

SECTOR ESTIMATES FOR HYPERBOLIC ISOMETRIES

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ABSTRACT. We prove various orbital counting statements for Fuchsian groups of the second kind. These are of independent interest, and also are used in the companion paper [BK09] to produce primes in the Affine Linear Sieve.

CONTENTS

1.	Introduction	1
2.	Preliminaries	6
3.	Proof of Theorem 1.5	19
4.	Proof of Theorem 1.13	23
5.	Proof of Theorem 1.14	28
6.	Proof of Theorem 1.15	29
	References	31

1. INTRODUCTION

Let $\Gamma < G = \mathrm{PSL}(2, \mathbb{R})$ be a non-elementary, geometrically finite group of isometries of the upper half plane \mathbb{H} . In the case when Γ has fundamental domain with finite hyperbolic area, much effort has gone into understanding the asymptotic behavior of the number of points in a Γ -orbit which lie in an expanding region inside G , e.g. [Sel65, LP82, Goo83a, DRS93, EM93, Kon09, KO09]. Such regions can be decomposed into their harmonics, and hence one can recover many counting statements from, say, taking the Cartan decomposition

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$G = KA^+K$ and studying the set of $\gamma \in \Gamma$ in a ball of expanding radius with a given harmonic on right and left K -types. Here

$$K = \text{SO}(2) = \left\{ k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

and

$$A^+ = \left\{ a_t = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}, t \geq 0 \right\}.$$

The state of the art in this direction in the finite-volume case is due to Good [Goo83a]. Let $0 \leq \theta_1(g), \theta_2(g) < \pi$ and $t(g) > 0$ be the functions in the Cartan decomposition of G , so that $g = \pm k_{\theta_1(g)} a_{t(g)} k_{\theta_2(g)}$.¹

Theorem 1.1 ([Goo83a], Thm 4). *Let $\Gamma < G$ be a lattice and fix integers n and k . Let*

$$0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_N < 1/4$$

be the eigenvalues of the Laplacian on $\Gamma \backslash \mathbb{H}$ below $1/4$. Write

$$\lambda_j = s_j(1 - s_j) \tag{1.2}$$

with $s_j > 1/2$. Then there are constants $c_0, \dots, c_N \in \mathbb{C}$ depending on n and k such that

$$\sum_{\substack{\gamma \in \Gamma \\ \|\gamma\| < T}} e^{2in\theta_1(\gamma)} e^{2ik\theta_2(\gamma)} = \mathbf{1}_{n=0} \mathbf{1}_{k=0} c_0 T^2 + \sum_{j=1}^N c_j(n, k) T^{2s_j} + O_{n,k}(T^{4/3}),^2 \tag{1.3}$$

as $T \rightarrow \infty$.

Our main goal is to give a version of the above in the case when Γ is again non-elementary and geometrically finite, but whose fundamental domain has infinite area. First we recall the spectral theory in this context. Given any fixed point $z \in \mathbb{H}$, the orbit Γz accumulates only on the boundary $\partial\mathbb{H} \cong \widehat{\mathbb{R}}$. The set Λ of accumulation points is called the limit set of Γ , and is a Cantor-like set having some Hausdorff dimension

$$0 < \delta < 1,$$

which is called the critical exponent of Γ . The spectrum above $1/4$ is purely continuous and there are finitely many discrete eigenvalues

¹ These are determined as follows. Assume that $g \notin K$. Define $t(g)$ as the solution to $\|g\|^2 = e^t + e^{-t} = 2 \cosh t$ with $t > 0$. Let $\theta_1(g)$ be defined as $\frac{1}{2} \arg \left(\frac{g \cdot i - i}{g \cdot i + i} \right)$ (that is, map g to the unit disk \mathbb{D} ; the argument of the image determines θ_1). Then $a_{t(g)}^{-1} k_{\theta_1(g)}^{-1} g$ is in K , whence θ_2 is determined.

² The exponent $4/3$ has never been improved, even for any specific Γ .

below $1/4$ [Pat75]. In fact, the spectrum contains no point eigenvalues at all unless $\delta > 1/2$, in which case the base eigenvalue is [Pat76]

$$\lambda_0 = \delta(1 - \delta).$$

Corresponding to λ_0 is the base eigenfunction φ_0 , which can be realized explicitly as the integral of a Poisson kernel against the so-called Patterson-Sullivan measure μ [Pat76, Sul84], supported on the limit set $\Lambda \subset \partial\mathbb{H}$. Roughly speaking, μ is the weak* limit as $s \rightarrow \delta^+$ of the measures

$$\mu_s(x) := \frac{\sum_{\gamma \in \Gamma} \exp(-s d(\mathfrak{o}, \gamma \cdot \mathfrak{o})) \mathbf{1}_{x=\gamma\mathfrak{o}}}{\sum_{\gamma \in \Gamma} \exp(-s d(\mathfrak{o}, \gamma \cdot \mathfrak{o}))}. \quad (1.4)$$

Here $d(\cdot, \cdot)$ is the hyperbolic distance, and \mathfrak{o} is the origin (or any base point) in \mathbb{D} .

Let $\widehat{\mu}$ denote the Fourier coefficients of the measure μ . Our first result is

Theorem 1.5. *Let Γ be a Fuchsian group of the second kind with critical exponent $\delta > 1/2$. Let*

$$0 < \delta(1 - \delta) = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_N < 1/4 \quad (1.6)$$

be the eigenvalues of the Laplacian on $\Gamma \backslash \mathbb{H}$ below $1/4$, and use the notation (1.2). Then for integers n and k , there are constants $c_1, \dots, c_N \in \mathbb{C}$ depending on n and k such that

$$\begin{aligned} \sum_{\substack{\gamma \in \Gamma \\ \|\gamma\| < T}} e^{2in\theta_1(\gamma)} e^{2ik\theta_2(\gamma)} &= \widehat{\mu}(2n)\widehat{\mu}(2k) \pi^{1/2} \frac{\Gamma(\delta - 1/2)}{\Gamma(\delta + 1)} T^{2\delta} + \sum_{j=1}^N c_j(n, k) T^{2s_j} \\ &\quad + O\left(T^{1 \cdot \frac{1}{4} + 2\delta \cdot \frac{3}{4}} (\log T)^{1/4} (1 + |n| + |k|)^{3/4}\right), \end{aligned}$$

as $T \rightarrow \infty$. Here $|c_j(n, k)| \ll |c_j(0, 0)|$, as n and k vary, and the implied constants depend only on Γ .

Remark 1.7. We make no attempt to obtain a best-possible error term; the above can surely be improved with some effort. The natural remainder term here would be the one which corresponds to Lax-Phillips [LP82] when $n = k = 0$, namely $T^{1 \cdot \frac{2}{3} + 2\delta \cdot \frac{1}{3}}$, ignoring logs.

Remark 1.8. It is crucial for our intended applications below and in [BK09] (for reasons of positivity) that the leading order term be recognized in terms of the Patterson-Sullivan measure; this is why we made the constant $c_0(n, k)$ completely explicit.

Remark 1.9. In the absence of an explicit spectral expansion into Maass forms and Eisenstein series, we control the above error term using representation theory, smoothing the counting function in two copies of $\Gamma \backslash G$ and appealing to the decay of matrix coefficients [HM79, Cow78]. This technique dates at least as far back as [DRS93].

In the intended applications in the companion paper [BK09], one requires the above with uniform control on cosets of congruence subgroups. Recall the spectral gap property in the infinite volume situation. Assume $\Gamma < \mathrm{SL}(2, \mathbb{Z})$. There is a fixed integer $\mathfrak{B} \geq 1$ called the *ramification number* which depends only on Γ and needs to be avoided. Let $q \geq 1$ with

$$q = q'q'', \quad \text{and} \quad q' \mid \mathfrak{B}.$$

Let $\Gamma(q)$ denote a “congruence” subgroup of Γ of level q , that is, a group which contains the set

$$\{\gamma \in \Gamma : \gamma \equiv I(q)\}.$$

The inclusion of vector spaces

$$L^2(\Gamma \backslash \mathbb{H}) \subset L^2(\Gamma(q) \backslash \mathbb{H})$$

induces the same inclusion on the spectrum:

$$\mathrm{Spec}(\Gamma \backslash \mathbb{H}) \subset \mathrm{Spec}(\Gamma(q) \backslash \mathbb{H}).$$

Definition 1.10. The *new spectrum*

$$\mathrm{Spec}_{new}(\Gamma(q) \backslash \mathbb{H})$$

at level q is defined to be the set of eigenvalues below $1/4$ which are in $\mathrm{Spec}(\Gamma(q) \backslash \mathbb{H})$ but not in $\mathrm{Spec}(\Gamma \backslash \mathbb{H})$.

Definition 1.11. We will call Θ a *spectral gap* for Γ if Θ is in the interval $1/2 < \Theta < \delta$ and

$$\mathrm{Spec}(\Gamma(q) \backslash \mathbb{H})_{new} \cap (0, \Theta(1 - \Theta)) \subset \mathrm{Spec}(\Gamma(q') \backslash \mathbb{H})_{new}.$$

That is, the eigenvalues below $\Theta(1 - \Theta)$ which are new for $\Gamma(q)$ are coming from the “bad” part q' of q . As the ramification number \mathfrak{B} is a fixed integer depending only on Γ , there are only finitely many possibilities for its divisors q' .

Infinite volume spectral gaps are known [Gam02, BG07, BGS09] for prime and square-free q . The method in [Gam02] applies also for arbitrary composite q , and in particular we have:

Theorem 1.12 ([Gam02]). *Let Γ be a Fuchsian group of the second kind with $\delta > 5/6$. Then there exists some ramification number \mathfrak{B} depending on Γ such that $\Theta = 5/6$ is a spectral gap for Γ .*

We also require the following Sobolev-type norm. Fix $T \geq 1$ and let $\{X_1, X_2, X_3\}$ be a basis for the Lie algebra \mathfrak{g} , cf. (2.5). Then define the $\mathcal{S}_{\infty, T}$ norm by

$$\mathcal{S}_{\infty, T} f = \max_{X \in \{0, X_1, X_2, X_3\}} \sup_{g \in G, \|g\| < T} |d\pi(X) \cdot f(g)|,$$

that is, the supremal value of first order derivatives of f in a ball of radius T in G .

Equipped with a spectral gap, we prove the following uniform counting statement.

