Lecture No 1
Introduction to Diffusion equations
The heat equation

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Outline of the lectures

We will discuss some basic models of diffusion equations and present their basic properties.

Such models include various geometric flows such as: the curve shortening flow, the Ricci flow on surfaces and the Yamabe flow.

We will discuss existence, regularity, a priori estimates, asymptotic behavior and classification of solutions.

- Lecture 1: Introduction and the heat equation
- Lecture 2: Slow diffusion and free-boundaries
- Lecture 3: Fast diffusion and the Ricci flow on surfaces
- Lecture 4: Ancient solutions to the curve shortening flow and the Ricci flow on surfaces

Introduction to diffusion

- The simplest model of linear diffusion is the familiar heat equation:

$$u_t = \Delta u, \qquad u = u(x,t), \quad x \in \mathbb{R}^n, \quad T > 0$$

where $\Delta u := \sum_{i=1}^n u_{x_i x_i} = \text{div}\,(\nabla u)$. (Diffusion of heat, chemical concentration).

- One of the simplest models of non-linear diffusion is the porous medium $(m > 1)$ or fast-diffusion $(m < 1)$ equation

$$u_t = \Delta u^m, \quad u \geq 0, \quad x \in \mathbb{R}^n, \quad T > 0.$$

(Diffusion of gas through a porous medium, population dynamics, gas kinetics, diffusion in plasma, geometry).

- Another simple model is the reaction-diffusion equation

$$u_t = \Delta u + u^p, \qquad u \geq 0, \quad p > 0.$$

(Population dynamics, geometry).

Other models of Diffusion

- Evolution p-Laplacian Equation (quasi-linear)

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\, \nabla u\right), \qquad p > 0.$$

- Motion of a curve $y = u(x,t)$ by its curvature

$$u_t = \frac{u_{xx}}{1+u_x^2}.$$

- Motion of a surface $z = u(x,y,t)$ in $\mathbb{R}^3$ by its mean curvature

$$u_t = \frac{(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy}}{1+u_x^2+u_y^2}$$

- Motion of a surface $y = u(x,y,t)$ in $\mathbb{R}^3$ by its Gaussian curvature (fully-nonlinear)

$$u_t = \frac{\det D^2 u}{(1+u_x^2+u_y^2)^{3/2}}.$$

Slow and fast-diffusion

Consider the equation $u_t = \Delta u^m$. Since

$$\Delta u^m = \mathrm{div}\,(\nabla u^m) = \mathrm{div}\,(m\,u^{m-1}\nabla u)$$

the diffusion is governed by $D(u) := m\,u^{m-1}$.

- $m = 1$: The diffusivity $D(u) := 1$ is constant in any direction and the equation is linear.
- $m > 1$: Since the diffusivity $D(u) := m\,u^{m-1} \downarrow 0$, as $u \downarrow 0$ and we have slow diffusion.
- $m < 1$: Since the diffusivity $D(u) := m\,u^{m-1} \uparrow +\infty$, as $u \downarrow 0$ and we have fast diffusion.
- $m = 0$: The equation takes the form $u_t = \Delta \log u$.
- $m < 0$: We have super-fast diffusion.

Remark: We will see in these lectures that slow and fast diffusion have very different properties.

The Heat Equation - Derivation

The heat equation $u_t = \Delta u$ describes the distribution of heat in a given region over time. The heat equation can be derived from the following principles:

- The amount of heat Q contained in a region Ω is proportional to the temperature T, the density ρ and the heat capacity κ_s of the material, i.e. if Ω is a region of the material:

$$Q = \int_\Omega \rho\, \kappa_s\, T(x,t)\, dx. \tag{1}$$

- The heat transfer through the boundary $\partial\Omega$ of the region is proportional to the heat conductivity σ, to the gradient of the temperature across the region and to the area of contact, i.e.

$$\frac{dQ}{dt} = \int_{\partial\Omega} \sigma\, \frac{\partial T}{\partial n}\, da. \tag{2}$$

If we differentiate (1) in time and apply the divergence Theorem in (2) we obtain:

$$\int_\Omega \rho\, \kappa_s \frac{\partial T}{\partial t}\, dx = \int_{\partial\Omega} \operatorname{div}(\sigma \nabla T)\, da.$$

Since Ω can be an arbitrary part of the material under study, we obtain the heat equation

$$\frac{\partial T}{\partial t} = \lambda \Delta T$$

if we assume that ρ, κ_s and σ are independent of the position x.

Remark: If the heat conductivity σ is taken to depend on the temperature T, then we obtain non-linear versions of the heat equation.

