P. Seidel — 24/1/07 — Homological algebra of Lefschetz fibrations

<u>Symplectic homology</u>: X (smooth complex) affine alg. var., $K_X \underset{top.}{\cong} \mathcal{O}_X$

Make X into a sympl. mfld: $\omega = -dd^c h$ (w/ contact boundary)

$\rightsquigarrow$ $SH_*(X)$ (∞-dim! usually) $\mathbb{Z}$-graded vector space

<u>Ex</u>:
- $SH_*(\mathbb{C}^n) = 0$
- $SH_*(X \times \mathbb{C}) = 0$ (subcritical case – Eliashberg)
- $SH_*((\mathbb{C}^*)^n) = H^{-*}(\mathcal{L}T^n)$ (cotangent bundle case – Viterbo etc... Schwarz ....)

$\uparrow$ free loop space

<u>NB</u>: not computable in general.

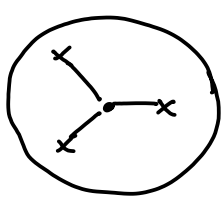
<u>Lefschetz fibrations</u>: $\pi : X \to \mathbb{C}$ affine Lefschetz fibration
(nondeg. critical pts, + condition on behavior at ∞)

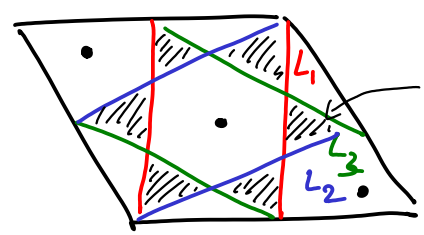
Fix a smooth fiber Y. For every critical point τ, choose a path
$\rightsquigarrow$ get a vanishing cycle = Lagrangian sphere $\subset Y$.

Fix a basis of such vanishing cycles, denote it by $(L_1, \dots, L_m) \subset Y$.

<u>Ex</u>: mirror of $\mathbb{CP}^2$: $X = (\mathbb{C}^*)^2$, $\pi(x,y) = x + y + x^{-1}y^{-1}$

Fiber Y = affine elliptic curve





Let $\mathcal{B}$ = subcat. of Fukaya category $\mathcal{F}(Y)$ with objects $L_1, \dots, L_m$.

$\mathcal{A} \subset \mathcal{B}$ directed subcategory

<u>So</u>: $\mathrm{Hom}_{\mathcal{A}}(L_i, L_j) = \begin{cases} 0 & i > j \\ \mathbb{C}.1_{L_i} & i = j \\ CF^*(L_i, L_j) & i < j \end{cases}$ vs. $\mathrm{Hom}_{\mathcal{B}}(L_i, L_j) = CF^*(L_i, L_j)$ $\forall i,j$

(⚠ case $i = j$ is tricky.)

<u>Ex</u>: quiver picture of $\mathcal{A}$:

$$\begin{tikzcd} L_1 \arrow[r, "V"] \arrow[rr, bend left, "\Lambda^2 V"] & L_2 \arrow[r, "V"] & L_3 \end{tikzcd}$$

$V \cong \mathbb{C}^3$

(compositions = the 6 discs in above picture
$\equiv$ each basis elt of $\Lambda^2 V$ can be written as a product in 2 ways, $e_i \wedge e_j = -e_j \wedge e_i$).

In this case, $\mathcal{A}$ is <u>formal</u>.

quiver picture of $\mathcal{B}$:

(symmetrization of the previous one)

<u>Actually</u> $H(\mathcal{B}) \simeq \Lambda^*(V) \rtimes \Gamma$ $\quad \Gamma = \mathbb{Z}/3$ acting diagonally on $V = \mathbb{C}^3$

(compositions are exterior products)

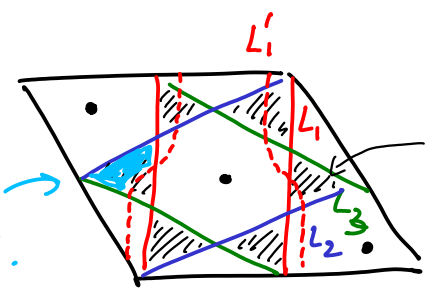
<u>But</u>: $\mathcal{B}$ is <u>NOT</u> formal — higher order products $\neq 0$ in $\mathcal{B}$

