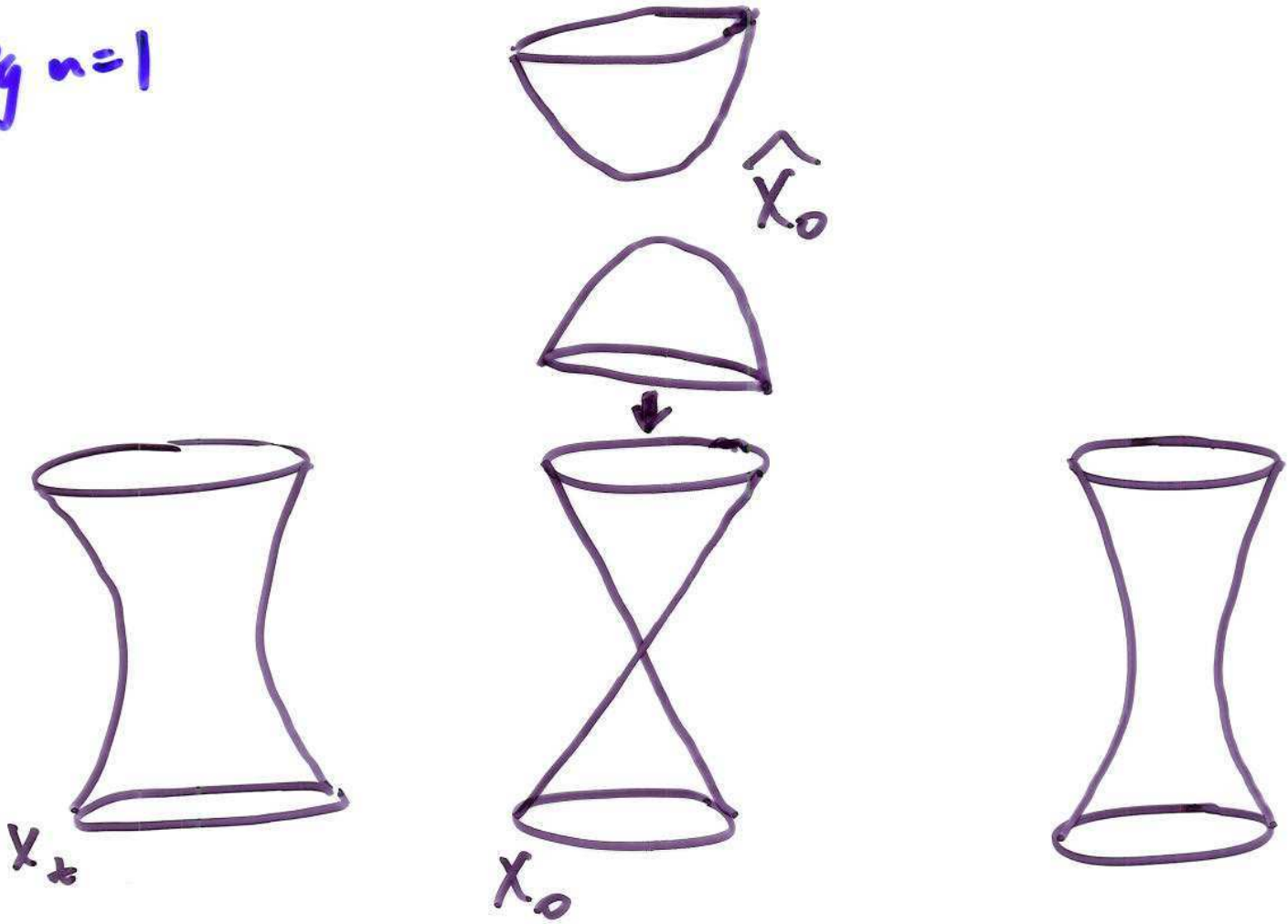
Simultaneous Resolution

Family $\begin{matrix} X \\ \downarrow \\ \mathbb{C} \end{matrix}$ of X_ts; simultaneous resolution is a family $\begin{tikzcd} \hat{X} \arrow[r] \arrow[d] & X \arrow[dl] \\ \mathbb{C} & \end{tikzcd}$ s.t. $\begin{matrix} \hat{X}_t \\ \downarrow \\ X_t \end{matrix}$ is a resol. of sings $\forall t$, and an isomorphism if X_t is smooth.

