Framed correspondences.

(Jussieu Apr. 23. 2003)

формула (*) $Hom(M(X), M(Y)) = \dots$

- постановка задачи. Finite correspondences. construction of the mot. cat. Почему работает — потому что инт. нас теории обладают тр. для кон. соотв.
- все работает для более общих теорий e.g. K-теории, W
- Фовен и я придумали решение SH. Не столь удовлетворительное описание т.к. нет понятной формулы для $Hom(\Sigma^\infty X, \Sigma^\infty Y[i])$.
- Задача т. framed cor. в том чтобы дать описание SH в духе описаний ДМ через finite cor. В частности хотим аналог (*)

Picture:

$$Fr(Sm/k) \longrightarrow PFr(Sm/k) \longrightarrow H_{\mathbb{A}^1}(PFr(Sm$$

$$X \longmapsto F(X)$$

$$\underset{H_{\mathbb{A}^1}}{Hom}(F(X), F(Y)) = Hom_{SH}(\Sigma^\infty_T X_+, \Sigma^\infty_T Y$$

$$\| $$

$$\mathbb{H}^0(\ , \ )$$

– <u>Опр. 1</u> $Z \hookrightarrow X$ closed subset. A (glob. framing of Z of level n is . . . .

<u>Опр. 2</u> An expl. framed cor. of level n

$$X \longrightarrow Y$$

is . . . .

<u>Опр. 3</u> Two expl. are called equiv. if. A framed cor.

Notation $\quad Fr_*(X,Y) := \coprod_n Fr_n(X,Y)$

$Fr_{-1}(X,Y) := *$. Note: $Fr_1(X,Y) =$
$= Hom(X,Y)$.

<u>Lemma</u> $Fr_n(X,Y) = Hom_{Shv_{nis}}\left((X_+ \wedge (\mathbb{P}^1,\infty))^n, Y_+ \wedge (\mathbb{A}^1/\mathbb{A}^1-0)^n\right)$

<u>Proof:</u> Construct a map $\longrightarrow$.

<u>Composition:</u> $\quad X_+ \wedge (\mathbb{P}^1)^n \longrightarrow Y_+ \wedge T^n$

$$Y_+ \wedge (\mathbb{P}^1)^m \longrightarrow Z_+ \wedge T^m$$

$$X_+ \wedge (\mathbb{P}^1)^n \wedge (\mathbb{P}^1)^m \longrightarrow Y_+ \wedge (\mathbb{P}^1)^m \wedge T^n \longrightarrow Z_+ \wedge T^m \wedge T^n$$

$$\cancel{Y_+ \wedge T^m \wedge T^n}$$

Projection $\mathbb{P}^{1,\infty} \to T$ defines for any X a framed cor. $\sigma_X : X \to X$ of level 1. Expl. given by

$$t=0 \quad X \times \mathbb{A}^1 \longrightarrow X$$

<u>Def:</u> A framed presheaf $Fr_*(Sm/k) \to$ sets.

Called <u>stable</u> if:

a. $F(X \amalg Y) = F(X) \times F(Y)$

b. $F(\sigma_X) = Id \quad \forall X.$

~~The cat. of stable framed presheaves $SF(Sm/k)$ $\simeq$ the cat. of presh. on Sm_{con}/k s.t. $F(\sigma_X) = Id$~~ ?

is ~~a topos.~~

Called h. inv. if.

<u>Def. of +</u>

<u>Def.</u> Called group-like if..

<u>Lemma</u> Consider $Fr(X)$

$$Fr(X) : Y \longrightarrow colim(Fr_n(Y, X))$$

where given $Y \xrightarrow{n} X$ it is mapped to
$$\begin{tikzcd} Y \arrow[r, "n"] \arrow[rr, bend right, "n+1"'] & X \arrow[r, "\sigma_X"] & X \end{tikzcd}$$

Consider $C_* Fr(X)$.

<u>Lemma:</u> $\pi_i(C_* Fr(X))$ are stable and g.l. for is

Need $\mathcal{B}(X \amalg Y) \simeq \mathcal{B}(X) \times \mathcal{B}(Y)$.

Have $Fr(X)(Y_1 \amalg Y_2) = Fr(X)(Y_1) \times Fr(X)(T_2)$

i.e. $F(X)$ is radditive.