(so information is <u>lost</u> when forgetting "half" of $\mathcal{B}$)

Higher product in $\mathcal{B}$: $\underset{\hom_{\mathcal{B}}(L_3,L_1)}{\overset{\shortparallel}{V}} \otimes \underset{\hom_{\mathcal{B}}(L_2,L_3)}{\overset{\shortparallel}{V}} \otimes \underset{\hom_{\mathcal{B}}(L_1,L_2)}{\overset{\shortparallel}{V}} \xrightarrow{\mu^3_{\mathcal{B}}} \underset{\hom_{\mathcal{B}}(L_1,L_1)}{\overset{\cap}{\Lambda^0 V}}$

$\forall v \in V$ basis elt, $\quad \mu^3_{\mathcal{B}}(v,v,v) = v_1 v_2 v_3$

<u>NB</u>: in gd $\mu^3_{\mathcal{B}}$ is ambiguous (choices, as in Massey products), but since $v \wedge v = 0$ this one is well-defined



$\exists!$ 4-gon with sides in order L_1, L_2, L_3, L_1'.

<u>Mirror symmetry interpretation</u>:

$$D(\mathcal{A}) = H^0(\text{mod-}\mathcal{A}) \simeq D^b\,\mathrm{Coh}(X^*) \qquad X^* = \mathbb{P}^2$$

$$D(\mathcal{B}) = H^0(\text{mod-}\mathcal{B}) \subseteq D^b\,\mathrm{Coh}(Y^*)$$

full subcat. (essentially, restrictions to Y^* of $\mathrm{Coh}(X^*)$)

$Y^* =$ Calabi-Yau hypersurf. in X^*

$Y^* = \{x_1 x_2 x_3 = 0\} \subset X^*$

Information is lost from $\mathcal{B}$ to $\mathcal{A}$ because $\mathcal{A}$ doesn't remember <u>which</u> CY hypersurface $Y^* \subset X^*$ we were looking at.

Back to homological algebra---

- we have a short exact sequence of $\mathcal{A}$-bimodules
  ($\triangle$ A_∞-bimodules)

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A} \to 0$$
$$\mathcal{B}/\mathcal{A} \cong \mathcal{A}^\vee[-\dim_{\mathbb{C}} Y]$$

(because of symmetry property of $\mathcal{B}$ – "it's a CY A_∞-cat.")

There is a <u>boundary map</u> $\mathcal{B}/\mathcal{A}[-1] \cong \mathcal{A}^\vee[-\dim_{\mathbb{C}} X] \xrightarrow{\delta} \mathcal{A}$

(take a splitting $\mathcal{B} \underset{\sigma}{\rightleftarrows} \mathcal{B}/\mathcal{A}$, it's not a module hom., but $d(\sigma)$ is a module hom. and lands in $\mathcal{A}$)

- Every $\mathcal{A}$-bimodule $\mathcal{P}$ gives rise to a <u>convolution functor</u>
  $\Phi_{\mathcal{P}} : D(\mathcal{A}) \to D(\mathcal{A})$
  - $\mathcal{P} = \mathcal{A} \rightsquigarrow \Phi_{\mathcal{P}} \cong Id$
  - $\mathcal{P} = \mathcal{A}^\vee \rightsquigarrow \Phi_{\mathcal{P}} \cong S$ Serre functor

We also have, for every morphism of $\mathcal{A}$-bimodules, a nat transf.

here, $\mathcal{B}/\mathcal{A}[-1] \xrightarrow{\delta} \mathcal{A}$ induces $\underline{\underline{\Phi_\delta : S[-\dim_{\mathbb{C}} X] \to Id}}$

(admits a natural interpretation in a Lefschetz fibration)

- <u>Mirror symm. interpretation:</u> $D(A) = D^b Coh(X^\alpha)$

$$S = (K_{X^\alpha} \otimes -)[\dim_{\mathbb{C}} X^\alpha]$$

Nat transform^{ns} $S[-\dim X] \to id \iff$ sections of $K_{X^\alpha}^{-1}$

In our case, $\Phi_\delta \longleftrightarrow x_1 x_2 x_3 \in H^0(K_{X^\alpha}^{-1})$.