Theorem 1.13. *Let $T \geq 1$ and $f : G \rightarrow \mathbb{C}$ be a smooth function with $|f| \leq 1$. Let Γ be as above with $\delta > 1/2$, ramification number \mathfrak{B} , and spectral gap Θ . Then for any $\gamma_0 \in \Gamma$ and integer $q \geq 1$ with $q = q'q''$ and $q' | \mathfrak{B}$,*

$$\begin{aligned} \sum_{\substack{\gamma \in \gamma_0 \cdot \Gamma(q) \\ \|\gamma\| < T}} f(\gamma) &= \frac{1}{[\Gamma : \Gamma(q)]} \left(\sum_{\substack{\gamma \in \Gamma \\ \|\gamma\| < T}} f(\gamma) + \mathcal{E}_{q'} \right) \\ &\quad + O\left((1 + \mathcal{S}_{\infty, T} f)^{6/7} T^{\frac{6}{7} \cdot 2\delta + \frac{1}{7} \cdot 2\Theta} \right). \end{aligned}$$

Here $\mathcal{E}_{q'} \ll T^{2\delta - \alpha_0}$, with $\alpha_0 > 0$, and all implied constants are independent of q'' and γ_0 .

The application in the companion paper [BK09] requires the following two estimates, which we derive as consequences of the above.

Theorem 1.14. *Assume Γ has critical exponent $\delta > 1/2$. Let v and w be vectors in \mathbb{Z}^2 , and $n \in \mathbb{Z}$. Let $0 < K < T < N$ be parameters with $K \rightarrow \infty$ and $N/K \rightarrow \infty$ (and a fortiori $N/T \rightarrow \infty$). Assume that $\frac{N}{K} < |n| < N$, $|w| < \frac{N}{T}$, $|v| \leq 1$, and $|n| < |v||w|T$. Then*

$$\sum_{\substack{\gamma \in \Gamma \\ \|\gamma\| < T}} \mathbf{1} \left\{ |\langle v\gamma, w \rangle - n| < \frac{N}{K} \right\} \gg \frac{T^{2\delta}}{K} + O\left(T^{\frac{3}{4} + 2\delta\frac{1}{4}} (\log T)^{1/4} \right).$$

Theorem 1.15. *Assume Γ has $\delta > 1/2$ and spectral gap Θ . Let N, K , and T be as above, and fix $q \geq 1$. Fix $y = (y_1, y_2)$ and (c, d) in \mathbb{Z}^2 such that $|y| < N$, $|(c, d)| < \frac{N}{T}$ and $|y| < T|(c, d)|$. Then*

$$\sum_{\substack{\gamma \in \Gamma \\ \|\gamma\| < T}} \mathbf{1} \left\{ |(c, d)\gamma - y| < \frac{N}{K} \right\} \mathbf{1} \left\{ (c, d)\gamma \equiv y \pmod{q} \right\} \ll \frac{T^{2\delta}}{K^{1+\delta}q^2} + T^{\frac{6}{7}2\delta + \frac{1}{7}2\Theta},$$

as $N, K, T \rightarrow \infty$.

In §2, we collect various preliminary pieces of information before proving Theorem 1.5 in §3 and Theorem 1.13 in §4. Finally, Theorem 1.14 is proved in §5 and Theorem 1.15 is proved in §6.

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2. PRELIMINARIES

2.1. Representation Theory.

Let $G = \mathrm{SL}(2, \mathbb{R})$, let Γ be a Fuchsian group of the second kind with critical exponent $\delta > 1/2$, use the notation (1.6) and (1.2), and let φ_j be a Laplace eigenfunction corresponding to $\lambda_j = s_j(1 - s_j)$. The decomposition into irreducibles of the right regular representation on the vector space $V = L^2(\Gamma \backslash G)$ is of the form [GGPS66]

$$V = V_{\varphi_0} \oplus V_{\varphi_1} \oplus \cdots \oplus V_{\varphi_N} \oplus V_{\mathrm{temp}}, \quad (2.1)$$

where each V_{φ_j} is the G -span of the eigenfunction φ_j , and is isomorphic as a G -representation to the complementary series representation with parameter s_j (in our normalization, the principal series representations lie on the critical line $\Re(s) = 1/2$). The reducible space V_{temp} consists of the tempered spectrum.

It will be convenient to use both the automorphic model above and, say, the line model, which we recall now. Fix $s > 1/2$ and let (π, V_s) denote the line model for the complementary series representation with parameter $s > 1/2$ [GGPS66]. That is, let G act on functions $f : \mathbb{R} \rightarrow \mathbb{C}$ with action given by

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} . f(x) = |-bx + d|^{-2s} f \left(\frac{ax - c}{-bx + d} \right).$$

The intertwining operator $\mathcal{I} : V_s \rightarrow V_{1-s}$ is defined by

$$\mathcal{I}.f(y) := \int_{\mathbb{R}} \frac{f(x)}{|x - y|^{2(1-s)}} dx. \quad (2.2)$$

For $f_1, f_2 \in V_s$, the pairing is

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}} f_1(x) \overline{\mathcal{I}.f_2(x)} dx. \quad (2.3)$$

Then V_s consists of functions f with $\langle f, f \rangle < \infty$.

2.2. Raising, Lowering, and Casimir Operators.

We return to the automorphic model and let \mathcal{H} be one such irreducible V_{φ_j} . The dense subspace \mathcal{H}^∞ of smooth vectors is infinite dimensional, and decomposes further into one-dimensional K -isotypic components:

$$\mathcal{H}^\infty = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}^{(2k)}, \quad (2.4)$$

each $\mathcal{H}^{(2k)} = \mathbb{C} \cdot v_{2k}$ consisting of functions of “weight $2k$ ”, that is, those functions $v_{2k} \in \mathcal{H}^\infty$ which transform as:

$$v_{2k}(gk_\theta) = e^{2ik\theta} v_{2k}(g).$$

Let

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (2.5)$$

denote a basis for the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. The ladder (raising and lowering) operators \mathcal{R} and \mathcal{L} in the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ are defined by:

$$\mathcal{R} = h + i(e + f), \quad \mathcal{L} = h - i(e + f).$$

Recall also the Casimir element \mathcal{C} , which generates the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, and acts on \mathcal{H} as scalar multiplication by $-2\lambda = -2s(1 - s)$:

$$\mathcal{C} = \frac{1}{2}h^2 + ef + fe.$$

We will require expressions for these operators in Cartan coordinates (θ_1, t, θ_2) , corresponding to $g = k_{\theta_1} a_t k_{\theta_2}$. For the Casimir operator, these can be found in many places, e.g. [Kna86, p. 216 and p. 700], [CM82, p. 884], or [KT00, §7]:

$$\frac{1}{2} \mathcal{C} = \frac{\partial^2}{\partial t^2} + (\coth t) \frac{\partial}{\partial t} + \operatorname{csch}^2 t \left(\frac{\partial^2}{\partial \theta_1^2} + \frac{\partial^2}{\partial \theta_2^2} \right) - \frac{2 \cosh t}{\sinh^2 t} \frac{\partial^2}{\partial \theta_1 \partial \theta_2}. \quad (2.6)$$

The raising and lowering operators are not as readily available in Cartan coordinates in the literature, so we derive their expression here.

Lemma 2.7. *In KA^+K coordinates, the raising and lowering operators are:*

$$\mathcal{R} = e^{2i\theta_2} \left(-i \operatorname{csch}(t) \frac{\partial}{\partial \theta_1} + 2 \frac{\partial}{\partial t} + i \operatorname{coth}(t) \frac{\partial}{\partial \theta_2} \right),$$

and

$$\mathcal{L} = e^{-2i\theta_2} \left(i \operatorname{csch}(t) \frac{\partial}{\partial \theta_1} + 2 \frac{\partial}{\partial t} - i \operatorname{coth}(t) \frac{\partial}{\partial \theta_2} \right).$$

Proof. Set $V = e + f$ and $Y = e - f$, so that $e = \frac{1}{2}V + \frac{1}{2}Y$ and $f = \frac{1}{2}V - \frac{1}{2}Y$. Let \mathcal{J} be the Cartan decomposition, that is, the injection

$$\mathcal{J} : K \times A^+ \times K \rightarrow G.$$

For a point $x = (\theta_1, t, \theta_2)$, we compute the derivation

$$D(\mathcal{J}) : T_x \rightarrow T_g = \mathfrak{g},$$

as follows. Any element T_x is of the form

$$T_x = \xi_1 Y_1 + \eta h + \xi_2 Y_2,$$

where $Y_1 = Y = Y_2 \in \mathfrak{k} = \mathfrak{so}(2)$, but each is interpreted as a vector field in its respective component. We compute keeping only first order terms:

$$\begin{aligned} & \mathcal{J}(k_1(I + \xi_1 Y_1), a_t(I + \eta h), k_2(I + \xi_2 Y_2)) \\ &= k_1(I + \xi_1 Y_1) a_t(I + \eta h) k_2(I + \xi_2 Y_2) \\ &= k_1 a_t k_2 (k_2^{-1} a_t^{-1} (I + \xi_1 Y_1) a_t k_2) (k_2^{-1} (I + \eta h) k_2) (I + \xi_2 Y_2) \\ &= k_1 a_t k_2 (I + \operatorname{ad}(k_2^{-1}) \operatorname{ad}(a_t^{-1}) \xi_1 Y_1) (I + \operatorname{ad}(k_2^{-1}) \eta h) (I + \xi_2 Y_2) \\ &\approx k_1 a_t k_2 (I + \operatorname{ad}(k_2^{-1}) \operatorname{ad}(a_t^{-1}) \xi_1 Y_1 + \operatorname{ad}(k_2^{-1}) \eta h + \xi_2 Y_2) \\ &= k_1 a_t k_2 \left(I + \operatorname{ad}(k_2^{-1}) \left\{ \operatorname{ad}(a_t^{-1}) \xi_1 Y_1 + \eta h + \operatorname{ad}(k_2) \xi_2 Y_2 \right\} \right). \end{aligned}$$

Hence

$$D(\mathcal{J}) : \xi_1 Y_1 + \eta h + \xi_2 Y_2 \mapsto \operatorname{ad}(k_2^{-1}) \left\{ \operatorname{ad}(a_t^{-1}) \xi_1 Y_1 + \eta h + \operatorname{ad}(k_2) \xi_2 Y_2 \right\}.$$

