The goal is to sktech the definition of the Fukaya category of a symplectic manifold. In the context of HMS this is the category of D-branes of type A in a σ model. Let us start without a B-field for now.

0.1. **Lagrangian Floer homology.**

Definition 0.1. *If (M^{2n}, ω) is a symplectic manifold then $L \subset M$ is Lagrangian if $\omega|_L = 0$. In other words it's maximal isotropic.*

The standard example is $\mathbb{R}^n \subset \mathbb{C}^n$, with $\omega = \sum dx_i \wedge dy_i$.

Another example is Σ a surface equipped with an area form. Any simple closed curve is a Lagrangian.

Let L and L' be lagrangians. Floer homology assigns to them a group $HF^*(L, L')$ which is $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ graded depending on the context. This group measures quantum intersections of L and L', corrected by pseudo-holomorphic disc.

The main property is invariance under Hamiltonian isotopies.

$$HF(L, \phi(L) \cong HF(L, L')$$

for $\phi \in \mathrm{Ham}(M, \omega)$.

For us, this will imply that Hamiltonian isotopic Lagrangians are quasi-isomorphic objects in the Fukaya category. So we'll only need to consider Lagrangians up to Hamiltonian equivalence.

0.1.1. *Assumptions.*

- L and L' intersect transversely
- $\pi_2(M, L) = \pi_2(M, L') = 0$. This was the setup of FLoer, which was weakened later by Oh, then finally by FOOO which is the most general setting in which we can do this.

Floer homology is the Morse homology of the space of path from L to L'. There are difficulties making sense of this rigourously. Rather, we consider the vector space with basis given by intersection points of L and L'.

$$CF^*(L, L') = \mathbb{C}^{L \cup L'}$$

This admits a relative grading ($\deg p - \deg q$ is well defined for any intersection points p and q) over $\mathbb{Z}/N\mathbb{Z}$ given by the Maslov index.

This complex carries a differential *partial* which increases the degree by 1, and which counts pseudo-holomorphic dics with respect to an almost complex structure which is compatible with ω. This is an extra structure that needs to be chosen, and it is often easiest to pick a generic such J.

We then consider

$$u\colon\ D^2 \to M$$

such that $du \circ j = J \circ du$ and, in addition $u(-1) = q$, $u(1) = p$, and the boundary of the disc is mapped to $L \cup L'$.

The space of such maps is goverened by a Fredholm operator, so, for a generic choice of J, together with some technical assumptions, we can guarantee that the space of such maps is a manifold

$$\mathcal{M}_{p,q} = \text{solutions}/\mathbb{R} = \text{Aut}(D^2 \pm 1)$$

of dimension $\deg(q) - \deg(p) - 1$. This dimension is the Maslov index of a disc uu:

In other words, we can consider $u^*(TM) \cong \mathbb{C}^n \times D^2$. Along ∂D^2, we have a Lagrangian subbunlde by tracing out the tangent space of L and L' along the boundary, then interpolating at the intersection points p and q. This yields an element of $\pi_1(LGR(n)$ which happens to be isomorphic to $\mathbb{Z}$.

Note that, in general, this index depends on ∂u as well as on p and q.

The definition is now

$$\partial(p) = \sum_{q \in L \cup L'} n_{p,q} q$$

where

$$n_p, d = \sum_{u \in \mathcal{M}_{p,q} | \dim(\mathcal{M}_{p,q})=1} \pm exp(-\int_{D^2} u^*(\omega)$$

The signs in $n_{p,q}$ only make sense if L and L' are oriented and relatively spin. Or, rather, if the choice of orientations and relative spin structure are chosen.

A priori, there is no reason for the series to converge, although there are no known counter examples. MAthematically, one may want to work with Novikov rings.

Thm(Floer) When $\pi_2(M, L) = \pi_2(M, L') = 0$, the $\partial^2 = 0$. We define

$$HF^*(L, L') = H^*(CF^*(L, L'), \partial)$$

0.1.2. *Counter example.* Given $\mathbb{R} \times S^1$. Consider an essential and an inessential circle which intersect in two poijnts p and q. One can check that there is a holomorphic disc from p to q, and one from q to p.. In particular, $\partial^2 \neq 0$.

