Lecture No 3
Fast Diffusion
The Ricci flow on surfaces

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Outline

We will discuss non-linear parabolic equations of fast diffusion.
Our model is the fast diffusion equation

$$u_t = \Delta u^m = \operatorname{div}\left(m\, u^{m-1}\, \nabla u\right), \qquad m < 1.$$

- It appears in physical applications such as diffusion in plasma, thin liquid film dynamics.
- The case $m = 0$ in dimension $n = 2$ corresponds to the Ricci flow on surfaces.
- The case $m = \frac{n-2}{n+2}$ in dimensions $n \geq 3$ corresponds to the Yamabe flow.
- Since, the diffusivity $D(u) = m\, u^{m-1} \uparrow +\infty$, as $u \downarrow 0$ the equation becomes singular at $u = 0$, resulting to the phenomenon of fast-diffusion.

Scaling and the Barenblatt solution

Scaling: If u solves the fast diffusion equation $u_t = \Delta u^m$, then

$$\tilde{u}(x,t) = \gamma^{-1}\, u(\alpha\, x, \beta\, t), \qquad \gamma = \left(\frac{\beta}{\alpha^2}\right)^{\frac{1}{1-m}}$$

also solves the same equation.

Self-Similar solution: There exists a self-similar solution

$$U(x,t) = t^{-\lambda}\left(C + k\,\frac{|x|^2}{t^{2\mu}}\right)^{-\frac{1}{1-m}}$$

with

$$\lambda^{-1} = \frac{2}{n} - (1-m), \quad \mu = \frac{\lambda}{n}, \quad k = \frac{\lambda\,(1-m)}{2mn}.$$

The above is a solution if $\frac{2}{n} - (1-m) > 0$, i.e. $m > \frac{n-2}{n}$.

The exponent $m = \frac{n-2}{n}$ is **critical**.

The Aronson-Bénilan inequality

Every solution u to the fast-diffusion equation

$$u_t = \Delta u^m, \qquad \frac{(n-2)_+}{n} < m < 1$$

satisfies the differential inequality

$$(*) \quad u_t \geq -\frac{\lambda\, u}{t}$$

with $\lambda^{-1} = \frac{2}{n} - (1-m) > 0$ iff $m > \frac{(n-2)_+}{n}$.
The *pressure* $v := \frac{m}{1-m}\, u^{-(1-m)}$ which evolves by the equation

$$v_t = (1-m)\, v\, \Delta v - |\nabla v|^2$$

satisfies the sharp differential inequality

$$(**) \qquad \Delta v \leq \frac{\lambda}{t}.$$

Remark: The Aronson-Bénilan $(*)$ inequality follows from $(**)$. The differential inequality $(**)$ becomes an equality when v is the Barenblatt solution.

The Li-Yau type Harnack Inequality

Combining the Aronson-Bénilan inequality $\Delta v \leq \frac{\lambda}{t}$ with the evolution equation for v gives the following Li-Yau type differential inequality:

$$-v_t + (1-m)\,\lambda\,\frac{v}{t} \geq |\nabla v|^2.$$

Integrating this inequality on optimal paths we obtain:
Harnack Inequality (Auchmuty-Bao and Hamilton)
If $0 < t_1 < t_2$, then

$$v(x_2, t_2) \leq \left(\frac{t_2}{t_1}\right)^{\mu} \left[v(x_1, t_1) + \frac{\delta}{4}\,\frac{|x_2 - x_1|^2}{t_2^{\delta} - t_1^{\delta}}\, t_1^{\mu} \right].$$

with $\mu = (1-m)\,\lambda > 0$ and $\delta = \frac{2\lambda}{n}$.

Application: Solutions of $u_t = \Delta u^m$ with $\frac{(n-2)_+}{n} < m < 1$ satisfy the lower bound:

$$u(x,t) \geq c(t)\,(1+|x|^2)^{-\frac{1}{1-m}}.$$

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The other Aronson-Bénilan inequality

A simple scaling argument shows that every solution u to the fast-diffusion equation $u_t = \Delta u^m$, $0 \leq m < 1$ satisfies the differential inequality

$$(*) \qquad u_t \leq \frac{u}{(1-m)\,t}.$$

Integrating $(*)$ on a time interval $[t_1, t_2]$ gives:

$$u(x,t_2) \leq u(x,t_1)\left(\frac{t_2}{t_1}\right)^{\frac{1}{1-m}}, \quad \forall x$$

i.e. the L^∞ norm of a solution doesn't blow up.