- <u>The curved A_∞-category $\mathcal{D}$:</u>

$Ob\, \mathcal{D} = \{L_1, \dots, L_m\}$

$$hom_{\mathcal{D}}(L_i, L_j) = hom_{\mathcal{A}}(L_i, L_j) \oplus t.hom_{\mathcal{B}}(L_i, L_j)[[t]]$$

$\deg t = 2$.

(i.e. formal power series w/ t^0-coeff$^t \in \mathcal{A}$)

extend $\mu^k_{\mathcal{B}}$ t-linearly to $\mathcal{D}$ (ok since $\mathcal{A}$ subcat. $\to t^0$-terms still $\in \mathcal{A}$)

+ add a <u>curvature term</u> $\mu^0_{\mathcal{D}} = t.1_{x_i} \in \hom^2_{\mathcal{D}}(L_i, L_i)$.

$\mathcal{D}$ is a curved A_∞-category.

Symplectically, "$\mathcal{D}$ represents the total space of the Lefschetz fibration"
Mirror-wise, "$\mathcal{D}$ represents the complement of the CY divisor Y^α inside X^α".

<u>Thm:</u> $D(\mathcal{D}) = D(\mathcal{A})/D(\mathcal{B})$

($\mathcal{A} \subset \mathcal{B} \Rightarrow \rho : D(\mathcal{B}) \to D(\mathcal{A})$ (view $\mathcal{B}$-mod. as $\mathcal{A}$-mod)
$\to$ Verdier quotient of tri. cats:
localize along $\mathrm{Im}(\rho)$).

<u>Thm:</u> $HH_\alpha(\mathcal{D}) = \varprojlim HH_\alpha(\mathcal{A}, (\mathcal{B}/\mathcal{A})^{\otimes_{\mathcal{A}} p})$

where connecting map $(\mathcal{B}/\mathcal{A})^{\otimes_{\mathcal{A}} p} \longrightarrow (\mathcal{B}/\mathcal{A})^{\otimes_{\mathcal{A}} p-1}$
is induced by $\mathcal{B}/\mathcal{A} \xrightarrow[\delta]{} \mathcal{A}$ on (any) one of the factors

<u>Hochschild homology</u>: (given chains, d_i: ⊖ collapse by inserting a $\mu^k_{\mathcal{D}}$
⊕ extend by inserting a $\mu^0_{\mathcal{D}} = t.1_{L_i}$)

<u>Conj:</u> $HH_\alpha(\mathcal{D}) = SH_\alpha(X)$.

- <u>Mirror:</u>

$$HH_\alpha(\mathcal{D}) = \varprojlim \mathrm{Ext}^*_{X^\alpha \times X^\alpha}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \otimes K_{X^\alpha}^{\otimes p})$$

$$= \ldots \text{ (Koszul duality) } \ldots$$

$$\cong \varinjlim H^{-*}(X^\alpha, \Lambda^{-*} T_{X^\alpha} \otimes K_{X^\alpha}^{1-p})$$

Connecting map $= \otimes$ section $x_1 x_2 x_3$

$$\cong \Gamma(U^*, \Lambda^{-*} T_{U^\alpha})^\vee$$

where $U^\alpha = X^\alpha \setminus Y^\alpha$.

($\varinjlim$ allows sections of $\Lambda^{-*}T$ with poles of arbitrarily high order along Y^α).

Computation on case of $\mathbb{P}^2 \rightsquigarrow$

get $HH_*(\mathcal{D}) = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \theta_1, \theta_2]^V$

$$\cong$$

$$SH^*((\mathbb{C}^*)^2) = H^{-*}(\mathcal{L}T^2)$$

✓