A reflexive coeq. of rad. is rad.
A limit of rad. is rad

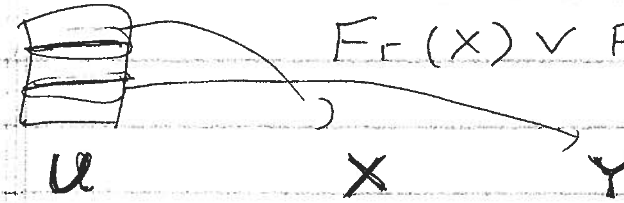
Need $F \amalg_r G$

$Fr(X)$ $\quad$ $Fr(X/Y)$.

$Fr : Sm/k \longrightarrow FPS$

$$\mathrm{Hom}(F \times G, H) = \mathrm{Hom}(F, H) \times \mathrm{Hom}(G,$$

$$Fr(X) \vee Fr(Y)$$



$$Fr_*(X) \vee Fr_*(Y)$$

$$Fr_*(X \amalg Y)$$

$$F$$

$$Sm/k \longrightarrow FPS$$

$$\mathbf{F} = \pi_0 \, GF$$

Picture: develope $\mathbb{A}^1$-h. category of FPS

There $Fr \approx Fr_*$ ~~and, for "connected"~~

We may extend $Fr: Sm/k \to FPS$ to $Fr: PreShv_\bullet(Sm/k) \to FPS$

This Fr commutes with limits and com. w. colimits up to an $\mathbb{A}^1$-equiv.

For F s.t. $C_*(Fr(F))$ is connected (or, more gen, π_0 is group-like) it is $\mathbb{A}^1$-local.

Hence: $Hom(\quad)$.

~~The funct~~ We have:

$$\begin{tikzcd} PreShv_\bullet(Sm/k) \arrow[r, "\Sigma^\infty_T"] \arrow[d, "Fr"'] & SH(k) \\ H_\bullet(FPS) \arrow[ur, dashed] & \end{tikzcd}$$

where $\dashrightarrow$ is a full emb.

~~Set : $\mathcal{Q}(F) = Fr(F)$ consid~~

If true then $Fr(F)$ as a pointed presheaf is a model for $\Omega^\infty_T \Sigma^\infty_T F$.

A corolary : .....

Jan. 21. 2002

Framed correspondences – a plan

1 composition of framed correspondences.

2 two framed cor. with the same trace are homotopic

3 two framed cor. with etale framings and equal determinants are homotopic

4. framed cor over a semi-local scheme is homotopic to ~~an eta~~ one with an etale framing

~~5. framed cor with an etale framing~~

5. A homotopy invariant framed functor takes values in the abelian groups.

6. Standard theorems on homotopy invariant framed functors

7. Cancellation theorem for framed correspondences

8. Construction of the functor $H(Fr(k)) \to SH(k)$ and a proof of the fact that it is a full embedding.

9. Construction of the adjoint pair $H(Fr(k)) \rightleftarrows D(Cor$

10.

11. $p_{*,0}$ - criterion or a more simple analog (should be.)

12. Th.1 ker $(1 \to \mathbb{Z})$ satisfies $p_{*,0}$ - criterion

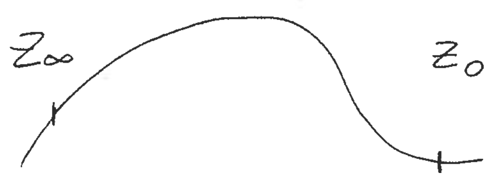
(?) 13. effective framed cor. and conn. theorem: for any X, $\mathcal{B}^{eff}(\Sigma^1_T X) \to \mathcal{B}(\Sigma^1_T X)$ is an $\mathbb{A}^1$- equivalence.

14. The geometry of ~~$\mathcal{E}$~~ $\mathcal{B}^{eff}(X)$ (= a fiber prod. over $Hilb(\mathbb{A}^\infty)$ of Fr and $A(X)$).

15. Th.2 $\mathcal{B}^{eff}(\Sigma^1_T X) \hookrightarrow \Sigma^1_T H_{\mathbb{A}^1}(Sm/k)$.

Nov. 20. 2001

$D(\varphi)|_{Z_\infty}$ $D(\varphi)|_{Z_0}$



$\mathcal{L} = \mathcal{L}(D(\varphi)|_{Z_0})$

s - the section giving $D(\varphi)|_{Z_0}$

Then s and $\frac{s}{\varphi}$ are trivializations of $\mathcal{L}$ at Z_∞ and Z_0 resp.