5.

$\nexists$ for our family and $n \neq 2$.

eg $n=1$



$\nexists$ for $n=2$ since $(T_L)_* \neq id$

<u>But</u> does exist $(n=2)$ after base change

$\underline{t \mapsto t^2}$

6.

$$\{x_1^2+x_2^2+x_3^2=t^2\} \subseteq \mathbb{C}^4$$ Total space 3-fold ODP.

$x=x_1+ix_2$
$y=x_1-ix_2$
etc

$$\{xy=zw\} \subseteq \mathbb{C}^4$$

Use ratios $\frac{x}{z}=w/y \in \mathbb{P}^1$ to resolve

le $$\hat{X}=\left\{\begin{array}{l} xy=zw \\ x\mu=z\lambda \\ w\mu=y\lambda \end{array}\right\} \subseteq \underset{xyzw}{\mathbb{C}^4} \times \underset{[\lambda:\mu]}{\mathbb{P}^1}$$ Atiyah

Not full blow up of singular point 0!

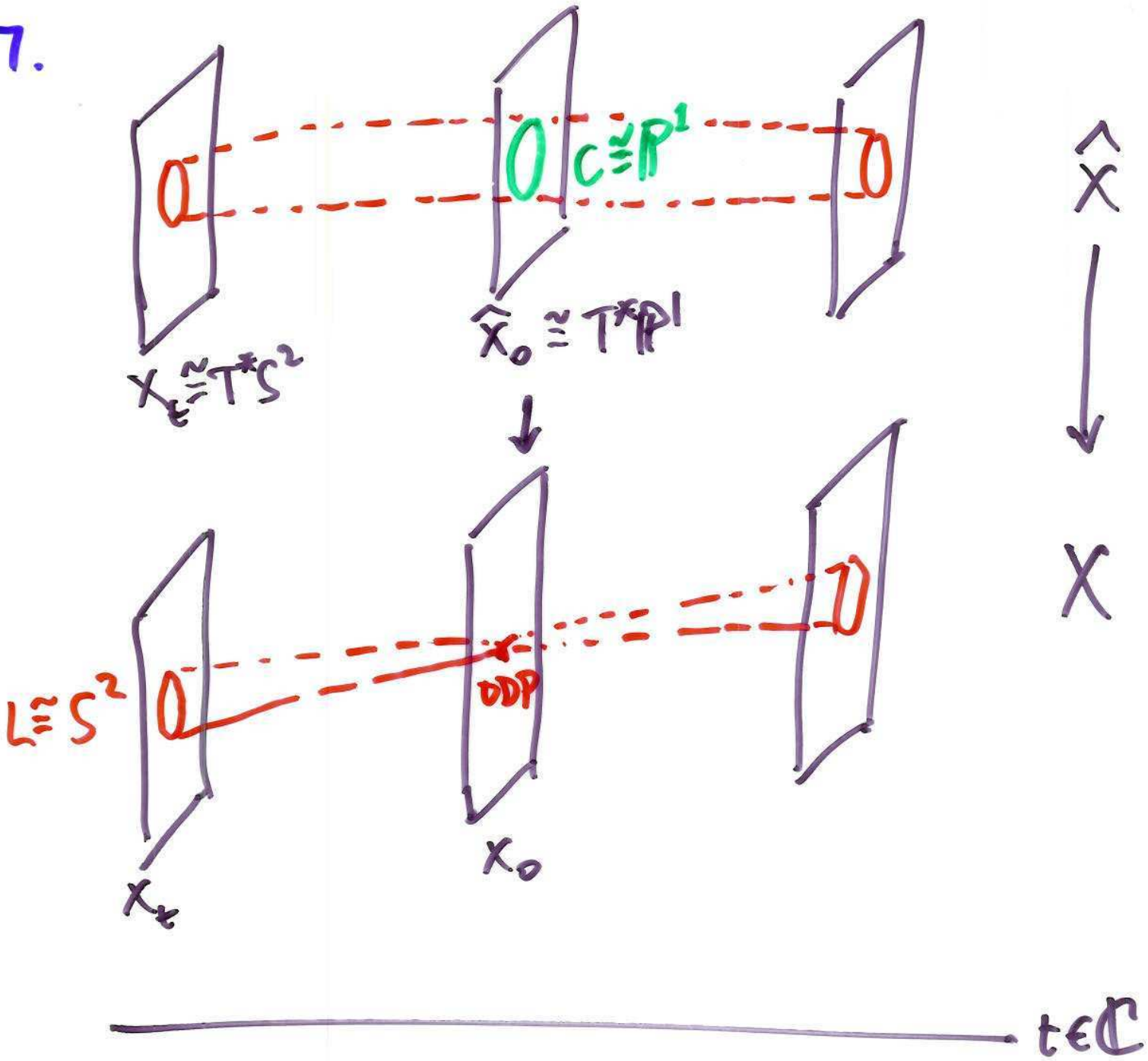
This would give $\mathbb{P}^1\times\mathbb{P}^1$ excep. divisor.