We compute

$$\begin{aligned}
\text{ad}(a_t^{-1})Y &= a_t^{-1}Y a_t = e^{-t}e - e^t f = e^{-t}\left(\frac{1}{2}V + \frac{1}{2}Y\right) - e^t\left(\frac{1}{2}V - \frac{1}{2}Y\right) \\
&= \cosh(t)Y - \sinh(t)V, \\
\text{ad}(k^{-1})Y &= Y, \\
\text{ad}(k_\theta^{-1})V &= \cos(2\theta)V - \sin(2\theta)h, \\
\text{ad}(k_\theta^{-1})h &= \sin(2\theta)V + \cos(2\theta)h.
\end{aligned}$$

Therefore

$$\begin{aligned}
D(\mathcal{J}) &: \xi_1 Y_1 + \eta h + \xi_2 Y_2 \\
&\mapsto \text{ad}(k_2^{-1}) \left\{ \text{ad}(a_t^{-1})\xi_1 Y_1 + \eta h + \text{ad}(k_2)\xi_2 Y_2 \right\} \\
&= \text{ad}(k_2^{-1}) \left\{ \xi_1 (\cosh(t)Y - \sinh(t)V) + \eta h + \xi_2 Y \right\} \\
&= \text{ad}(k_2^{-1}) \left\{ -\xi_1 \sinh(t)V + \eta h + (\xi_2 + \xi_1 \cosh(t))Y \right\} \\
&= -\xi_1 \sinh(t) \left(\cos(2\theta_2)V - \sin(2\theta_2)h \right) \\
&\quad + \eta \left(\sin(2\theta_2)V + \cos(2\theta_2)h \right) + (\xi_2 + \xi_1 \cosh(t))Y \\
&= \left(-\xi_1 \sinh(t) \cos(2\theta_2) + \eta \sin(2\theta_2) \right) V \\
&\quad + \left(\xi_1 \sinh(t) \sin(2\theta_2) + \eta \cos(2\theta_2) \right) h + \left(\xi_2 + \xi_1 \cosh(t) \right) Y.
\end{aligned}$$

To determine which (ξ_1, η, ξ_2) give $\mathcal{R} = h + iV$, we simply solve the linear system of equations:

$$\begin{aligned}
-\xi_1 \sinh(t) \cos(2\theta_2) + \eta \sin(2\theta_2) &= i \\
\xi_1 \sinh(t) \sin(2\theta_2) + \eta \cos(2\theta_2) &= 1 \\
\xi_2 + \xi_1 \cosh(t) &= 0,
\end{aligned}$$

which has the solution:

$$\begin{aligned}
\xi_1 &= -ie^{2i\theta_2} \text{csch}(t) \\
\eta &= e^{2i\theta_2} \\
\xi_2 &= ie^{2i\theta_2} \text{coth}(t).
\end{aligned}$$

Of course $Y_1 = \frac{\partial}{\partial \theta_1}$, $h = 2\frac{\partial}{\partial t}$, and $Y_2 = \frac{\partial}{\partial \theta_2}$, whence

$$D(\mathcal{J}) : \left(-ie^{2i\theta_2} \operatorname{csch}(t) \frac{\partial}{\partial \theta_1} + 2e^{2i\theta_2} \frac{\partial}{\partial t} + ie^{2i\theta_2} \operatorname{coth}(t) \frac{\partial}{\partial \theta_2} \right) \mapsto \mathcal{R}.$$

The formula for \mathcal{L} is derived in the same way. \square

2.3. Polar Coordinates in the Disk Model.

At times it will also be convenient to use polar coordinates (θ_1, r, θ_2) in the unit tangent bundle of the disk model \mathbb{D} , obtained from Cartan coordinates (θ_1, t, θ_2) by the change of variables

$$r = \frac{e^t - 1}{e^t + 1} = \tanh(t/2), \quad \frac{1+r}{1-r} = e^t, \quad (2.8)$$

with

$$\frac{\partial}{\partial t} = \frac{\partial r}{\partial t} \frac{\partial}{\partial r} = \frac{2e^t}{(e^t + 1)^2} \frac{\partial}{\partial r} = \frac{1}{2} \operatorname{sech}^2(t/2) \frac{\partial}{\partial r} = \frac{1}{2} (1 - r^2) \frac{\partial}{\partial r},$$

and

$$\operatorname{csch}(t) = \frac{1 - r^2}{2r}, \quad \operatorname{coth}(t) = \frac{1 + r^2}{2r}.$$

In the (θ_1, r, θ_2) , coordinates, the Casimir operator becomes:

$$\begin{aligned} \frac{1}{2} \mathcal{C} &= \frac{(1 - r^2)^2}{4} \frac{\partial^2}{\partial r^2} + \frac{(1 - r^2)^2}{4r} \frac{\partial}{\partial r} \\ &\quad + \frac{(1 - r^2)^2}{16r^2} \left(\frac{\partial^2}{\partial \theta_1^2} + \frac{\partial^2}{\partial \theta_2^2} \right) - \frac{1 - r^4}{8r^2} \frac{\partial^2}{\partial \theta_1 \partial \theta_2}, \end{aligned} \quad (2.9)$$

and the ladder operators are

$$\mathcal{R} = e^{2i\theta_2} \left(-i \frac{1 - r^2}{2r} \frac{\partial}{\partial \theta_1} + (1 - r^2) \frac{\partial}{\partial r} + i \frac{1 + r^2}{2r} \frac{\partial}{\partial \theta_2} \right), \quad (2.10)$$

and

$$\mathcal{L} = e^{-2i\theta_2} \left(i \frac{1 - r^2}{2r} \frac{\partial}{\partial \theta_1} + (1 - r^2) \frac{\partial}{\partial r} - i \frac{1 + r^2}{2r} \frac{\partial}{\partial \theta_2} \right).$$

2.4. K -isotypic Vectors in the Line Model. Turning now to the line model, let $\mathcal{H} = V_s$ with grading as in (2.4).

Lemma 2.11. *In the line model, if $f_{2k,s} \in V_s^{(2k)}$ then*

$$f_{2k,s}(x) = c(x - i)^{k-s} (x + i)^{-k-s} \quad (2.12)$$

for some $c \in \mathbb{C}$.

Proof. For any vector $f \in V_s^{(2k)}$, the action of $Y = e - f \in \mathfrak{g}$ is $Y.v = \frac{\partial}{\partial \theta_2} v = 2ikv$. We compute in the line model that the element Y acts on V_s by:

$$Y.f(x) = 2sxf(x) + (1 + x^2)f'(x).$$

Hence $f \in V_s^{(2k)}$, if f satisfies

$$2sxf(x) + (1 + x^2)f'(x) = 2ikf(x).$$

It is elementary to verify that (2.12) is the unique solution up to constant. \square

2.5. Fourier Expansions.

Let $v_{2k} \in \mathcal{H}^{(2k)}$ be a K -isotypic vector, and recall that the Casimir operator acts as $\frac{1}{2}D(\mathcal{C}) + \lambda = 0$, with $\lambda = s(1 - s)$ and $s > 1/2$. Then as a function in (θ_1, r, θ_2) coordinates, one has the Fourier expansion:

$$v_{2k}(\theta_1, r, \theta_2) = e^{2ik\theta_2} \sum_{n \in \mathbb{Z}} v_{2n, 2k}(r) e^{2in\theta_1}.$$

(Only even frequencies appear, since the representation factors through $\mathrm{PSL}(2, \mathbb{R})$.) The differential equation induced from (2.9) on $v_{2n, 2k}$ is:

$$\begin{aligned} & \frac{(1 - r^2)^2}{4} \frac{\partial^2}{\partial r^2} v_{2n, 2k}(r) + \frac{(1 - r^2)^2}{4r} \frac{\partial}{\partial r} v_{2n, 2k}(r) \\ & + \left(-\frac{(1 - r^2)^2}{4r^2} (n^2 + k^2) + \frac{1 - r^4}{2r^2} nk + s(1 - s) \right) v_{2n, 2k}(r) = 0. \end{aligned} \quad (2.13)$$

Before solving this equation, we note that v_{2k} is regular everywhere, in particular at the origin, and hence so is $v_{2n, 2k}$. Expansion in series of (2.13) about the origin $r = 0$ gives

$$(1 + O(r)) \frac{\partial^2}{\partial r^2} + \frac{1 + O(r)}{r} \frac{\partial}{\partial r} - \frac{1 + O(r)}{r^2} (n - k)^2 = 0,$$

which has asymptotic solution of the form:

$$\begin{cases} (c_1 r^{n-k} + c_2 r^{k-n})(1 + O(r)) & \text{if } n \neq k, \\ (c_1 + c_2 \log r)(1 + O(r)) & \text{if } n = k. \end{cases}$$

In the case when $n > k$ (respectively, $n < k$), regularity at the origin forces c_2 (respectively, c_1) to vanish. If $n = k$, then $c_2 = 0$. Either way, there is a multiplicity one principle, and $v_{2n, 2k}$ is a constant multiple of the unique solution $\Phi_{2n, 2k}$ to (2.13) having 1 in its first non-zero Taylor

coefficient in r (note that $\Phi_{2n,2k}$ vanishes to order $|n-k|$ at the origin $r=0$). One can explicitly compute:

$$\Phi_{2n,2k}(r) = (1-r^2)^s r^{|n-k|} {}_2F_1(s - \epsilon_{n,k}k, s + \epsilon_{n,k}n; 1 + |n-k|; r^2), \quad (2.14)$$

where

$$\epsilon_{n,k} = \begin{cases} 1 & \text{if } n \geq k \\ -1 & \text{otherwise,} \end{cases} \quad (2.15)$$

and ${}_2F_1$ is the standard Gauss hypergeometric series. We will use the same name $\Phi_{2n,2k}$ for the function on G defined by

$$\Phi_{2n,2k}(k_{\theta_1} a_t k_{\theta_2}) = e^{2in\theta_1} \Phi_{2n,2k}(r) e^{2ik\theta_2},$$

where r is related to t by (2.8).

2.6. Choice of a Basis from Ladder Operators.

Take a K -fixed vector $v_0 \in \mathcal{H}$ (it is unique up to scalar), and assume that it has unit norm under the inner product $\langle \cdot, \cdot \rangle$ in \mathcal{H} (whence it is unique up to a scalar of norm one). The raising (respectively, lowering) operator takes vectors in $\mathcal{H}^{(2k)}$ to ones in $\mathcal{H}^{(2k+2)}$ (respectively, $\mathcal{H}^{(2k-2)}$), but the images no longer have unit norm. The normalization is as follows.