Parabolic scaling and the Fundamental Solution

Parabolic Scaling: If $u(x,t)$ solves the heat equation, then

$$\tilde{u}(x,t) = u(\gamma\, x, \gamma^2\, t), \qquad \forall \gamma > 0$$

also solves the heat equation.

Self-Similar solution: The above scaling suggests that we search for a special radially symmetric solution of the form

$$\Phi(x,t) = \alpha(t)\, v\left(\frac{|x|^2}{t}\right).$$

for some functions $\alpha(t)$ and $v(r)$. The above representation leads to the fundamental solution

$$\Phi(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\, e^{-\frac{|x|^2}{4t}}, \qquad t > 0.$$

Properties of the Fundamental solution

A simple calculation leads to:

$$\int_{\mathbb{R}^n} \Phi(x,t)\,dx = 1, \qquad \forall t > 0.$$

Also, $\lim_{t\to 0} \Phi(x,t) = 0$, for all $x \neq 0$. Hence,

$$\lim_{t\to 0} \Phi(\cdot,t) = \delta_0.$$

We observe the following two characteristic properties of the fundamental solution:

- Infinite speed of propagation: $\Phi(\cdot,t) > 0$, for all $t > 0$.
- Smoothing effect: Φ is C^∞ smooth in x and t, for all $t > 0$.

Remark: All solutions of the heat equation have the above two properties. Non-linear equations don't in general.

The Cauchy problem

Solutions u of the Cauchy problem

$$(*) \quad \begin{cases} u_t = \Delta u & \text{in} \quad \mathbb{R}^n \times [0, T) \\ u(\cdot, 0) = g & \text{on} \quad \mathbb{R}^n \end{cases}$$

are given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\, g(y)\, dy.$$

Indeed, we have:

$$u_t - \Delta_x u = \int_{\mathbb{R}^n} [(\Phi_t - \Delta_x \Phi)(x-y,t)]\, g(y)\, dy = 0$$

and (since $\lim_{t \to 0} \Phi(x-y,t) = \delta_x(y)$):

$$\lim_{t \to 0} u(x,t) = \lim_{t \to 0} \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy = g(x).$$

Non-homogeneous problem

Consider now solutions u of the non-homogeneous problem

$$(**) \quad \begin{cases} u_t - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u(\cdot,0) = 0 & \text{on} \quad \mathbb{R}^n. \end{cases}$$

Let $u(x,t;s)$ be the solution of the Cauchy problem

$$\begin{cases} u_t = \Delta u & \text{in} \quad \mathbb{R}^n \times (s,\infty) \\ u(\cdot,s) = f(\cdot,s) & \text{on} \quad \mathbb{R}^n \end{cases}$$

Duhamel's principle asserts that the solution u of $(**)$ is given by:

$$u(x,t) = \int_0^t u(x,t;s)\, ds, \quad x \in \mathbb{R}^n, \ t \geq 0.$$

Hence,

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\, f(y,s)\, dy\, ds.$$

The Mean-value formula

The well known mean value formula for harmonic functions: $\Delta u = 0$ in $\Omega \subset \mathbb{R}^n$, asserts that

$$u(x_0) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u(x)\, dx, \quad \text{if } B_r(x_0) \subset \Omega.$$

A similar formula holds for solutions of the heat equation $u_t = \Delta u$ in $Q_T = \Omega \times (0, T]$. Considering the *parabolic balls*:

$$E(x_0, t_0; r) := \left\{ (x,t) \in \mathbb{R}^{n+1} | t \leq t_0,\ \Phi(x - x_0, t_0 - t) \geq \frac{1}{r^n} \right\}$$

then

$$u(x_0, t_0) = \frac{1}{4r^n} \int_{E(x_0, t_0; r)} u(x,t) \frac{|x - x_0|^2}{(t - t_0)^2}\, dx\, dt$$

if $E(x_0, t_0; r) \subset Q_T$.

The Strong Maximum Principle

The parabolic boundary $\partial_p Q_T$ of the cylinder $Q_T := \Omega \times (0, T]$ is defined as:

$$\partial_p Q_T = (\partial\Omega \times (0, T]) \cup (\Omega \times \{0\}).$$

Strong maximum principle: If u solves the heat equation in Q_T then:

- We have:

$$\max_{\bar{Q}_T} u = \max_{\partial_p Q_T} u.$$

- Furthermore, if Ω is *connected* and there exists a point $(x_0, t_0) \in Q_T$ such that

$$u(x_0, t_0) = \max_{\bar{Q}_T} u$$

then u must be **constant** in Q_T.