So central fibre has 2 irred. cpts.

Blow down one ruling $\Rightarrow \hat{X}$

other ruling $\Rightarrow$ flop of $\hat{X}$ using $\frac{x}{w}=\frac{z}{y}$.

7.



So $T_L^2 \cong id$ as diffeo.

$(T_L)_*^2 = id$

But $T_L^2 \in Aut(X_t, \omega)$ <u>not</u> $\cong id$. Seidel

$\hat{X}$ has <u>degenerate</u> symplectic form $\omega|_{\mathbb{P}^1} = 0$.

8.

Perturb pullback of $\omega|_L$ to genuine Kähler symplectic form

($\Rightarrow L \cong S^2$ no longer Lagrangian)

Get <u>hyperkähler rotation</u> of X_t as $\hat{X}_0$.

[X_t holomorphic symplectic $\omega = \frac{i}{2}\sum dz_i \wedge d\bar{z}_i$

$\Omega = \sum dz_{2i} \wedge dz_{2i+1}$

$\omega|_L \equiv 0$, $\operatorname{Im}\Omega|_L \equiv 0$, $\operatorname{Re}\Omega|_L = \mathrm{vol}_L$ wlog

HK rot^n

Sim. on $T^*\mathbb{P}^1$, $\operatorname{Re}\Omega|_C \equiv 0$, $\omega|_{\mathbb{P}^1} = \mathrm{vol}_{\mathbb{P}^1}$.

$\operatorname{Im}\Omega|_C \equiv 0$,]

HK mfds <u>self mirror</u> (at diffeo level)

In some limit/regime?? MS is HK rot^n??

$D^b(\mathrm{Fuk}(X)) \ni \mathrm{SLags} \leftrightarrow \mathrm{cxLags} \in D^b(\mathrm{coh}(X))$

In our case we pass to $\lim_{t \to 0}$ and get in central fibre $\hat{X}_0$ the graph $\underline{\underline{\Delta_{\hat{X}_0} \cup (C \times C)}}$

Use $\underline{\mathcal{O}_{\Delta \cup C \times C} \in D^b(\hat{X}_0 \times \hat{X}_0)}$ as

Fourier-Mukai kernel.

$$\Rightarrow T_C : D^b(\hat{X}_0) \longrightarrow D^b(\hat{X}_0)$$

$$T_C(\mathcal{E}) = \pi_{2*}\left((\pi_1^* \mathcal{E}) \otimes \mathcal{O}_{\Delta \cup C \times C}\right)$$

$$\begin{tikzcd} & \hat{X}_0 \times \hat{X}_0 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \hat{X}_0 & & \hat{X}_0 \end{tikzcd}$$

Acts as id on $\mathcal{E}$ s.t. $\mathrm{supp}(\mathcal{E}) \cap C = \phi$.

