The Mixed Boundary Value Problem in Lipschitz Domains

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Classical Boundary Value Problems for the Laplacian

Dirichlet Problem:

$$(D) \quad \begin{cases} u \in \mathcal{C}^2(\Omega), \\ \triangle u = 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = f \in L^p(\partial\Omega). \end{cases}$$

Neumann Problem:

$$(N) \quad \begin{cases} u \in \mathcal{C}^2(\Omega), \\ \triangle u = 0 \ \text{ in } \ \Omega, \\ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = f \in L^p_0(\partial\Omega), \end{cases}$$

where ν denotes the outward unit normal vector.

Function Spaces

Definition. $L^p(\partial\Omega)$, $1 < p < \infty$ is the Lebesgue space of p-integrable functions on $\partial\Omega$,

$$L^p(\partial\Omega) := \left\{ f : \left(\int_{\partial\Omega} |f|^p d\sigma \right)^{1/p} < +\infty \right\},$$

where $d\sigma$ denotes surface measure on $\partial\Omega$.

Further, define

$$L^p_0(\partial\Omega) := \left\{ f \in L^p(\partial\Omega);\ \int_{\partial\Omega} f d\sigma = 0 \right\},$$

and

$$L^p_1(\partial\Omega) := \left\{ f \in L^p(\partial\Omega);\ \partial_\tau f \in L^p(\partial\Omega) \right\}.$$

Lipschitz Domains

- A function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is **Lipschitz** if there exists a constant $M > 0$ such that for any x, y in the domain of ϕ,

$$|\phi(x) - \phi(y)| < M|x - y|.$$

- Ω is a **Lipschitz domain** if $\partial\Omega$ locally given by the graph of a Lipschitz function ϕ.

History

- B. Dahlberg [1977,1979], E. Fabes, M. Jodeit, N. Riviere [1978]: (D) is well-posed $\forall\, p \in (1, \infty)$ in the class of **smooth** domains.
- B. Dahlberg [1977,1979]: (D) is well-posed $\forall\, p \in [2, \infty)$ in the class of **Lipschitz** domains. This range is sharp.
- B. Dahlberg, C. Kenig [1987]: (N) is well-posed $\forall\, p \in (1, 2]$ in the class of **Lipschitz** domains. This range is sharp.
- C. Kenig [1984]: Counterexamples.

The Mixed Problem for the Laplacian

- Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.
- Split the boundary of the domain $\partial\Omega$ into a Dirichlet and Neumann portion so that

$$\partial\Omega = D \cup N, \quad D \subset \partial\Omega \text{ and } N = \partial\Omega \setminus \overline{D}.$$

- Assume $D \subset \partial\Omega$ is relatively open, denote by Λ the boundary of D (with respect of $\partial\Omega$).

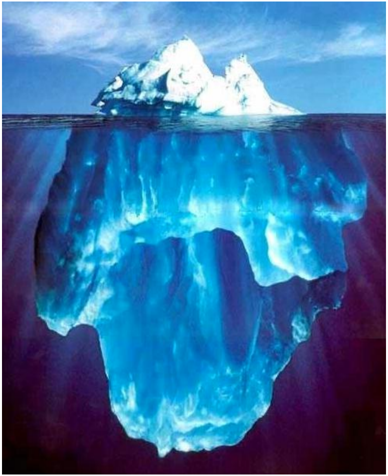
$$(MP) \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f_D & \text{on } D \\ \frac{\partial u}{\partial \nu} = f_N & \text{on } N \end{cases}$$

where, as before, ν denotes the outward unit normal vector on $\partial\Omega$.

Motivation for Studying (MP)

Example 1: Iceberg

- Consider an iceberg Ω partially submerged in water.
- Solution to (MP), $u(x)$, is the temperature at each point $x \in \Omega$.
- D is the portion of $\partial\Omega$ *underneath* the waterline. Here, Ω behaves like a thermostat so **Dirichlet** boundary conditions are imposed.
- N is the portion of $\partial\Omega$ *above* the waterline. Here, Ω acts like an insulator so **Neumann** boundary conditions are imposed.



Motivation for Studying (MP)

Example 2: Metallurgical Melting

- Ω is the cross section of an infinitely long solid with thermal sources located within.
- $u(x)$ is the temperature of the solid at each point $x \in \Omega$.
- $\partial\Omega = \Gamma_1 \cup \Gamma_2$.
- On Γ_1, u is cooled to 0 by a distribution of heat sinks.
- On Γ_2, the heat u is leaving through Γ_2 at a steady rate g.