Remark: In the range of exponents $\frac{(n-2)_+}{n} < m < 1$, solutions u exhibit a regularizing effect from L^1_{loc} to L^∞_{loc}:

$$\sup_{|x|\leq R} u(x,t) \leq F\left(t, R, \int_{B_{2R}} u_0(x)\,dx\right).$$

The Cauchy problem in the super-critical case

Consider the fast-diffusion equation

$$(*) \qquad u_t = \Delta u^m, \qquad \frac{(n-2)_+}{n} < m < 1.$$

In the super-critical case no growth conditions need to be imposed on the initial data for existence. More precisely:

- For any nonnegative continuous weak solution u of $(*)$, there exists a unique locally finite Borel measure μ_0 on $\mathbb{R}^n$ such that

$$\lim_{t\downarrow 0} u(\cdot, t) = \mu_0 \qquad \text{in} \quad D'(\mathbb{R}^n).$$

- The trace μ_0 determines the solution uniquely.
- For any locally finite Borel measure μ_0 on $\mathbb{R}^n$ there exists a continuous weak solution u of $(*)$ in $S_\infty = \mathbb{R}^n \times (0, \infty)$ with initial trace μ_0.
- Remark: Solutions exist for all time $t > 0$ independently of their initial data.

The sub-critical case $m < (n-2)_+/n$

- In the sub-critical case $m < \frac{(n-2)_+}{n}$ the analogues of the above results do not hold true. In particular, there exists no solution with initial data the Dirac mass.
- One particularly interesting case of critical fast diffusion is the case where $u_t = \Delta log u$, corresponding to the limiting case of $u_t = \frac{\Delta u^m}{m} = \mathrm{div}(u^{-1}\nabla u)$, when $m \to 0$. In dim $n = 2$ it corresponds to the Ricci flow on surfaces.
- The sub-critical fast-diffusion equation $u_t = \Delta u^{\frac{n-2}{n+2}}$ corresponds to the evolution of a conformally flat metric $g_{ij} = u^{\frac{4}{n+2}}\, dx_i dx_j$ by the Yamabe flow $(g_{ij})_t = -R\, g_{ij}$, where $R = -u^{-1}\,\Delta u^{\frac{n-2}{n+2}}$ denotes the scalar curvature of g.
- The Yamabe flow was used to give an alternative proof of the well known Yamabe conjecture.

Logaritmic fast-diffusion

We will now consider the logarithmic fast-diffusion equation

$$(*) \quad u_t = \Delta \log u, \quad \text{in } \mathbb{R}^n \times [0, T), \;\; T > 0.$$

- Lions and Toscani: $(*)$ arises as a singular limit for finite velocity Boltzmann kinetic models.
- Kurtz: $(*)$ describes the limiting density distribution of two gases moving against each other and obeying the Boltzmann equation.
- In dimension $n = 2$ equation $(*)$ arises as a model for long Va-der-Wals interactions in thin films of a fluid spreading on a solid surface, if certain nonlinear fourth order effects are neglected.
- Angenent and L. Wu: In dim $n = 2$, equation $(*)$ represents the evolution of the conformal metric $g_{ij} = u\, dx_i dx_j$ under the Ricci flow, $(g_{ij})_t = -2\, R_{ij}$.

The Ricci flow on $\mathbb{R}^2$

- In 1982 **R. Hamilton** introduced the Ricci flow, namely the evolution of a Riemannian metric g_{ij} by

$$\text{(RF)} \qquad \frac{\partial g_{ij}}{\partial t} = -2\, R_{ij}$$

where R_{ij} denotes the Ricci curvature of the metric g_{ij}.

- In dimension $n = 2$ we have:

$$g_{ij} = u\, dx_i dx_j, \quad R_{ij} = \frac{1}{2} R\, g_{ij}, \quad R = -\frac{\Delta \log u}{u}$$

where R denotes the scalar curvature of g_{ij}.