Exact sequence on $\hat{X}_0 \times \hat{X}_0$

$$0 \to \mathcal{O}_\Delta \to \mathcal{O}_{\Delta \cup (C \times C)}(C,0) \to \mathcal{O}_{C \times C}(C,0) \to 0$$

$$\mathcal{O}_{C \times C}(C,0) = \mathcal{O}_C^\vee \boxtimes \mathcal{O}_C[1]$$

gives exact triangle

$$R\mathcal{H}om(\mathcal{O}_C, \mathcal{E}) \otimes \mathcal{O}_C \xrightarrow{ev} \mathcal{E} \to T_C \mathcal{E} \xrightarrow{[1]} R\mathcal{H}om(\mathcal{O}_C, \mathcal{E}) \otimes \mathcal{O}_C$$

Mirror to T_L. $\quad (T_C)_* [\mathcal{E}] = [\mathcal{E}] + ([\mathcal{E}].[C])[C]$

<u>Kontsevich</u>: mirror of T_L is auto of $D^b(\mathrm{coh}\text{-}\hat{X}_0)$, not auto of $\hat{X}_0$.

Configuration Space

$C_m = \{ m$ distinct unordered pts in $\mathbb{C} \}$

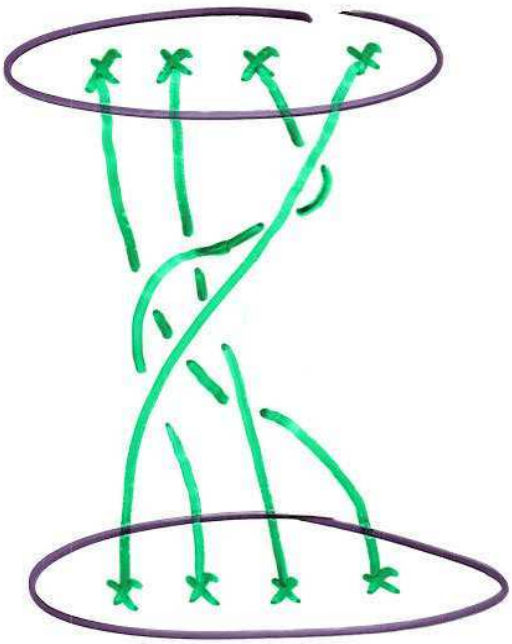
$\underline{\lambda} = (\lambda_1, \dots, \lambda_m)$

$= \{$ monic deg m polys w/ distinct roots $\}$

$P_{\underline{\lambda}}(t) = \prod_{i=1}^{m} (t - \lambda_i)$

$$= \frac{\mathbb{C}^m \setminus \text{diagonals}}{\sigma_m}$$

$\pi_1(C_m) = B_{m-1}$



$\exists$ spaces $X_{\underline{\lambda}}$ parametrised by pts $\underline{\lambda} \in C_m$.

12.

Eg (Donaldson, Khovanov-Seidel, Shapere-Vafa)

$$X_{\underline{\lambda}} = \left\{ \sum_{i=1}^{n} x_i^2 = p_{\underline{\lambda}}(t) = \prod_{i=1}^{m} (t - \lambda_i) \right\} \subseteq \underset{x_i}{\mathbb{C}^n} \times \underset{t}{\mathbb{C}}$$

Together form a family $\underline{X} = \bigcup_{\underline{\lambda} \in C_m} X_{\underline{\lambda}}$ over C_m. <u>Symplectic</u>.