- Mathematical model takes the form

$$\begin{cases} \Delta u = \rho \quad \text{in } \Omega \\ u|_{\Gamma_1} = 0 \\ \frac{\partial u}{\partial \nu}|_{\Gamma_2} = g \end{cases}$$

- Above, ρ is a source function capturing the input of energy into Ω.

The Mixed Problem for the Laplacian

$$(MP) \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f_D & \text{on } D \\ \frac{\partial u}{\partial \nu} = f_N & \text{on } N \end{cases}$$

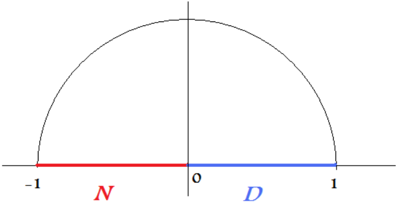
- **Goal**. Given than f_D is in a certain function space on D, and $\frac{\partial u}{\partial \nu} = f_N$ is in a certain function space on N, deduce information about ∇u on the whole boundary $\partial\Omega$.
- Via trace theorems obtain results for ∇u on Ω.

An Example

- **Expectation.**

$$(MP)\quad \begin{cases} \Delta u = h \quad \text{in } \Omega \\ u|_D = f_D \ \in L^2_1(\partial\Omega) \\ \frac{\partial u}{\partial \nu}|_N = f_N \ \in L^2(\partial\Omega) \end{cases} \quad \Rightarrow \quad \nabla u \in L^2(\partial\Omega).$$

- In the setting of (MP), our intuition that a smooth boundary is better does not hold.
- **Counterexample**. Let $\Omega \subset \mathbb{R}^2$, $\Omega := \{(x, y) : x^2 + y^2 < 1, y > 0\}$.



- Take $u(x, y) = \text{Im}\,(x + iy)^{1/2}$.
- In polar coordinates, $u(x, y) = U(r, \theta) = r^{1/2} \sin(\theta/2)$.

An Example, continued

$$u(x, y) = U(r, \theta) = r^{1/2} \sin(\theta/2)$$

- Calculus:

$$\begin{aligned}\Delta u(x, y) &= \tfrac{\partial^2 u}{\partial x^2} + \tfrac{\partial^2 u}{\partial y^2}, \\ &= \tfrac{\partial^2 U}{\partial r^2} + \tfrac{1}{r}\tfrac{\partial U}{\partial r} + \tfrac{1}{r^2}\tfrac{\partial^2 U}{\partial \theta^2}.\end{aligned}$$

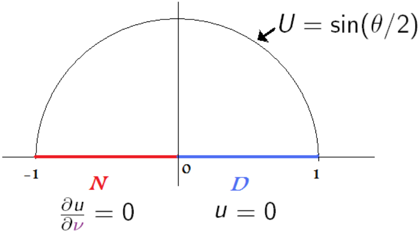
- Then $\Delta u = 0$ in Ω.
- More calculus: $\frac{\partial u}{\partial \nu} = \frac{\partial U}{\partial \theta} \cdot \frac{1}{r}$, so

$$\frac{\partial u}{\partial \nu}\Big|_N = r^{-1/2} \cos(\frac{\theta}{2}) \cdot \frac{1}{2}\Big|_N = 0.$$

An Example, continued

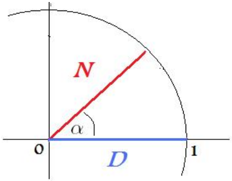
$$u(x,y) = U(r,\theta) = r^{1/2}\sin(\theta/2)$$

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u|_D = 0 \\ \frac{\partial u}{\partial \nu}|_N = 0 \end{cases}$$



- u satisfies $u|_D \in L^2_1(D)$, $\frac{\partial u}{\partial \nu}|_N \in L^2(N)$.
- **However.** $\nabla u \sim r^{-1/2}$ which is not in $L^2(\partial\Omega)$. Problem is at the origin.