$CF^*(L, L')$ can be defined using singular chains on L and modifing the differential by counting holomorphic discs passing through a chain, or by doing a similar modification of the Morse complex. Finally, we can compute $CF(L, \phi(L))$ for a Hamiltonian perturbation ϕ.

0.2. **Operations in Floer Homology:**

0.2.1. *Product.*

$$CF(L_0, L_1) \otimes CF(L_1, L_2) \to CF(L_0, L_2)$$

$$p \otimes q \mapsto \sum_r n_{p,q,r} r$$

where $n_{p,q,r}$ is a count of holomorphic maps from the 2-disc with 3 marked poijnts which go respectively to p, q and r, and the boundaries are constrained to map to L_0, L_1 and L_2. One can check that, ignoring issues of non-well definedness of degree that

$$\dim \mathcal{M}_{p,q,r} = \deg r - (\deg p + \deg q)$$

Theorem 0.2. *In favourable cases, the product is associative at the cohomology level, and the LEibniz rule holds on the chain level.*

0.2.2. *Higher Products.* Say$L_1, \cdots L_k$ are transverse. We define a mpa

$$CF(L_0, L_1) \otimes \cdots \otimes CF(L_{k-1,k}) \to^{m_k} CF(L_0, L_k)$$

by counting pseudo-holomorphic maps from a disc with "arbitrary marked points on the boundary" to M, such that the boundary is mapped to the LAgrangians L_i.

From this, we get an A_∞ algebra on $CF(L, L)$: This means that a sequence of equation of the form

$$\sum_{k+l=n} (-1) m_k(a_0, \cdots, a_l, m_l(a_{l+1}, \cdots, a_{i+l}), a_{i+l+1}, \cdots, a_n)$$

where $m_1 = \partial$ and m_2 is the product defined above.

If there are holomorphic discs with boundary on L or L', then we get a "curvature" term $m_0 \in CF^*(L, L)$. This counts the space of discs with boundary on L and L'.

For example, $m_1^2 = m_2(m_{0L}, \_) - m_2(\_, m_{OL'})$

0.3. **Twisted Floer Theory.** We may also choose a B-field, i.e. a closed real 2-form.

If B vanishes, we equip L with a flat unitary complex vector bundle (E, ∇). Otherwise, we require that the curvature $R^\nabla = -iBId$.

This allows us to modify the theory by defining

$$CF^*((L, E, \nabla), (L', E', \nabla')) = \bigoplus_{p \in L \cup L'} (E_p^*, \otimes E'_p)$$

In addition, we change the differential by weighting the contribution of each disc u by the complexified area and the holonomy around the boundary.

For example, having chosen p, q, and r intersection between L_0, L_1, and L_2, together with choices $\phi_p \in {E_0}_p^*, \otimes E_{1p}$ and $\phi_q \in {E_1}_p^*, \otimes E_{2q}$

then the contribution of a disc u passing through p, q, and r to $m_2(p,q)$

$$\pm exp(i \int_{D^2} u^*(B + i\omega)) \cdot (\text{hol}^{\nabla_2}\, \phi_q\, \text{hol}^{\nabla_1}\, \phi_p\, \text{hol}^{\nabla_0})$$

where the holonomy maps correspond to parallel transport along the boundary of u.

In good cases, we therefore get the Fukaya category; an A_∞ category with objects $\mathcal{L} = (L, E, \nabla)$. We then take the derived category:

1) Make an additive enlargement (i.e: add formal direct sums of objects, although direct sum of an object with itself corresponds to a rank 2 trivial vector bundle)

2) Add twisted complexes $\mathcal{E} = \oplus_{finite}\mathcal{L}_i[n_i]$ together with a choice of δ which is a degree 1 morphism

3) The derived category is the homology of "this category." This is written $H^0(Tw)$ and is an honest triangulated category.

Morally, a cone between two smooth Lagrangians corresponds to a smoothing of the corresponding intersection.

4) Karoubi completion: One may want to look at direct summands of objects in the Fukaya category. Formally, whenever we have an idempotent we add a Kernel anda cokernal.