Locally trivial symplectic fibre bundle

$$\Rightarrow \pi_1(C_m) \longrightarrow \mathrm{Aut}(X_{\underline{\lambda}}, \omega)$$

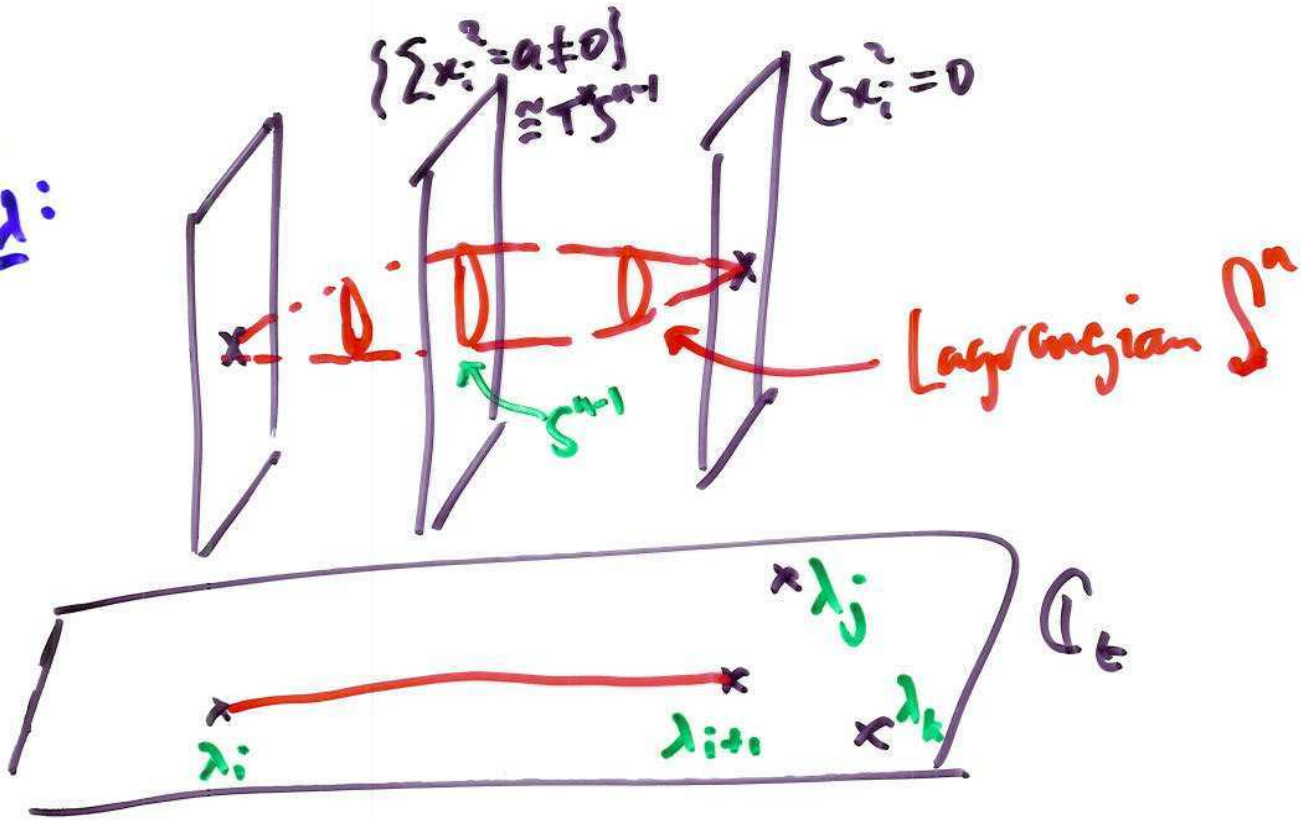
$$\pi_1(C_m) = B_{m-1}$$

$$\mathrm{Aut}(X_{\underline{\lambda}}, \omega) \xrightarrow{\text{grading...}} \mathrm{Aut}(\mathrm{Fuk}(X_{\underline{\lambda}}, \omega))$$

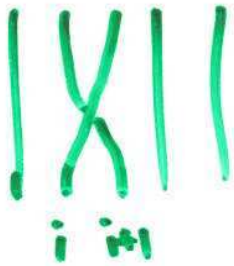
$$\mathrm{Aut}(\mathrm{Fuk}(X_{\underline{\lambda}}, \omega)) \xrightarrow{??} \mathrm{Aut}(D^b(\text{coh-mirror}))$$