More General Example



- $u(x,y) = U(r,\theta) = r^{\beta} sin(\beta\theta)$. Then $\Delta u = 0 \in \Omega$ and $u|_{D} = 0$.
- $\frac{\partial u}{\partial \nu}|_{N} = (\frac{\partial U}{\partial \theta} \cdot \frac{1}{r})|_{N}$, so $\frac{\partial u}{\partial \nu}|_{N} = r^{\beta-1}\cos(\beta\alpha)\beta$. In order for $\frac{\partial u}{\partial \nu}|_{N} = 0$, need $\beta\alpha = \frac{\pi}{2} \Rightarrow \beta = \frac{\pi}{2\alpha}$.
- Further, $\nabla u \sim r^{\beta-1} = r^{\frac{\pi}{2\alpha}-1}$, so $\nabla u \in L^2(\partial\Omega)$ whenever $(\frac{\pi}{2\alpha} - 1)2 > -1$. In other words, when $\pi > \alpha$.
- Leads to the study of (MP) is **creased domains**.

Some History of (MP)

- Sevare [1997]: Ω a smooth domain, then solution u of (MP) lies in the Besov space $B_{\infty}^{3/2,2}(\Omega)$.
- Brown [1994]: Ω a *creased domain*, $f_D \in L^2_1(\partial\Omega)$, $f_N \in L^2(\partial\Omega)$, then there exists a unique solution u with $(\nabla u)^* \in L^2(\partial\Omega)$. Results extended with J. Sykes [1999] to $L^p(\partial\Omega)$, $1 < p < 2$.
- Brown, Capgona and Lanzani [2008]: Ω a Lipschitz graph domain in two dimensions with Lipschitz constant $M < 1$, solutions in $L^p(\partial\Omega)$ for $1 < p < p_0$ with $p_0 = p_0(M) > 1$.
- Venouziou and Verchota [2008]: $L^2(\partial\Omega)$ results for (MP) for certain polyhedra in $\mathbb{R}^3$.

The Mixed Problem with Atomic Data

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain.

$$(MP_a) \qquad \begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u = 0 \text{ on } D, \\ \frac{\partial u}{\partial \nu} = a \text{ atom for } N. \end{cases}$$

- a is an **atom for** $\partial\Omega$ if:
  - $\operatorname{supp} a \subset \Delta_r(x)$ for some $x \in \partial\Omega$, where $\Delta_r(x) = B_r(x) \cap \partial\Omega$,
  - $||a||_\infty \leq 1/\sigma(\Delta_r(x))$,
  - $\int_{\partial\Omega} a d\sigma = 0$.
- a is an *atom for* N if a is the restriction to N of a function $\bar{a}$ which is an atom for $\partial\Omega$.
- $H^1(N)$ is the collection of functions f which can be represented as $\Sigma_j \lambda_j a_j$, where each a_j is an atom for N and $\Sigma_j |\lambda_j| < \infty$.

The Fundamental Estimate

Recall $\Delta_r(x) = B_r(x) \cap \partial\Omega$ and let $\Sigma_k = \Delta_{2^k r}(x) \setminus \Delta_{2^{k-1} r}(x)$.

Theorem 1, R. Brown, KO

Let u be a weak solution of (MP_a) with data $f_N = a$ an atom for N which is supported in $\Delta_r(x)$ and $f_D = 0$. There exists $q > 1$ such that the following estimates hold

$$\left(\int_{\Delta_r(x)} |\nabla u|^q d\sigma \right)^{1/q} \leq C\sigma(\Delta_{8r}(x))^{-1/q'},$$

$$\left(\int_{\Sigma_k} |\nabla u|^q d\sigma \right)^{1/q} \leq C2^{-\alpha k}\sigma(\Sigma_k)^{1/q'}, \quad k \geq 3.$$

Here, C, q and α depend only on the Lipschitz character of Ω.

$L^1(\partial\Omega)$ Estimates for (MP)

Theorem 2, R. Brown, KO

Let u be a weak solution of the mixed problem with $f_D = 0$ and $f_N = a$, where a is an atom for the Hardy space $H^1(N)$. Then u satisfies

$$||(\nabla u)^*||_{L^1(\partial\Omega)} \leq C.$$

Theorem 3, R. Brown, KO

Let u be a weak solution of (MP) with $f_D \in H^1_1(D)$ and $f_N \in H^1(N)$. Then u satisfies

$$||(\nabla u)^*||_{L^1(\partial\Omega)} \leq C\left(||f_D||_{H^1_1(D)} + ||f_N||_{H^1(N)}\right).$$

Results for (MP) in Other Function Spaces

- **Goal**. Extend Theorem 3 to $L^p(\partial\Omega)$, $p \in [1, 1+\epsilon)$.
- That is, wish to prove an estimate of the form

$$||(\nabla u)^*||_{L^p(\partial\Omega)} \leq C \left(||f_D||_{L^p_1(D)} + ||f_N||_{L^p(N)} \right).$$