Assume that we have a function p such that $Z_\infty = p^{-1}(\infty)$, $Z_0 = p^{-1}(0)$.

Consider the rational section of $\mathcal{L}$ of the form

$$\tilde{s} = s \cdot \frac{p^N - \frac{1}{\varphi}}{p^N - 1} = s \frac{p^N}{p^N-1} + \left(1 - \frac{p^N}{p^N-1}\right)\frac{s}{\varphi}$$

where $N \gg 0$.

We have: $\tilde{s}|_{Z_\infty} = s$

$$\tilde{s}|_{Z_0} = \frac{s}{\varphi}$$

$D(\check{\gamma})$ is a zero cycle of degree $\deg \mathcal{L} = \deg D(\varphi)|_{Z_0}$ which does not intersect Z_0 and Z_∞.

Let's go several steps back. Consider a smooth curve C with a compactification (smooth) $\bar{C}$ and points at ∞ C_∞.

Let φ be an invertible function: $C \xrightarrow{\varphi} \mathbb{G}_m$

Then we have:

a. Canonical map $C_\infty \otimes \mathbb{G}_m \to h_0 C$

b. map $h_0(C) \to h_0(C) \otimes \mathbb{G}_m$ defined by φ and $\Delta : h_0 C \to h_0 C \otimes h_0 C$.

The composition is ~~an~~ a map

$$\alpha : C_\infty \otimes \mathbb{G}_m \to h_0 C \to h_0 C \otimes \mathbb{G}_m$$

which should correspond to ~~th~~ a well-defined map $\mathbb{Z}(C_\infty) \to h_0 C$.

~~We have th following hypo~~ ~~The~~ Consider the map $u = u(\varphi) : \mathbb{Z}(C_\infty) \to h_0 C$ given as follows.

~~Assume first that points at C_∞ are rational. Then for any $x \in C_\infty$ we need to define $u(x) \in h_0 C$.~~

~~Let m~~

~~Let E be a divisor supported in C_∞ together and let $(\mathcal{L}(E), s_E)$ be the corresponding line bundle and a section.~~

~~Consider in every point x of~~

Let $E = \sum n_i p_i$ be a divisor supported in C_∞. ~~and let $(\mathcal{L}_E, s_E)$ be the corresponding line bundle and a section.~~ Let further m_i ~~$= m_{\varphi p_i}$~~ be the multiplicity of φ in p_i. Set $E_\varphi := \sum n_i m_i p_i$ and let $(\cancel{\mathcal{L}_E}\, \mathcal{L}, s)$ be the cor. line bundle and a section. Then s has singularity of order $n_i m_i$ in p_i. Consider ~~for each p_i~~ around each p_i the section $s_i = \frac{s}{\varphi^{n_i}}$. This section is a trivialization in p_i and we get a line bundle $\mathcal{L}$ on $\bar{C}$ and a collection of its trivializations s_i at C_∞. This gives us $u(E) \in h_0(C) = \mathrm{Pic}(\bar{C}, C_\infty)$.

We want to show that

$$u \otimes Id_{\mathbb{G}_m} : \mathbb{Z}(C_\infty) \otimes \mathbb{G}_m \to h_0 C \otimes \mathbb{G}_m$$

is homotopic to α.

An explicit description of α:

$$(x \otimes f \in k_x^*) \longmapsto (\vartheta, f) \longmapsto$$

given $E = \sum n_i p_i$ and $f_i \in k_{p_i}^*$

we first find a meromorphic function

ψ such that $\psi = f_i^{n_i}$ in p_i

Then consider $D(\psi) = \sum \ell_j q_j$

Assume for a moment that q_j are rational. Then we take:

$$\alpha(E, f_i) = \sum_j \ell_j (q_j \otimes \varphi(q_j))$$

If q_i are not rational we have to write

$$\alpha(E, f_i) = \sum_j \ell_j \, tr_{k_{q_j}/k} (q_j \otimes \varphi(q_j))$$

On the other hand $u \otimes Id_{G_m}$ gives the following.

~~$(u \otimes Id_{G_m})(E, f_i) =$~~

Given $E = \sum n_i p_i$ we need to find a rational function

z ~~which equals~~ φ^{-n_i} such that

s/z equals $\frac{s}{\varphi^{n_i}}$ in p_i.