Each $X_{\underline{\lambda}}$ fibred over $\mathbb{C}_t$ with fibres affine quadrics $\{\sum x_i^2 = p_{\underline{\lambda}}(t)\} \subseteq \mathbb{C}^n \times \mathbb{C}$

$X_{\underline{\lambda}}$:



ith generator T_i of B_{n-1}



acts by Dehn twist T_{L_i} about Lagrangian S^n fibred over path between λ_i and λ_{i+1}. Monodromy about X_{sdp} with $\lambda_i = \lambda_{i+1}$

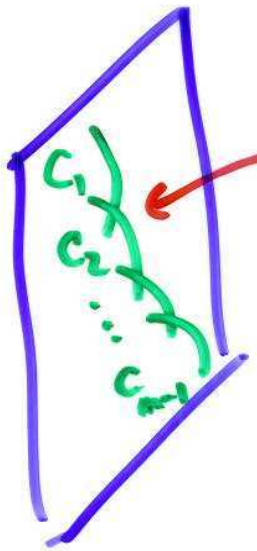
Dim $\underline{n=2}$. Hyperkähler!

Bring all λ_i together

$\Rightarrow$ Most singular fibre $X_0 = \{\sum x_i^2 = t^m\}$

A_{m-1} surface sing.

Resolution



A_{m-1}-chain of (-2) $\mathbb{P}^1$'s.

Fits into simultaneous resolution with $X_{\underline{\lambda}}$s after σ_m base change

(ie base = {distinct <u>ordered</u> $\lambda_i \in \mathbb{C}$})

using various ratios of factors in $xy = \prod_{i=1}^{m}(t-\lambda_i)$

$\hat{X}_0$ HK vol^n of $X_{\underline{\lambda}}$.

MCI

15. <u>Thm</u> $T_{\mathcal{O}_{c_i}} : D^b(\hat{X}_0) \circlearrowleft$ give repn:

$$B_{m-1} \to \mathrm{Auteq}(D^b(\hat{X}_0)).$$ (faithful)

ie $\begin{cases} T_i T_{i+1} T_i \cong T_{i+1} T_i T_{i+1} \\ T_i T_j \cong T_j T_i \quad |i-j|>1 \end{cases}$

<u>Eg</u> Matrices with distinct e/vals.

$$\begin{array}{ccc} \mathfrak{sl}(n,\mathbb{C}) & \supset & U = \{\text{distinct e/vals}\} \\ \downarrow & & \downarrow \\ \mathbb{C}^n/\sigma_n & \supset & C_0^n \end{array}$$

adjoint action orbits

characteristic polynomial

Fibres of U are regular orbits.

Symplectic. $B_{m-2} \to \mathrm{Aut}(U_{\geq}, \omega)$. cf. Seidel-Smith

Mirror Cautis-Kamnitzer

Elvals collide $\Rightarrow$ singular fibres / orbits.

Worst / central fibre = "<u>nilpotent cone</u>"

matrices w/ elvals 0.

Again, HK, $\exists$ simultaneous resolution,

Grothendieck

after σ_n basechange.

$$\begin{tikzcd}
\left\{ \begin{array}{l} A \in \mathfrak{sl}(n,\mathbb{C}) \\ \text{Flag } (V_0 \subseteq V_1 \subseteq \dots \subseteq V_n) \end{array} : AV_i \subseteq V_i \right\} \arrow[r, "A"] \arrow[d, "\text{elvals}"] & \mathfrak{sl}(n,\mathbb{C}) \arrow[d, "\text{char. poly.}"] \\
\mathbb{C}^n \arrow[r] & \mathbb{C}^n/\sigma_n
\end{tikzcd}$$

$\dim V_i = i$

smooth!

Over $U \subset \mathfrak{sl}(n,\mathbb{C})$, horiz. map is iso:

elvals distinct (+ ordered) so V_i uniquely

determined as $\langle v_1, \dots, i\text{th elvec} \rangle$.