- Hence, in dimension $n = 2$ the Ricci flow (RF) is equivalent to the equation:

$$(*) \qquad u_t = \Delta \log u.$$

Examples of solutions of $u_t = \Delta \log u$ on $\mathbb{R}^2$

- Contracting spheres: $u(x,t) = \frac{8\,\lambda\,(T-t)}{(\lambda+|x|^2)^2}, \;\; \lambda > 0.$

  They are *ancient solutions* which vanish at time $t = T$ and:

$$\frac{d}{dt}\int_{\mathbb{R}^2} u\,dx = \int_{\mathbb{R}^2} R\,u = -4\pi.$$

- Cigar solution: $u(x,t) = \frac{1}{\lambda\,|x|^2 + e^{4\lambda t}}.$

  They are *eternal complete non-compact* solutions which look like cigars and have infinite area, i.e.

$$\int_{R^2} u\,dx = \infty.$$

- Cusp solution: $u(x,t) = \frac{2t}{|x|^2\,\log^2|x|}, \;\; |x| > 1.$

  They are *complete non-compact* solutions which look like cusps and have finite area.

Solutions on orbifolds

- Cylindrical change of coordinates:
  If $u(r,t)$ is a **radial** solution of $u_t = \Delta \log u$ in $\mathbb{R}^2 \times (0,T)$, then $v(s,t) := r^2\, u(r,t)$, $s = \log r$ is a solution of

$$(*) \qquad v_t = (\log v)_{ss}, \qquad \text{on } \mathbb{R} \times (0,T).$$

- We seek for solutions $v^\mu(s,t)$ of $(*)$ with $T = 1$ in the form $v^\mu(s,t) = (1-t)\, g(s - \gamma_\mu \log(1-t))$. It follows that for each $\mu > 0$, there exists γ_μ and solution v^μ such that

$$\frac{d}{dt} \int_{-\infty}^{\infty} v^\mu(s,t)\, ds = -(2+\mu).$$

  Then, $u^\mu := |x|^{-2}\, v(\log|x|, t)$ defines a radial solution of $u_t = \Delta \log u$ on in $\mathbb{R}^2 \times (0,T)$ such that

$$\frac{d}{dt} \int_{\mathbb{R}^2} u^\mu(x,t) dx = -2\pi\,(2+\mu).$$

  The u^μ, $\mu \neq 2$ define metrics on orbifolds. When $\mu = 2$ we recover the contracting spheres.

The Cauchy problem

Consider the Cauchy problem

$$(*) \quad \begin{cases} u_t = \Delta \log u & \text{in} \quad \mathbb{R}^2 \times [0, T) \\ u(x,0) = f & \text{on} \quad \mathbb{R}^2 \end{cases}$$

with initial data $f \geq 0$.

- If $\int_{\mathbb{R}^2} f\, dx < \infty$, then $\forall \mu \geq 0$, $\exists\, u_\mu$ solution of $(*)$ on $\mathbb{R}^2 \times (0, T_\mu)$ with $T_\mu = \frac{1}{2\pi(2+\mu)} \int_{\mathbb{R}^2} f(x)\, dx$ satisfying

$$\frac{d}{dt} \int_{\mathbb{R}^2} u^\mu(x,t)\, dx = -2\pi\,(2+\mu).$$

- If $\int_{\mathbb{R}^2} f\, dx = \infty$, $\exists\, u$ solution of $(*)$ on $\mathbb{R}^2 \times (0, \infty)$.

Remarks on the Cauchy problem

- If $\int_{\mathbb{R}^2} f\,dx < \infty$, then every solution vanishes at time $T \le T_{\max}$, with $T_{\max} = \frac{1}{4\pi} \int_{\mathbb{R}^2} f(x)\,dx$.
- The maximal solution ($\mu = 0$) defines **complete** metrics on $\mathbb{R}^2$ of finite area which evolve by the Ricci flow and behave as cusps near infinity.
- The intermediate solution u_μ with $\mu = 2$ corresponds to smooth metrics on S^2 evolving by the Ricci flow.
- All other solutions u_μ ($\mu \ne 0, 2$) correspond to metrics on orbifolds evolving by the Ricci flow.
- Radial u_μ, $\mu > 0$ are characterized by the outgoing flux at infinity:
$$\lim_{r \to \infty} r\,(\log u_\mu)_r = -(2+\mu).$$
- Conclusion: A strong non-uniqueness phenomenon takes place. This is in contrast to the uniqueness of solutions to equations $u_t = \Delta u^m$, $m > 0$.