Lemma 2.16. *Let \mathcal{X} be a ladder operator, $\mathcal{X} = \mathcal{R}$ or $\mathcal{X} = \mathcal{L}$. Then for any $k \geq 0$,*

$$\langle \mathcal{X}^k.v_0, \mathcal{X}^k.v_0 \rangle = 2^{2k} \frac{\Gamma(s+k)\Gamma(1-s+k)}{\Gamma(s)\Gamma(1-s)} =: b_{k,s}. \quad (2.17)$$

Proof. We will exhibit the computation for $\mathcal{X} = \mathcal{R}$, the case of $\mathcal{X} = \mathcal{L}$ being similar. A standard calculation (recall that $Y = e - f$ in the basis (2.5) and acts as $\frac{\partial}{\partial \theta_2}$) shows that

$$\mathcal{L}\mathcal{R} = 2\mathcal{C} + Y^2 + 2iY, \quad (2.18)$$

and hence acts on $\mathcal{H}^{(2k)}$ as scalar multiplication by

$$-4\lambda + (2ik)^2 + 2i(2ik) = -4(s+k)(1-s+k).$$

Using

$$\langle \mathcal{R}.v, w \rangle = -\langle v, \mathcal{L}.w \rangle \quad (2.19)$$

then gives

$$\begin{aligned}
\langle \mathcal{R}^k.v_0, \mathcal{R}^k.v_0 \rangle &= (-1)^k \langle \mathcal{L}^k \mathcal{R}^k.v_0, v_0 \rangle \\
&= (-1)^k (-4(s)(1-s))(-4(s+1)(1-s+1)) \cdots \\
&\quad \times (-4(s+k-1)(1-s+k-1)) \langle v_0, v_0 \rangle \\
&= (-1)^k (-1)^k 4^k \frac{\Gamma(s+k)\Gamma(1-s+k)}{\Gamma(s)\Gamma(1-s)},
\end{aligned}$$

as claimed. \square

Now given a fixed vector $v_0 \in \mathcal{H}^{(0)}$ of unit norm, we make once and for all the following choice for an orthonormal basis for the space \mathcal{H}^∞ of smooth vectors.

Definition 2.20. For $k \neq 0$, let

$$v_{2k} := \frac{1}{\sqrt{b_{|k|,s}}} \begin{cases} \mathcal{R}^k.v_0 & \text{if } k > 0, \\ \mathcal{L}^{|k|}.v_0 & \text{if } k < 0, \end{cases} \quad (2.21)$$

where $b_{k,s}$ is defined in (2.17).

2.7. Basis for V_s in the Line Model. In the line model V_s , we have the un-normalized functions $f_{2k,s}$ given in (2.12). Hence to determine the relationship between $f_{2k,s}$ and v_{2k} in (2.21), we need only to normalize $f_{2k,s}$. First we compute the action of the intertwining operator.

Lemma 2.22. For $f_{2k,s}$ as in (2.12), we have

$$\mathcal{I}.f_{2k,s} = \frac{4^{1-s}\pi(-1)^k\Gamma(2s-1)}{\Gamma(s-k)\Gamma(s+k)} f_{2k,1-s}. \quad (2.23)$$

Proof. The identity to be verified is

$$\int_{\mathbb{R}} \frac{(y-i)^{k-s}(y+i)^{-k-s}}{|x-y|^{2(1-s)}} dy = \frac{4^{1-s}\pi(-1)^k\Gamma(2s-1)}{\Gamma(s-k)\Gamma(s+k)} (x-i)^{k-(1-s)}(x+i)^{-k-(1-s)}.$$

The intertwining operator preserves the group action, so $\mathcal{I}.f_{2k,s} \in V_{1-s}^{(2k)}$, and hence is a multiple of $f_{2k,1-s}$. To determine the multiple, we may simply set $x = 0$ in the above and compute the integral. \square

With this computation at hand, we may determine the norms of $f_{2k,s}$.

Lemma 2.24. *For $f_{2k,s}$ as in (2.12), we have*

$$\langle f_{2k,s}, f_{2k,s} \rangle = \frac{4^{1-s}\pi^2(-1)^k\Gamma(2s-1)}{\Gamma(s-k)\Gamma(s+k)} =: \tilde{b}_{k,s} > 0. \quad (2.25)$$

Proof. By definition, the left hand side of (2.25) is

$$\begin{aligned} &= \int_{\mathbb{R}} f_{2k,s}(x) \overline{\mathcal{I}.f_{2k,s}(x)} dx \\ &= \frac{4^{1-s}\pi(-1)^k\Gamma(2s-1)}{\Gamma(s-k)\Gamma(s+k)} \int_{\mathbb{R}} (x-i)^{k-s}(x+i)^{-k-s} \overline{(x-i)^{k-(1-s)}(x+i)^{-k-(1-s)}} dx \\ &= \frac{4^{1-s}\pi(-1)^k\Gamma(2s-1)}{\Gamma(s-k)\Gamma(s+k)} \int_{\mathbb{R}} (x^2+1)^{-1} dx \\ &= \frac{4^{1-s}\pi(-1)^k\Gamma(2s-1)}{\Gamma(s-k)\Gamma(s+k)} \pi, \end{aligned}$$

using (2.23). Note that this value is real and positive. \square

2.8. Ladder Operators on Fourier Expansions.

Fix some $v_0 \in \mathcal{H}$ and let v_{2k} be the basis defined by (2.21). From §2.5, v_0 in coordinates (θ_1, r, θ_2) has Fourier expansion:

$$v_0(\theta_1, r, \theta_2) = \sum_n c_{2n} \Phi_{2n,0}(r) e^{2in\theta_1}, \quad (2.26)$$

with some Fourier coefficients $c_{2n} \in \mathbb{C}$. In this subsection, we express the Fourier expansions of all the v_{2k} in terms of the coefficients c_{2n} . For this we require the following

Lemma 2.27. *The ladder operators act on $\Phi_{2n,2k}$ by:*

$$\mathcal{R}.\Phi_{2n,2k} = -2 \Phi_{2n,2k+2} \times \begin{cases} -(n-k) & \text{if } n > k, \\ \frac{(s+k)(1-s+k)}{1-n+k} & \text{if } n \leq k, \end{cases}$$

and

$$\mathcal{L}.\Phi_{2n,2k} = -2 \Phi_{2n,2k-2} \times \begin{cases} \frac{(s-k)(1-s-k)}{1+n-k} & \text{if } n \geq k, \\ -(k-n) & \text{if } n < k. \end{cases}$$

Proof. We will demonstrate the case of acting by \mathcal{R} with $n > k$, the other cases being similar. From (2.10), we have that \mathcal{R} acts on $\Phi_{2n,2k}$ by

$$\frac{1-r^2}{r}n + (1-r^2)\frac{\partial}{\partial r} - \frac{1+r^2}{r}k$$

Recall from (2.14) and $n > k$ that

$$\Phi_{2n,2k}(r) = (1 - r^2)^s r^{n-k} {}_2F_1(s - k, s + n; 1 - k + n; r^2),$$

and hence a computation yields

$$\begin{aligned} \mathcal{R}.\Phi_{2n,2k}(r) &= -2 (1 - r^2)^s r^{n-k-1} \\ &\quad \times \left\{ \begin{aligned} &(r^2(k + n) + s - n) {}_2F_1(s - k, s + n; 1 - k + n; r^2) \\ &- (1 - r^2) (s - k) {}_2F_1(s - k + 1, s + n; 1 - k + n; r^2) \end{aligned} \right\}. \end{aligned}$$

Using the series expansion of the Gauss hypergeometric series, it is a matter of combinatorics to verify that the above expression is the same as

$$2(n - k) (1 - r^2)^s r^{n-k-1} {}_2F_1(s - k - 1, s + n; n - k; r^2),$$

as claimed. \square

From Lemma 2.27, it follows after a calculation that for $k \geq 0$,

$$\begin{aligned} \mathcal{R}^k.v_0(\theta_1, r, \theta_2) &= (-1)^k 2^k e^{2ik\theta_2} \sum_n c_{2n} \Phi_{2n,2k}(r) e^{2in\theta_1} \\ &\quad \times \begin{cases} (-1)^k \frac{\Gamma(n + 1)}{\Gamma(n - k + 1)} & \text{if } n \geq k, \\ (-1)^n \frac{\Gamma(n + 1)\Gamma(s + k)\Gamma(1 - s + k)}{\Gamma(k - n + 1)\Gamma(s + n)\Gamma(1 - s + n)} & \text{if } 1 \leq n \leq k - 1, \\ \frac{\Gamma(|n| + 1)\Gamma(s + k)\Gamma(1 - s + k)}{\Gamma(k + |n| + 1)\Gamma(s)\Gamma(1 - s)} & \text{if } n \leq 0. \end{cases} \end{aligned}$$

A similar identity holds for $\mathcal{L}^k.v_0$. Recall that $\Phi_{2n,2k}$ vanishes at the origin $r = 0$ unless $n = k$, in which case it takes the value 1. We have proved

Proposition 2.28. *For $k \geq 0$, the value at the origin of v_0 acted on by ladder operators is related to its Fourier coefficients by*

$$\mathcal{R}^k.v_0(e) = c_{2k} 2^k \Gamma(k + 1) \quad (2.29)$$

and

$$\mathcal{L}^k.v_0(e) = c_{-2k} 2^k \Gamma(k + 1).$$

2.9. Matrix Coefficients.

Fix integers n and k . We first record here the asymptotic growth rate of $\Phi_{2n,2k}$ at infinity.

Lemma 2.30. *As $t \rightarrow \infty$,*

$$\Phi_{2n,2k}(a_t) = 4^{1-s} e^{-t(1-s)} \frac{\Gamma(1 + |n - k|)\Gamma(2s - 1)}{\Gamma(s - \epsilon_{n,k} k)\Gamma(s + \epsilon_{n,k} n)} (1 + O(nk e^{-t})), \quad (2.31)$$

with $\epsilon_{n,k}$ defined in (2.15).