Uniqueness of Solutions

Uniqueness on bounded domains: If $g \in C(\partial_p Q_T)$, then there exists **at most one** solution $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ of:

$$\begin{cases} u_t = \Delta u & \text{in } Q_T \\ u = g & \text{on } \partial_p Q_T \end{cases}$$

Maximum principle for the Cauchy problem: If u solves

$$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^n \times (0, T) \\ u(\cdot, 0) = g & \text{on } \mathbb{R}^n \end{cases}$$

$u \in C^{2,1}(\mathbb{R}^n \times (0, T]) \cap C(\mathbb{R}^n \times [0, T])$ and satisfies the growth estimate $\mathbf{u(x, t) \leq A\, e^{a\,|x|^2}}$, then $\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g$.

Uniqueness: Solutions to the Cauchy problem which satisfy the above growth condition are **unique.**

Remark: The growth assumption is necessary !

Regularity

Let $Q_r(x_0, t_0)$ denote the parabolic cylinder of size r around a point (x_0, t_0), namely:

$$Q_r(x_0, t_0) := B_r(x_0) \times [t_0 - r^2, t_0].$$

Derivative estimates: For each $k, l = 0, 1, \cdots$, there exists a constant $C_{k,l}$ such that

$$\max_{Q_{\frac{r}{2}}(x_0,t_0)} |D_x^k D_t^l u| \leq \frac{C_{k,l}}{r^{k+2l+n+2}} \|u\|_{L^1(Q_r(x_0,t_0))}$$

for all cylinders $Q_r(x_0, t_0) \subset Q_T$ and all solutions to the heat equation in Q_T.

Conclusion: Local solutions to the heat equation are C^∞-smooth.

Proof: You use the exact representation of solutions to the heat equation in terms of the initial data and cut-off functions.

Schauder Estimate

Parabolic Hölder spaces:

- We say that $u \in C^{\alpha}(Q_T)$ if

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C\,\left(|x_1 - x_2| + \sqrt{|t_1 - t_2|}\right)^{\alpha}.$$

- We say that $u \in C^{2+\alpha}(Q_T)$ if

$$u, u_t, D_x u, D_x^2 u \in C^{\alpha}(Q_T).$$

Schauder estimate: If u is a smooth solution of

$$\begin{cases} u_t - \Delta u = f & \text{in } Q_T \\ u = g & \text{on } \partial_p Q_T \end{cases}$$

then

$$\|u\|_{C^{2+\alpha}(Q_T)} \leq C\,\left(\|u\|_{C^0(Q_T)} + \|f\|_{C^{\alpha}(Q_T)} + \|g\|_{C^{2,\alpha}(\partial Q_T)}\right).$$

Harnack Inequality

Assume that u is a **nonnegative** solution of the heat equation

$$u_t = \Delta u, \qquad t > 0.$$

Then, u satisfies the Harnack inequality:

$$u(x_1, t_1) \leq \left(\frac{t_2}{t_1}\right)^{\frac{n}{2}} u(x_2, t_2)\, e^{\frac{|x_2 - x_1|^2}{4(t_2 - t_1)}}$$

for all $0 < t_1 < t_2$ and $x_1, x_2 \in \mathbb{R}^n$.

Application: If $u(0, T) \leq M < \infty$, then for all $t < T - \epsilon$ we have

$$u(x, t) \leq C(M, T)\, t^{-\frac{n}{2}}\, e^{\frac{|x|^2}{4\epsilon}}$$

i.e. nonnegative solutions of the heat equation grow at most exponentially as $|x| \to \infty$.

Widder theory for the heat equation

Let u be a nonnegative distributional solution of the heat equation $u_t = \Delta u$ in $S_T := \mathbb{R}^n \times (0, T]$.

- The initial trace μ exists; there exists a nonnegative Borel measure μ_0 on $\mathbb{R}^n$ such that $\lim_{t\downarrow 0} u(\cdot, t) = d\mu_0$ and satisfies the growth condition

$$(*) \quad \int e^{-\frac{C\,|x|^2}{T}}\, d\mu_0 \; < \infty$$

  where C is an absolute constant.
- The trace μ_0 determines the solution uniquely.
- For each nonnegative Borel measure μ_0 on $\mathbb{R}^n$ satisfying the $(*)$, there is a nonnegative continuous weak solution u the heat equation in S_T with trace μ_0 and

$$u(x,t) = \frac{C_n}{t^{n/2}} \int e^{-\frac{|x-y|^2}{4t}}\, d\mu_0(y)$$

  for an absolute constant C_n depending only on dimension n.