The Ricci flow on surfaces

- Consider the intermediate solution u of $u_t = \Delta \log u$ with area decay: $\frac{d}{dt} \int_{\mathbb{R}^2} u(x,t)\, dx = -8\,\pi$.
- An example of such a solution is the contracting spheres

$$u_s(x,t) = \frac{8\,(T-t)}{(1+|x|^2)^2}.$$

- Any such solution can be lifted on S^2 defining smooth metrics $ds^2 = \bar{u}\, ds_p^2$ evolving by the Ricci flow. This solution will exist up to time $T = \frac{1}{8\pi} \int_{S^2} \bar{u}(\cdot, 0)\, da$.
- The special solution u_s when lifted on S^2 is $\bar{u}_s = 2\,(T-t)$.
- Hamilton: Any solution $ds^2 = \bar{u}\, ds_p^2$ of the Ricci flow on S^2 will converge (after re-scaling) as $t \to T$ to the round sphere solution $\bar{u}_s$.

Vanishing behavior of the maximal solution

- Consider the maximal solution u of $u_t = \Delta \log u$ with compactly supported initial data u_0. Its area decays as:

$$\frac{d}{dt} \int_{\mathbb{R}^2} u(x,t)\, dx = -4\,\pi.$$

- Hence u will vanish at time $T = \frac{1}{4\pi} \int_{\mathbb{R}^2} u_0(x)\, dx.$
- Any such solution will enjoy the asymptotic behavior:

$$u(x,t) = \frac{2t}{|x|^2 \log^2 |x|}(1 + o(1)), \qquad \forall t < T$$

  however the bound deteriorates as $t \to T$.
- Question: What is the vanishing behavior of u as $t \to T$?

Vanishing behavior of the maximal solution

Jointly with M. del Pino and N. Sesum we have shown the vanishing asymptotic behavior for the maximal solution u:

- On the outer region: $(T-t)\log|x| > T$, we have

$$u(x,t) \approx \frac{2T}{|x|^2 \log^2|x|}, \quad \text{as } t \to T^-.$$

- On the inner region: $(T-t)\log|x| > T$, or $|x| = O(e^{\frac{T}{T-t}})$, u has the self-similar profile:

$$u(x,t) \approx (T-t)^2 \, e^{-\frac{2T}{T-t}} \, \phi(e^{-\frac{T}{T-t}}|x|)$$

  where $\phi(r) = \frac{2T^{-1}}{r^2+b}$.
  This shows that in the inner region, after re-scaling, the solution u converges to one of the cigar solutions.

- Remark: The vanishing of the maximum solutions if of type II.

Selected References

- Aronson, D.G. and Bénilan P., Régularité des solutions de l'équation de milieux poreux dans $\mathbf{R}^n$, C.R. Acad. Sci. Paris, 288, 1979, pp 103-105.
- Auchmuty, G. and Bao, D., Harnack-type inequalities for evolution equations. Proc. Amer. Math. Soc. 122 (1994), no. 1, 117–129.
- B.E.J. Dahlberg, and C.E. Kenig, Nonnegative solutions to fast diffusions, Rev. Mat. Iberoamericana 4 (1988), no. 1, 11–29.
- Daskalopoulos,P., del Pino M.A., On a Singular diffusion equation, Comm. in Analysis and Geometry, Vol. 3, 1995, pp 523-542.
- Daskalopoulos, P.; Hamilton, R. Geometric estimates for the logarithmic fast diffusion equation. Comm. Anal. Geom. 12 (2004), no. 1-2, 143–164.

Selected References

- R. Hamilton, The Ricci flow on surfaces, *Contemp. Math., 71, Amer. Math. Soc., Providence, RI*, 1988, pp. 237-262.
- M. Herrero and M. Pierre, The Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$, Trans. Amer. Math. Soc., 291, 1985, pp. 145-158.
- Lions, Pierre Louis; Toscani, Giuseppe , Diffusive limit for finite velocity Boltzmann kinetic models. Rev. Mat. Iberoamericana 13 (1997), no. 3, 473–513.
- Rodriguez, Ana; Vázquez, Juan L.; Esteban, Juan R. The maximal solution of the logarithmic fast diffusion equation in two space dimensions, *Adv. Differential equations* **2** (1997), no. 6, 867–894.
- Wu, L.-F., A new result for the porous medium equation derived from the Ricci flow, *Bull. Amer. Math. Soc., 28*, 1993, pp 90-94.