Proof. Recall the well-known identity

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b, a+b-c+1; 1-z) \\ &\quad + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} {}_2F_1(c-a, c-b, c-a-b+1; 1-z). \end{aligned} \quad (2.32)$$

Applied to the series in (2.14) for the case $n \geq k$, (2.32) gives

$$\begin{aligned} {}_2F_1(s-k, s+n, 1-k+n; r^2) &= \frac{\Gamma(1-k+n)\Gamma(1-2s)}{\Gamma(1-s+n)\Gamma(1-s-k)} {}_2F_1(s-k, s+n, 2s; 1-r^2) \\ &\quad + (1-r^2)^{1-2s} \frac{\Gamma(1-k+n)\Gamma(2s-1)}{\Gamma(s-k)\Gamma(s+n)} {}_2F_1(1-s+n, 1-s-k, 2-2s; 1-r^2). \end{aligned} \quad (2.33)$$

Using (cf. Good [Goo83b])

$${}_2F_1(a, b, c; z) = 1 + O\left(\left|\frac{abz}{c}\right|\right) \quad \text{for} \quad |z| \max_{\ell \geq 0} \left| \frac{(a+\ell)(b+\ell)}{(c+\ell)(\ell+1)} \right| \leq \frac{1}{2},$$

the second term in (2.33) is the only one growing as $r \rightarrow 1$, and hence

$$\Phi_{2n,2k}(r) = (1-r^2)^s r^{n-k} (1-r^2)^{1-2s} \frac{\Gamma(1-k+n)\Gamma(2s-1)}{\Gamma(s-k)\Gamma(s+n)} (1 + O(nk(1-r^2))).$$

Simplifying and changing variables (2.8) gives (2.31). The case $n < k$ is similar. \square

Let π denote the right-regular representation on the irreducible \mathcal{H} . Take the K -isotypic vectors v_{2n} and v_{2k} in the basis (2.21), and form the matrix coefficient:

$$M_{2n,2k}(g) := \langle \pi(g).v_{2k}, v_{2n} \rangle.$$

Note that $M_{2n,2k}$ is an eigenfunction of the Casimir operator, and transforms by

$$M_{2n,2k}(k_{\theta_1} g k_{\theta_2}) = e^{2in\theta_1} M_{2n,2k}(g) e^{2ik\theta_2},$$

whence is a scalar multiple of $\Phi_{2n,2k}$. For instance, if $n = k$, then we instantly have $M_{2n,2n}(g) = \Phi_{2n,2n}(g)$. In the sequel, we require knowledge of this constant in the general case.

Lemma 2.34. *For integers n and k ,*

$$\begin{aligned} \langle \pi(g).v_{2k}, v_{2n} \rangle &= \Phi_{2n,2k}(g) (\tilde{b}_{k,s} \tilde{b}_{n,s})^{-1/2} \\ &\times \frac{(-1)^k 4^{1-s} \pi^2 \Gamma(2s-1)}{\Gamma(1+|n-k|) \Gamma(s-\epsilon_{n,k} n) \Gamma(s+\epsilon_{n,k} k)}. \end{aligned} \quad (2.35)$$

Proof. We carry out this computation by switching to the line model. For $g = a_t$, and relating t to r via (2.8), the left hand side of (2.35) is

$$\begin{aligned} &= (\tilde{b}_{k,s} \tilde{b}_{n,s})^{-1/2} \langle \pi(a_t).f_{2k,s}, f_{2n,s} \rangle \\ &= (\tilde{b}_{k,s} \tilde{b}_{n,s})^{-1/2} \int_{\mathbb{R}} \left(\frac{1+r}{1-r} \right)^s \left(\left(\frac{1+r}{1-r} \right) x - i \right)^{k-s} \left(\left(\frac{1+r}{1-r} \right) x + i \right)^{-k-s} \\ &\quad \times \frac{4^{1-s} \pi (-1)^n \Gamma(2s-1)}{\Gamma(s-n) \Gamma(s+n)} \frac{1}{(x-i)^{n-(1-s)} (x+i)^{-n-(1-s)}} dx, \end{aligned}$$

using (2.23). Qua a function of r , the above integral is a multiple of (2.14). To determine the multiple, we may simply study the asymptotics at infinity, corresponding to $r \rightarrow 1$, using Laplace's method, and compare it to (2.31).

In this way, one obtains

$$\begin{aligned} &\int_{\mathbb{R}} \left(\frac{1+r}{1-r} \right)^s \left(\left(\frac{1+r}{1-r} \right) x - i \right)^{k-s} \left(\left(\frac{1+r}{1-r} \right) x + i \right)^{-k-s} \\ &\quad \times \frac{1}{(x-i)^{n-(1-s)} (x+i)^{-n-(1-s)}} dx \\ &= \frac{(-1)^{k+n} \pi}{\Gamma(1+|n-k|)} \frac{\Gamma(s+\epsilon_{n,k} n)}{\Gamma(s+\epsilon_{n,k} k)} \Phi_{2k,2n}(r). \end{aligned}$$

Combining the constants completes the proof. \square

2.10. The Patterson Sullivan measure.

Recall from (1.4) the Patterson-Sullivan measure μ , supported on the limit set $\Lambda \subset \partial\mathbb{H}$, which has Hausdorff dimension δ with $1/2 < \delta < 1$. The eigenfunction φ_0 corresponding to the base eigenvalue $\lambda_0 = \delta(1-\delta)$

is expressed explicitly in disk coordinates (θ_1, r, θ_2) as the integral of a Poisson kernel against μ as follows:

$$\varphi_0(\theta_1, r, \theta_2) = \int_0^\pi \left(\frac{1-r^2}{|re^{2i\theta_1} - e^{2i\alpha}|^2} \right)^\delta d\mu(\alpha). \quad (2.36)$$

Recall from (2.26) that φ_0 has Fourier development:

$$\varphi_0(\theta_1, r, \theta_2) = \sum_n c_{2n} \Phi_{2n,0}(r) e^{2in\theta_1}. \quad (2.37)$$

The $2n$ -th Fourier coefficient of μ is given by:

$$\widehat{\mu}(2n) = \int_0^\pi e^{2in\alpha} d\mu(\alpha).$$

Lemma 2.38. *The relationship between the coefficient c_{2n} and $\widehat{\mu}$ is given explicitly by*

$$c_{2n} = \frac{1}{\Gamma(\delta)} \frac{\Gamma(\delta + |n|)}{\Gamma(1 + |n|)} \widehat{\mu}(-2n). \quad (2.39)$$

Proof. Equating the two expressions (2.36) and (2.37), inserting (2.14) with $s = \delta$, and dividing both sides by $(1-r^2)^\delta$ gives

$$\sum_n c_{2n} r^{|n|} {}_2F_1(\delta, \delta + |n|, 1 + |n|, r^2) e^{2in\theta_1} = \int_0^\pi (r^2 - r(e^{i(2\theta_1 - 2\alpha)} + e^{i(2\alpha - 2\theta_1)}) + 1)^{-\delta} d\mu(\alpha).$$

Expanding both sides in series and collecting terms yields (2.39). \square

2.11. Decay of Tempered Matrix Coefficients.

We end this section by recalling the well-known strong mixing property for matrix coefficients [HM79, Cow78, CHH88, Ven05].

Lemma 2.40. *Let (π, V) be a tempered unitary representation of G . Then for any vectors $v, w \in V$ whose K -span is one-dimensional,*

$$|\langle \pi(k_{\theta_1} a_t k_{\theta_2}).v, w \rangle| \ll te^{-t/2} \|v\|_2 \|w\|_2, \text{ as } t \rightarrow \infty, \quad (2.41)$$

where implied constant is absolute.

Define the Sobolev norm

$$\mathcal{S}v = \|v\|_2 + \|d\pi(h).v\|_2 + \|d\pi(e).v\|_2 + \|d\pi(f).v\|_2,$$

where $h, e,$ and f are an orthonormal basis for \mathfrak{g} , cf. (2.5).

Lemma 2.42. *Let $\Theta > 1/2$ and (π, V) be a unitary representation of G which does not weakly contain any complementary series representation with parameter $s > \Theta$. Then for any smooth vectors $v, w \in V^\infty$,*

$$|\langle \pi(k_{\theta_1} a_t k_{\theta_2}).v, w \rangle| \ll e^{-\Theta t} (\|v\| \|w\|)^{1/2} (\mathcal{S}v \mathcal{S}w)^{1/2}, \text{ as } t \rightarrow \infty, \quad (2.43)$$

where implied constant is absolute.

3. PROOF OF THEOREM 1.5

Let $\Gamma < G = \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian group of the second kind, and let $K = \mathrm{SO}(2)$ be the maximal compact. Assume the limit set of Γ has Hausdorff dimension $\delta > 1/2$. Let

$$0 < \delta(1 - \delta) = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_N < 1/4$$

be the point spectrum of the Laplacian acting on $L^2(\Gamma \backslash \mathbb{H})$, with

$$\lambda_j = s_j(1 - s_j)$$

and $s_j > 1/2$.

Fix integers n and k . Our goal is to evaluate

$$\mathcal{N}(T) := \sum_{\substack{\gamma \in \Gamma \\ \|\gamma\| < T}} e^{2in\theta_1(\gamma)} e^{2ik\theta_2(\gamma)}.$$

For $g \in G$, let

$$f_T(g) := e^{2in\theta_1(g)} e^{2ik\theta_2(g)} \mathbf{1}_{\|g\| < T},$$

and define $\mathcal{F}_T : \Gamma \backslash G \times \Gamma \backslash G \rightarrow \mathbb{C}$ via

$$\mathcal{F}_T(g, h) := \sum_{\gamma \in \Gamma} f_T(g^{-1}\gamma h). \quad (3.1)$$

Clearly $\mathcal{F}_T(e, e) = \mathcal{N}(T)$.

For a fixed η (to be chosen later depending on T), let

$$\psi : \Gamma \backslash G \rightarrow \mathbb{R}$$

be a smooth test function with unit mass, $\int_{\Gamma \backslash G} \psi = 1$, and compact support in a ball of radius η about the identity $e \in G$. Then the integral

$$\mathcal{H}(T) := \langle \mathcal{F}_T, \psi \otimes \psi \rangle = \int_{\Gamma \backslash G} \int_{\Gamma \backslash G} \mathcal{F}_T(g, h) \psi(g) \psi(h) dg dh \quad (3.2)$$

approximates $\mathcal{N}(T)$ as follows.