17.

Fibre over (Nilpotent cone) $\to \{0\}$

is $\underline{T^*\text{Flag}}$, HK resol of nil. cone

and HK rotn of general fibre $U_{\underline{\lambda}}$.

Mirror!

Monodromy action $B_{n-2} \to \mathrm{Aut}(U_{\underline{\lambda}}, \omega)$

generated by $\underline{\text{fibred Dehn twists}}$

Locally $\mathbb{C}^k \times (T^*S^2)$

$\circlearrowleft$ Dehn twist

As $\lambda_i \to \lambda_{i+1}$ collapse to $\underline{\text{fibred odp}}$

Locally $\mathbb{C}^k \times$ (surface odp)

$\underline{\text{Mirror}}$: take limits of graphs as $\lambda_i \to \lambda_{i+1}$

(and then all $\lambda_i \to 0$)

18.

Get, in $T^*\mathrm{Flag} \times T^*\mathrm{Flag}$,

$$\underline{\Delta_{T^*\mathrm{Flag}} \cup (N_i \times_{K_i} N_i)}$$

Do FmT with structure sheaf

Here: $Fl_i = \{\text{Flags } V_1 \leq \dots \leq V_{i-1} \leq V_{i+1} \leq \dots \leq V_n = \mathbb{C}^n\}$

$\dim V_j = j$

$$\begin{array}{c} Fl \\ \downarrow \pi_i \\ Fl_i \end{array} \quad \text{gives} \quad \begin{array}{c} \pi_i^* T^* Fl_i \subset T^*\mathrm{Flag} \\ \downarrow \\ T^* Fl_i \end{array}$$

Call this
$$\begin{array}{c} N_i \xhookrightarrow{\iota_i} T^*\mathrm{Flag} \\ \downarrow p_i \\ K_i \end{array}$$

p_i is a $\mathbb{P}^1$ bdle, $\nu_{N_i}|_{\mathbb{P}^1} = \mathcal{O}(-2)$

K_i - locus of odps when contract $\mathbb{P}^1$'s

ie locally about N_i, $T^*\mathrm{Flag}$ looks like $\underline{T^*\mathbb{P}^1 \times K_i}$.

19. Do fibered version of $D^b(\text{surface})$ Dehn twist $T_{\mathcal{O}_C}$ about (-2)-$\mathbb{P}^1$ $C \subseteq$ Surface.

Horja, Szendrői

In that case
$$\begin{array}{ccc} C & \xhookrightarrow{\iota} & S \\ \downarrow p & & \\ \text{point} & & \end{array}$$

$$R\mathrm{Hom}(\mathcal{O}_C, \mathcal{E}) \otimes \mathcal{O}_C \xrightarrow{ev} \mathcal{E}$$

is $\iota_* p^* p_* \iota^! \mathcal{E} \longrightarrow \mathcal{E}$

Find that FMT with $\mathcal{O}_{\Delta \cup (N_i \times_{K_i} N_i)}(N_i, 0)$ is the functor

$$\mathcal{E} \longmapsto \mathrm{Cone}\left(\iota_{i*} p_i^* p_{i*} \iota_i^! \mathcal{E} \longrightarrow \mathcal{E}\right)$$

Call this $T_i : D^b(T^* \mathrm{Flag}) \circlearrowleft$

Thm

- T_i invertible — essentially 2d result
- $T_i T_j \cong T_j T_i \quad |i-j|>1$ — also easy
- $T_i T_{i+1} T_i \cong T_{i+1} T_i T_{i+1}$ — hard

Can extend to action of <u>affine</u> braid group.

Also to other groups - <u>Bezrukavnikov</u>, <u>Mirković</u>, <u>Rumynin</u>.