Proposition 3.3.

$$\mathcal{H}(T) = \mathcal{N}(T) + O(\eta(1 + |n| + |k|)T^{2\delta}). \quad (3.4)$$

Proof. Writing

$$\mathcal{F}_T(g, h) = \mathcal{F}_T(e, e) + (\mathcal{F}_T(g, h) - \mathcal{F}_T(e, e))$$

and using $\int \psi = 1$ gives

$$\mathcal{H}(T) = \mathcal{N}(T) + \mathcal{E}(T),$$

where

$$\begin{aligned} \mathcal{E}(T) &:= \int_{\Gamma \backslash G} \int_{\Gamma \backslash G} (\mathcal{F}_T(g, h) - \mathcal{F}_T(e, e)) \psi(g) \psi(h) dg dh \\ &= \sum_{\gamma \in \Gamma} \int_{\Gamma \backslash G} \int_{\Gamma \backslash G} (f_T(g^{-1}\gamma h) - f_T(\gamma)) \psi(g) \psi(h) dg dh. \end{aligned}$$

Recall that ψ has support in a ball of radius η about the identity, and let $g, h \in \text{supp } \psi$. For $\gamma \in \Gamma$, we consider three ranges of $\|\gamma\|$ separately:

- (1) If $\|\gamma\| > \frac{T}{1-\eta}$, then both $f_T(g^{-1}\gamma h)$ and $f_T(\gamma)$ vanish.
- (2) If $\frac{T}{1+\eta} < \|\gamma\| \leq \frac{T}{1-\eta}$, then we estimate trivially

$$|f_T(g^{-1}\gamma h) - f_T(\gamma)| \leq 2. \quad (3.5)$$

- (3) Lastly, if $\|\gamma\| \leq \frac{T}{1+\eta}$, then

$$\mathbf{1}_{\|g^{-1}\gamma h\| < T} = \mathbf{1}_{\|\gamma\| < T} = 1,$$

and from

$$|e^{2in\theta_1(g^{-1}\gamma h)} - e^{2in\theta_1(\gamma)}| \ll |n|\eta$$

(using KA^+K coordinates), it follows that

$$|f_T(g^{-1}\gamma h) - f_T(\gamma)| \ll (|n| + |k|)\eta. \quad (3.6)$$

Combining (3.5) and (3.6) gives

$$\begin{aligned} \mathcal{E}(T) &\ll \sum_{\substack{\gamma \in \Gamma \\ \frac{T}{1+\eta} < \|\gamma\| \leq \frac{T}{1-\eta}}} 1 + (|n| + |k|)\eta \sum_{\substack{\gamma \in \Gamma \\ \|\gamma\| \leq \frac{T}{1+\eta}}} 1 \\ &\ll \eta T^{2\delta} + (|n| + |k|)\eta T^{2\delta}, \end{aligned}$$

by Lax-Phillips [LP82]. This completes the proof. \square

It remains to evaluate $\mathcal{H}(T)$. First we rewrite it, as follows.

Lemma 3.7. *The inner product $\mathcal{H}(T)$ can be expressed as the follows:*

$$\mathcal{H}(T) = \int_G f_T(g) \langle \pi(g).\psi, \psi \rangle dg. \quad (3.8)$$

Proof. Insert the definition of \mathcal{F}_T (3.1) into (3.2), interchange summation and integration, change variables $g = x^{-1}\gamma h$, and use the left Γ -invariance of ψ :

$$\begin{aligned} \mathcal{H}(T) &= \int_{x \in \Gamma \backslash G} \int_{h \in \Gamma \backslash G} \sum_{\gamma \in \Gamma} f_T(x^{-1}\gamma h) \psi(h) \psi(x) dh dx \\ &= \int_{x \in \Gamma \backslash G} \left(\sum_{\gamma \in \Gamma} \int_{h \in \Gamma \backslash G} f_T(x^{-1}\gamma h) \psi(h) dh \right) \psi(x) dx \\ &= \int_{x \in \Gamma \backslash G} \left(\sum_{\gamma \in \Gamma} \int_{g \in \gamma^{-1}x(\Gamma \backslash G)} f_T(g) \psi(\gamma^{-1}xg) dg \right) \psi(x) dx \\ &= \int_{x \in \Gamma \backslash G} \left(\int_{g \in G} f_T(g) \psi(xg) dg \right) \psi(x) dx, \end{aligned}$$

since for x fixed, $x(\Gamma \backslash G)$ is a fundamental domain for Γ , and hence $\sum_{\gamma \in \Gamma} \int_{\gamma^{-1}x(\Gamma \backslash G)} = \int_G$. Interchanging integrals gives

$$\mathcal{H}(T) = \int_G f_T(g) \int_{\Gamma \backslash G} \psi(xg) \psi(x) dx dg,$$

as desired. \square

At this point, we could expand the matrix coefficient $\langle \pi(g).\psi, \psi \rangle$ spectrally, but the error term would then contain more harmonics than necessary, leading to worse bounds (essentially requiring the Sobolev norms arising in (2.43), in place of the \mathcal{L}^2 norms in (2.41)). So we first remove the immaterial harmonics. To this end, decompose ψ into its Fourier series with respect to θ_2 ,

$$\psi(g) = \sum_{m \in \mathbb{Z}} \psi_{2m}(g), \quad (3.9)$$

where ψ_{2m} transforms on the right by

$$\psi_{2m}(g k_\theta) = \psi_{2m}(g) e^{2im\theta}.$$

Insert (3.9) twice into (3.8):

$$\mathcal{H}(T) = \sum_m \sum_\ell \int_G f_T(g) \langle \pi(g).\psi_{2m}, \psi_{2\ell} \rangle dg.$$

Note that the matrix coefficient above transforms on the left and right by

$$\langle \pi(k_{\theta_1} g k_{\theta_2}).\psi_{2m}, \psi_{2\ell} \rangle = e^{2i\ell\theta_1} \langle \pi(g).\psi_{2m}, \psi_{2\ell} \rangle e^{2im\theta_2},$$

whence

$$\int_G f_T(g) \langle \pi(g).\psi_{2m}, \psi_{2\ell} \rangle dg = 0,$$

unless $m = -k$ and $\ell = -n$. Having dispensed with extraneous harmonics, we write

$$\mathcal{H}(T) = \int_G f_T(g) \langle \pi(g).\psi_{-2k}, \psi_{-2n} \rangle dg, \quad (3.10)$$

and now expand the matrix coefficient spectrally. Recall that $\lambda_0 = \delta(1 - \delta)$ is the base frequency with corresponding eigenfunction φ_0 , and assume at first that it is the sole discrete eigenvalue, the rest of the spectrum being tempered. Let V denote the vector space consisting of the closure of the G -span of φ_0 , and use the notation $v_0 = \varphi_0$ and (2.21).

As the matrix coefficient in (3.10) is bi- K -isotypic, only one mode is excited in each expansion. Hence

$$\langle \pi(g).\psi_{-2k}, \psi_{-2n} \rangle = \langle \psi_{-2k}, v_{-2k} \rangle \langle v_{-2n}, \psi_{-2n} \rangle \langle \pi(g).v_{-2k}, v_{-2n} \rangle + \langle \pi(g).\psi_{-2k}^\perp, \psi_{-2n}^\perp \rangle, \quad (3.11)$$

where the K -spans of ψ_1^\perp and ψ_2^\perp are one-dimensional and the last matrix coefficient is tempered. Note that

$$\langle \psi_{-2k}, v_{-2k} \rangle = \langle \psi, v_{-2k} \rangle. \quad (3.12)$$

Proposition 3.13. *As $T \rightarrow \infty$,*

$$\begin{aligned} \mathcal{H}(T) &= \langle \psi, v_{-2k} \rangle \langle v_{-2n}, \psi \rangle \int_{t=0}^{2 \log T} \langle \pi(a_t).v_{-2k}, v_{-2n} \rangle \sinh t dt \\ &\quad + O(\|\psi\|_2^2 T \log T). \end{aligned} \quad (3.14)$$

Proof. The main term is simply a combination of (3.10), (3.11), and (3.12). It remains to estimate

$$\int_G f_T(g) \langle \pi(g).\psi_{-2k}^\perp, \psi_{-2n}^\perp \rangle.$$

Take absolute values and apply mixing (2.41). The Haar measure on da_t is $\sinh t dt$, giving

$$\|\psi\|^2 \int_{A^+} \mathbf{1}_{\|a_t\| < T} t e^{-t/2} \sinh t dt \ll \|\psi\|_2^2 T \log T, \quad (3.15)$$

as claimed. \square

Recall that ψ has unit mass, and is compactly supported in a ball of radius η about the origin. This fact has two implications: the first is that

$$\langle v_{-2n}, \psi \rangle = v_{-2n}(e) + O(\eta),$$

and the second is that, since G is a 3-dimensional space, we have

$$\|\psi\|^2 \ll \eta^{-3}. \quad (3.16)$$

Combining these facts with (3.4), we now have

$$\begin{aligned} \mathcal{N}(T) &= \bar{v}_{-2k}(e)v_{-2n}(e) \int_{t=0}^{2\log T} \langle \pi(a_t).v_{-2k}, v_{-2n} \rangle \sinh t \, dt \quad (3.17) \\ &\quad + O\left(\eta(1 + |n| + |k|)T^{2\delta} + \eta^{-3} T \log T\right). \end{aligned}$$

The optimal choice

$$\eta = T^{(1-2\delta)/4}(\log T)^{1/4}(1 + |n| + |k|)^{-1/4}$$

in (3.17) leads to the error term

$$O\left(T^{1\cdot\frac{1}{4}+2\delta\cdot\frac{3}{4}}(\log T)^{1/4}(1 + |n| + |k|)^{3/4}\right),$$

as claimed.

Returning to (3.17), it remains only to evaluate the main term. This is simply a matter of combining (2.29), (2.21), and (2.39), together with (2.35) and (2.31). This completes the proof of Theorem 1.5.

4. PROOF OF THEOREM 1.13

As before, let $\Gamma < G = \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian group of the second kind and assume the limit set of Γ has Hausdorff dimension $\delta > 1/2$. Assume $\Gamma < \mathrm{SL}(2, \mathbb{Z})$ with ramification number \mathfrak{B} , and let Θ be a spectral gap for Γ . For $q \geq 1$, write $q = q'q''$ with $q' \mid \mathfrak{B}$, and let

$$0 < \delta(1 - \delta) = \lambda_0^q < \lambda_1^q \leq \cdots \leq \lambda_{N(q)}^q < 1/4$$

be the point spectrum of the Laplacian acting on $L^2(\Gamma(q)\backslash\mathbb{H})$. The eigenvalues below $\Theta(1 - \Theta)$ are all oldforms coming from level 1, with the possible exception of finitely many eigenvalues coming from level q' .

Fix a function $f(g) = f(\theta_1(g), t(g), \theta_2(g))$ in KA^+K coordinates, and fix any $\gamma_0 \in \Gamma$. Assume that $|f| \leq 1$. Our goal in this section is to evaluate

$$\mathcal{N}_q(T) := \sum_{\substack{\gamma \in \gamma_0 \cdot \Gamma(q) \\ \|\gamma\| < T}} f(\gamma).$$

For $g \in G$, let

$$f_T(g) := f(g) \mathbf{1}_{\|g\| < T},$$

and define $\mathcal{F}_{T,q} : \Gamma(q) \backslash G \times \Gamma(q) \backslash G \rightarrow \mathbb{C}$ via

$$\mathcal{F}_{T,q}(g, h) := \sum_{\gamma \in \Gamma(q)} f_T(g^{-1}\gamma h). \quad (4.1)$$

Clearly $\mathcal{F}_{T,q}(\gamma_0^{-1}, e) = \mathcal{N}_q(T)$.

Now for a fixed η (to be chosen later depending on T), let

$$\psi : G \rightarrow \mathbb{R}$$

be a smooth test function with unit mass, $\int_G \psi = 1$, and compact support in a ball of radius η about the identity $e \in G$. Average over the group:

$$\Psi_q(g) := \sum_{\gamma \in \Gamma(q)} \psi(\gamma g). \quad (4.2)$$

Let

$$\psi_{\gamma_0}(g) = \psi(\gamma_0 g),$$

and similarly average

$$\Psi_{q,\gamma_0}(g) := \Psi_q(\gamma_0 g) = \sum_{\gamma \in \Gamma(q)} \psi_{\gamma_0}(\gamma g).$$

The integral

$$\begin{aligned} \mathcal{H}_q(T) &:= \langle \mathcal{F}_T, \Psi_{q,\gamma_0} \otimes \Psi_q \rangle = \int_{\Gamma(q) \backslash G} \int_{\Gamma(q) \backslash G} \mathcal{F}_T(g, h) \Psi_{q,\gamma_0}(g) \Psi_q(h) dg dh \\ &= \int_{\Gamma(q) \backslash G} \int_{\Gamma(q) \backslash G} \mathcal{F}_T(\gamma_0^{-1}g, h) \Psi_q(g) \Psi_q(h) dg dh \end{aligned} \quad (4.3)$$

again approximates $\mathcal{N}_q(T)$ as follows. Recall that

$$\mathcal{S}_{\infty,T} f = \max_{X \in \{0, X_1, X_2, X_3\}} \sup_{g \in G, \|g\| < T} |d\pi(X) \cdot f(g)|,$$

Proposition 4.4.

$$\mathcal{H}_q(T) = \mathcal{N}_q(T) + O(\eta(1 + \mathcal{S}_{\infty,T} f)T^{2\delta}). \quad (4.5)$$

Proof. This is the same argument as in the proof of Proposition 3.4, except using the bound

$$|f_T(g^{-1}\gamma_0\gamma h) - f_T(\gamma_0\gamma)| \ll \eta \mathcal{S}_{\infty, T} f, \quad (4.6)$$

for $\|\gamma_0\gamma\| < T/(1 + \eta)$. \square

The argument leading to (3.8) also gives

Lemma 4.7. *The inner product $\mathcal{H}_q(T)$ can be expressed as:*

$$\mathcal{H}_q(T) = \int_G f_T(g) \langle \pi(g) \cdot \Psi_q, \Psi_{q, \gamma_0} \rangle_{\Gamma(q) \backslash G} dg. \quad (4.8)$$

For ease of exposition, assume the spectrum below $\Theta(1 - \Theta)$ consists of only the base eigenvalue $\lambda_0 = \delta(1 - \delta)$ corresponding to $\varphi^{(q)}$, and one newform $\tilde{\varphi}^{(q)}$ from the “bad” level $q' \mid \mathfrak{B}$. The general case is a finite sum of such terms. The normalizations are such that

$$\varphi^{(q)} = \frac{1}{\sqrt{[\Gamma : \Gamma(q)]}} \varphi^{(1)}, \quad (4.9)$$

and

$$\tilde{\varphi}^{(q)} = \frac{1}{\sqrt{[\Gamma(q') : \Gamma(q)]}} \tilde{\varphi}^{(q')} = \frac{1}{\sqrt{[\Gamma : \Gamma(q)]}} \sqrt{[\Gamma : \Gamma(q')]} \tilde{\varphi}^{(q')},$$

with $\varphi^{(1)}$ a normalized newform in $\mathcal{L}^2(\Gamma \backslash G)$, and $\tilde{\varphi}^{(q')} \in \mathcal{L}^2(\Gamma(q') \backslash G)$. Let V and \tilde{V} be the irreducible vector subspaces of $\mathcal{L}^2(\Gamma(q) \backslash G)$ generated by the G -spans of $\varphi^{(q)}$ and $\tilde{\varphi}^{(q)}$, respectively. The space V has a dense subspace spanned by the K -fixed vector $\varphi^{(q)}$ and its translates $\varphi_{2k}^{(q)}$ under ladder operators, and similarly with \tilde{V} . Write

$$\Psi_q = \Psi_q|_V + \Psi_q|_{\tilde{V}} + \Psi_q^\perp,$$

and similarly with Ψ_{q, γ_0} , where the projections are

$$\Psi_q|_V := \sum_{k \in \mathbb{Z}} \langle \Psi_q, \varphi_{2k}^{(q)} \rangle \varphi_{2k}^{(q)}, \quad (4.10)$$

etc. Using (4.8), we can now write

$$\mathcal{H}_q(T) = W_q(T) + \tilde{W}_q(T) + W_q^\perp(T), \quad (4.11)$$

where

$$W_q(T) := \int_G f_T(g) \langle \pi(g) \cdot \Psi_q|_V, \Psi_{q, \gamma_0}|_V \rangle_{\Gamma(q) \backslash G} dg, \quad (4.12)$$

and similarly with the other two pieces.

Lemma 4.13.

$$\left\langle \Psi_q, \varphi_{2k}^{(q)} \right\rangle_{\Gamma(q)\backslash G} = \frac{1}{\sqrt{[\Gamma : \Gamma(q)]}} \left\langle \Psi_1 \varphi_{2k}^{(1)} \right\rangle_{\Gamma\backslash G},$$

where Ψ_1 is defined by averaging over all of Γ , as in (4.2). The same equality holds for Ψ_{q,γ_0} .

Proof. Using (4.9) and (4.2), unfold and refold the sum:

$$\begin{aligned} \left\langle \Psi_q, \varphi_{2k}^{(q)} \right\rangle_{\Gamma(q)\backslash G} &= \frac{1}{\sqrt{[\Gamma : \Gamma(q)]}} \int_{\Gamma(q)\backslash G} \sum_{\gamma \in \Gamma(q)} \psi(\gamma g) \varphi_{2k}^{(1)}(g) dg \\ &= \frac{1}{\sqrt{[\Gamma : \Gamma(q)]}} \int_G \psi(g) \varphi_{2k}^{(1)}(g) dg \\ &= \frac{1}{\sqrt{[\Gamma : \Gamma(q)]}} \int_{\Gamma\backslash G} \sum_{\gamma \in \Gamma} \psi(\gamma g) \varphi_{2k}^{(1)}(g) dg \\ &= \frac{1}{\sqrt{[\Gamma : \Gamma(q)]}} \left\langle \Psi_1(g) \varphi_{2k}^{(1)} \right\rangle_{\Gamma\backslash G}, \end{aligned}$$

as claimed. Replacing Ψ_q by Ψ_{q,γ_0} , one makes the additional change of variables $g \mapsto \gamma_0 g$, and uses the Γ -invariance of $\varphi_{2k}^{(1)}$. \square

Lemma 4.14. For any $k, k' \in \mathbb{Z}$,

$$\left\langle \pi(g) \varphi_{2k}^{(q)}, \varphi_{2k'}^{(q)} \right\rangle_{\Gamma(q)\backslash G} = \left\langle \pi(g) \varphi_{2k}^{(1)}, \varphi_{2k'}^{(1)} \right\rangle_{\Gamma\backslash G}.$$

Proof. Using (4.9) on each function gives a factor of $\frac{1}{\sqrt{[\Gamma : \Gamma(q)]}}$, which is cancelled by the fact that the integral over the space $\Gamma(q)\backslash G$ is $[\Gamma : \Gamma(q)]$ times larger than that over $\Gamma\backslash G$. \square

Lemma 4.15.

$$W_q(T) = \frac{1}{[\Gamma : \Gamma(q)]} W_1(T),$$

where

$$W_1(T) := \int_G f_T(g) \left\langle \pi(g) \cdot \Psi_1|_V, \Psi_1|_V \right\rangle_{\Gamma\backslash G} dg.$$

Proof. This is simply a combination of (4.10), (4.12), and Lemmata 4.13 and 4.14. \square

In the same way, one proves

Lemma 4.16.

$$\tilde{W}_q(T) = \frac{1}{[\Gamma : \Gamma(q)]} \mathcal{E}_{q'}(T),$$

where

$$\mathcal{E}_{q'}(T) := [\Gamma : \Gamma(q')] \int_G f_T(g) \langle \pi(g) \cdot \Psi_{q'}|_{\tilde{V}}, \Psi_{q', \tilde{\gamma}_0}|_{\tilde{V}} \rangle_{\Gamma(q') \backslash G} dg.$$

Here $\tilde{\gamma}_0$ is a representative for γ_0 in $\Gamma(q') \backslash \Gamma$.

The term W_q^\perp is handled using the spectral gap in a similar way as (3.15).

Lemma 4.17.

$$W_q^\perp(T) \ll T^{2\Theta} \eta^{-6}.$$

Proof. The bound (2.43) gives

$$W_q^\perp(T) \ll T^{2\Theta} \|\Psi_q\|_2 \mathcal{S}\Psi_q.$$

We estimate

$$\|\Psi_q\| \ll \eta^{-3/2}, \quad \text{and} \quad \mathcal{S}\Psi_q \ll \eta^{-9/2},$$

since Ψ_q is a bump function in a 3-dimensional ball of radius η . □

Putting everything together gives

Proposition 4.18.

$$\mathcal{N}_q(T) = \frac{1}{[\Gamma : \Gamma(q)]} \left(\mathcal{N}_1(T) + \mathcal{E}_{q'}(T) \right) + O\left(T^{2\Theta} \eta^{-6} + \eta(1 + \mathcal{S}_{\infty, Tf}) T^{2\delta} \right)$$

Proof. Combining (4.5), (4.11), and Lemmata 4.15, 4.16, and 4.17, gives

$$\mathcal{N}_q(T) = \frac{1}{[\Gamma : \Gamma(q)]} \left(W_1(T) + \mathcal{E}_{q'}(T) \right) + O\left(T^{2\Theta} \eta^{-6} + \eta(1 + \mathcal{S}_{\infty, Tf}) T^{2\delta} \right).$$

On the other hand, the same argument gives

$$\mathcal{N}_1(T) = W_1(T) + O\left(T^{2\Theta} \eta^{-6} + \eta(1 + \mathcal{S}_{\infty, Tf}) T^{2\delta} \right).$$

Combining the two completes the proof. □

The optimal choice of

$$\eta = (1 + \mathcal{S}_{\infty, Tf})^{-1/7} T^{-2(\delta-\Theta)/7}$$

gives the error term claimed in Theorem 1.13.