<u>Mirror Symmetry</u>

Smoothing of singularity $\leftrightarrow$ Resolution of (dual) singularity

vanishing cycle L — $\int_L \Omega$

exceptional set E — $\int_E e^{\omega}$

Hyperkähler case - self dual

Hyperkähler rotation: Slags $\rightsquigarrow$ Complex Lags

$D^b(\text{Fuk}) \rightsquigarrow$ Part of $D^b(\text{coh})$

21.

Braid cobordisms

$\exists$ more structure on {Functors $D^b(X) \circlearrowleft$}

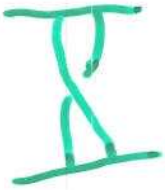
– natural transformations between them.

Homs between Fm kernels

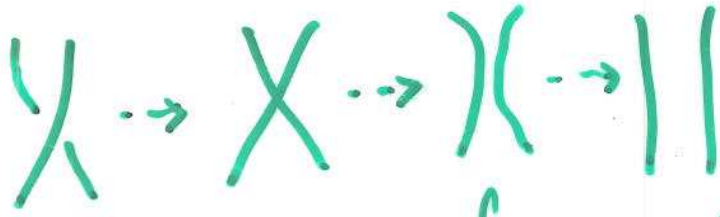
in $D^b(X \times X)$

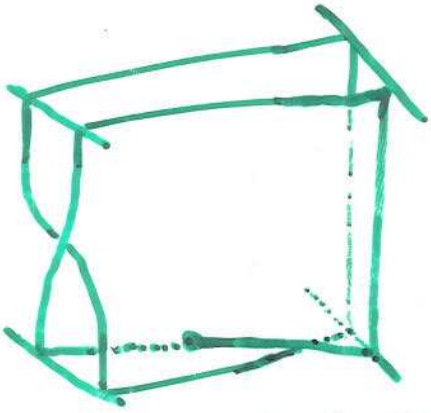
Can define a categorification of B_n action (eg on $D^b(X)$ or $K(X)$ $H^*(X)$) to be an action of the braid cobordism category

cf Rouquier

Objects: braids 

Morphisms: cobordisms





such simple

$\xrightarrow{\text{nat. transf.}}$ id

T_i defined as cone

$$\left(\iota_{i*} p_i^* p_{i*} \iota_i^! \longrightarrow \mathrm{id} \right)$$

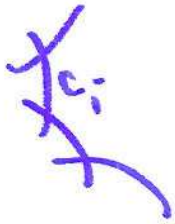
so has natural morphism from id.
nat. transf.

$\mathcal{O}_{\Delta \cup (N_i \times_{K_i} N_i)}(N_i, 0)$ has natural morphism

from $\mathcal{O}_\Delta$: $\quad 0 \to \mathcal{O}_\Delta \to \mathcal{O}_{\Delta \cup (N_i \times_{K_i} N_i)}(N_i, 0) \to \mathcal{O}_{N_i \times_{K_i} N_i}(N_i, 0) \to$

To positive simple braid cobordism $\| \to \times$
associate this nat. transf. $\mathrm{id} \to T_i$.

Negative nat. transf. trickier.

Describe in surface case  $T_i \to \mathrm{id}$

(T^*Flag is family version)

23.

$$\begin{tikzcd} T_i = \left(R\mathrm{Hom}(\mathcal{O}_c, \cdot) \otimes \mathcal{O}_c \arrow[r] \arrow[d] & \mathrm{id} \right) \arrow[d, "X_{i-1} - X_{i+1}"] \\ \mathrm{id} & \mathrm{id}[2] \end{tikzcd}$$

where $X_j \in R\mathrm{Hom}^2(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong H^2(X;\mathbb{C})$

is <u>dual</u> to $[C_j] \in H^2(X;\mathbb{C})$ <u>not</u> $[C_j]$!

(C_is span vector space $V \leq H^2(X;\mathbb{C})$ with non-degen bilinear form by intersection, giving iso $V \cong V^*$)

Show this defines nat. transf. and that all <u>Carter-Saito</u> movie-moves correspond to relns between these nat. transfs.

<u>Thm</u> The above defines an action of braid cobordism category on $D^b(T^*\mathrm{Flag})$.