5. PROOF OF THEOREM 1.14

Assume $\Gamma < \mathrm{SL}(2, \mathbb{Z})$ has critical exponent $\delta > 1/2$. Recall that N is a parameter going to infinity, T and K are small positive powers of N , $v, w \in \mathbb{Z}^2$, $n \in \mathbb{Z}$, $\frac{N}{K} < |n| < N$, $|w| < \frac{N}{T}$, $|v| \leq 1$, and $|n| < |v||w|T$. We wish to give a lower bound for the number of $\gamma \in \Gamma$, $\|\gamma\| < T$ such that

$$|\langle v\gamma, w \rangle - n| < \frac{N}{K}.$$

Decompose γ in KA^+K coordinates,

$$\begin{aligned} \gamma &= k_u a_\rho k_v = \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix} \begin{pmatrix} \rho & \\ & 1/\rho \end{pmatrix} \begin{pmatrix} \cos v & \sin v \\ -\sin v & \cos v \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\cos(u) \cos(v)} - \frac{\sin(u) \sin(v)}{\rho} & \frac{\cos(v) \sin(u)}{\rho} + \rho \cos(u) \sin(v) \\ -\rho \cos(v) \sin(u) - \frac{\cos(u) \sin(v)}{\rho} & \frac{\cos(u) \cos(v)}{\rho} - \rho \sin(u) \sin(v) \end{pmatrix}. \end{aligned}$$

As $1 < \rho \approx \|\gamma\|$, we have

$$\rho < T.$$

Let $v = (a, b)$ and $w = (c, d)$. The condition

$$|\langle v\gamma, w \rangle - n| < \frac{N}{K}$$

becomes in (u, ρ, v) coordinates

$$\begin{aligned} |\langle v\gamma, w \rangle - n| &= |(a, b)k_u a_\rho k_v^t(c, d) - n| \\ &= \left| \left(\rho(a \cos u - b \sin u), \frac{1}{\rho}(a \sin u + b \cos u) \right) \cdot \left(c \cos v + d \sin v, -c \sin v + d \cos v \right) - n \right| \\ &= \left| \rho(a \cos u - b \sin u)(c \cos v + d \sin v) \right. \\ &\quad \left. + \frac{1}{\rho}(a \sin u + b \cos u)(-c \sin v + d \cos v) - n \right| \\ &\approx \left| \rho(a \cos u - b \sin u)(c \cos v + d \sin v) - n \right| \\ &< \frac{N}{K}. \end{aligned}$$

Let \mathbf{u} be the angle between the vectors (a, b) and $(\cos u, -\sin u)$. Similarly, let \mathbf{v} be the angle between (c, d) and $(\cos v, \sin v)$. Then the above becomes

$$\left| \rho |v| |w| \cos \mathbf{u} \cos \mathbf{v} - n \right| < \frac{N}{K},$$

or

$$\left| \frac{\rho}{T} \cos \mathbf{u} \cos \mathbf{v} - \frac{n}{|v| |w| T} \right| < \frac{N}{|v| |w| T K}.$$

Set

$$\mathcal{A} := \frac{n}{T |v| |w|}, \quad \text{and} \quad \mathcal{B} := \frac{N}{T |v| |w|}.$$

Both $\cos \mathbf{u}$ and $\cos \mathbf{v}$ can range in intervals independent of K , and hence so do u and v . By an obvious approximation argument, divide these intervals into sectors $u \in \Psi_\alpha$ and $v \in \Phi_\beta$. An application of Theorem 1.5 gives (using a smooth function to capture the lower bound)

$$\begin{aligned} & \sum_{\substack{\gamma \in \Gamma \\ T(\mathcal{A} - \frac{N}{K}) < \|\gamma\| < T(\mathcal{A} + \frac{N}{K})}} \mathbf{1}\{u \in \Psi_\alpha\} \mathbf{1}\{v \in \Phi_\beta\} \\ & \gg \frac{1}{K} \left(\mu(\Psi_\alpha) \mu(\Phi_\beta) c_0 T^{2\delta} + \sum_j c_j T^{2s_j} \right) + O\left(T^{\frac{3}{4} + 2\delta \frac{1}{4}} (\log T)^{1/4}\right). \end{aligned}$$

As $\Psi := \cup \Psi_\alpha$ and $\Phi := \cup \Phi_\beta$ are intervals independent of K , we have $\mu(\Psi) \gg 1$ and same with $\mu(\Phi)$. This completes the proof.

6. PROOF OF THEOREM 1.15

Recall that we wish to give an upper bound for the number of $\gamma \in \Gamma$, $\|\gamma\| < T$, with

$$|(c, d)\gamma - y| < \frac{N}{K}$$

and

$$(c, d)\gamma \equiv y \pmod{q}.$$

Here $y = (y_1, y_2) \in \mathbb{Z}^2$ with $|y| < N$, $|(c, d)| < \frac{N}{T}$ and $|y| < T|(c, d)|$.

Let $\Gamma_0(q)$ be the subgroup of Γ (of level q) which stabilizes (c, d) modulo q , that is $\gamma_0 \in \Gamma_0(q)$ iff $(c, d)\gamma_0 \equiv (c, d) \pmod{q}$. Then we decompose

$\gamma \in \Gamma$ as $\gamma = \gamma_0 \gamma_1$ with $\gamma_0 \in \Gamma_0(q)$ and $\gamma_1 \in \Gamma_0(q) \setminus \Gamma$. The count becomes

$$\begin{aligned} & \sum_{\gamma_1 \in \Gamma_0(q) \setminus \Gamma} \mathbf{1}\{(c, d)\gamma_1 \equiv y(q)\} \sum_{\substack{\gamma \in \Gamma_0(q) \cdot \gamma_1 \\ \|\gamma\| < T}} \mathbf{1}\{|(c, d)\gamma - y| < \frac{N}{K}\} \\ & \ll \frac{1}{[\Gamma : \Gamma_0(q)]} \sum_{\substack{\gamma \in \Gamma \\ \|\gamma\| < T}} \mathbf{1}\{|(c, d)\gamma - y| < \frac{N}{K}\} + O\left(T^{\frac{6}{7}2\delta + \frac{1}{7}2\Theta}\right), \end{aligned} \quad (6.1)$$

where we used Theorem 1.13 on the inner sum, and estimated

$$\sum_{\gamma_1 \in \Gamma_0(q) \setminus \Gamma} \mathbf{1}\{(c, d)\gamma_1 \equiv y(q)\} \ll 1.$$

It remains to analyze

$$\mathcal{N} := \sum_{\substack{\gamma \in \Gamma \\ \|\gamma\| < T}} \mathbf{1}\left\{|(c, d)\gamma - y| < \frac{N}{K}\right\}.$$

Writing γ in KA^+K coordinates, we have $\gamma = k_u a_\rho k_v$ with $1 < \rho \approx \|\gamma\| < T$. The condition $|(c, d) \cdot \gamma - y| < \frac{N}{K}$ becomes

$$\begin{aligned} \left(\frac{N}{K}\right)^2 & > |(c, d) \cdot \gamma - y|^2 \\ & = \left(\rho \cos v (c \cos u - d \sin u) - \frac{1}{\rho} \sin v (c \sin u + d \cos u) - y_1\right)^2 \\ & \quad + \left(\rho \sin v (c \cos u - d \sin u) + \frac{1}{\rho} \cos v (c \sin u + d \cos u) - y_2\right)^2 \\ & \approx \left(\rho \cos v (c \cos u - d \sin u) - y_1\right)^2 + \left(\rho \sin v (c \cos u - d \sin u) - y_2\right)^2 \\ & = \left(\rho |(c, d)| \cos \mathbf{u} - |y| \cos \mathbf{v}\right)^2 + |y|^2 (1 - \cos^2 \mathbf{v}), \end{aligned} \quad (6.2)$$

after a calculation. In the above, we set

$$c \cos u - d \sin u = (c, d) \cdot (\cos u, -\sin u) = |(c, d)| \cos \mathbf{u},$$

where \mathbf{u} is the angle between the vectors (c, d) and $(\cos u, -\sin u)$, and

$$y_1 \cos v + y_2 \sin v = (y_1, y_2) \cdot (\cos v, \sin v) = |y| \cos \mathbf{v},$$

where \mathbf{v} is the angle between those two vectors.

By positivity, we break (6.2) into two pieces. The piece

$$\left(\frac{N}{K}\right)^2 > |y|^2(1 - \cos^2 \mathbf{v})$$

requires $|\mathbf{v}| \ll \frac{N}{|y|K}$. This forces v to be contained in an interval, say Φ , of length $\ll \frac{N}{|y|K}$.

The second piece simplifies to

$$\left| \frac{\rho}{T} \cos \mathbf{u} - \mathcal{A} \right| \ll \frac{1}{K},$$

where

$$\mathcal{A} := \frac{|y|}{T|(c, d)|} \cos \mathbf{v}.$$

As $\rho < T$, \mathbf{u} ranges in a constant interval, say Ψ . We break into sectors as before and bound using Theorem 1.5:

$$\begin{aligned} \mathcal{N} &\ll \sum_{\substack{\gamma \in \Gamma \\ \mathcal{AT}(1 - \frac{c_1}{K}) < \|\gamma\| < \mathcal{AT}(1 + \frac{c_1}{K})}} \mathbf{1}\{u \in \Psi\} \mathbf{1}\{v \in \Phi\} \\ &\ll \frac{1}{K} \mu(\Psi) \mu(\Phi) T^{2\delta}. \end{aligned}$$

Since $|\Phi| \ll \frac{1}{K}$, we have $\mu(\Phi) \ll \frac{1}{K^\delta}$. Inserting the above into (6.1) and using $[\Gamma : \Gamma(q)] \gg q^2$ completes the proof.

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