~~Let's try to get rid of s. We need to find a rational function σ such that in a neighbourhood of Z we have $D(\sigma) = \sum n_i m_i p_i$.~~

i.e. such that $z/\varphi^{n_i} \equiv 1$ in p_i.

Then $D(s/z) = \sum n_i m_i p_i - D(z)$ is $u(E)$.

We can further choose z_i for each p_i such that
$$\begin{cases} z_i/\varphi \equiv 1 & \text{in } p_i \\ z_i \equiv 1 & \text{in } p_j \quad j \neq i \end{cases}$$

Then:
$$u\left(\sum n_i p_i\right) = \sum n_i m_i p_i - \sum n_i D(z_i)$$
$$= \sum n_i \left(m_i p_i - D(z_i)\right).$$

Unfortunately this description of u ~~only~~ is not universal in the base field. Choosing z_i's for all points p_i over k is not enough to get u over an extention which

can split one of p_i's.
We may try instead to get a description of α which provides the answer in terms of $(\mathcal{L}_i, s_i) \otimes \varphi_i$ where $\mathcal{L}_i$ – a line bundle
s_i – trivialization at ∞
φ_i – invertible function on $D(s_i)$.

We need to change the description of u slightly. For ~~easch~~ each p_i we need to choose z_i – a rational function on X <u>over</u> <u>k_{p_i}</u> such that

$$\begin{cases} z_i/\varphi = 1 & \text{in } p_i \\ z_i = 1 & \text{in all oth } p \end{cases}$$

Then $u(\sum n_i p_i) = \sum n_i (m_i p_i - D(N(z_i)))$

Now for $f_i \in k_{p_i}^*$ we have:

$$(u \otimes \mathrm{Id})(\sum n_i (p_i, f_i)) =$$

$$= \sum n_i \, \mathrm{tr}_{k_{p_i}/k}(m_i p_i - D(z_i), f_i)$$

Hence we need to compare:

$$\alpha\left(\sum n_i (p_i, f_i)\right) = \sum_j \ell_j \,(q_j, \varphi(q_j))$$

and

$$(u \otimes Id)\left(\sum n_i (p_i, f_i)\right) = \sum_i n_i \, tr_{k p_i / k} (m_i p_i - D(\mathfrak{z}_i), f$$

~~Let's again consider the case when all p_i's are rational. Then we have on the one hand~~

It is suf. to consider one p among p_i's and to assume it is rational. Then for α we have:
(let m be the mult. of φ at p)

for each $f \in k^* = k_p^*$ choose ψ_f such that $\psi_f = f$ in p and $\psi_f = 1$ in all other p_j.

- - - - .

or u:

choose $\mathfrak{z}$ such that $\mathfrak{z} = \varphi$ in p and $\mathfrak{z} = 1$ in all other p_j. Then we set

$$f \longmapsto (mp - D(\mathfrak{z}), f)$$

~~Again~~ For α we can proceed as follows.
To choose ψf choose φ such that
$\varphi = 0$ in all $p_j \neq p$ and
$\varphi = 1$ in p. Then

$$\psi f = \varphi \cdot (f - 1) + 1$$

line bundle $\mathcal{L}$ cor. to the part of the divisor of φ which is in zero's of p.

It comes with an s.

~~Looking at~~ $\cancel{\frac{s}{\varphi}}$ ~~we have~~

~~Look at~~ $\cancel{\frac{p+1}{p+\varphi}}$.

We now need to find a rational function φ such that $s \cdot \varphi$ is s where $p = \infty$ and $\frac{s}{\varphi}$ where $p = 0$.

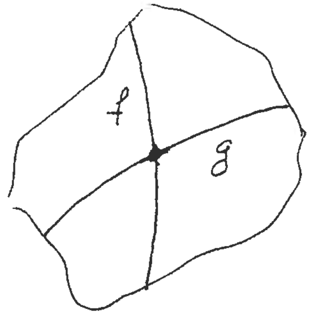
$$\frac{p}{p+1} = \begin{cases} 1 & \text{for } p = \infty \\ 0 & \text{for } p = 0 \end{cases}$$

$$s \cdot \frac{p}{p+1} + \left(1 - \frac{p}{p+1}\right) \cdot \frac{s}{\varphi}$$

$$= s \cdot \left(\frac{p}{p+1} + \frac{1/\varphi}{p+1} \right)$$

Nov. 15. 2001

$\mathcal{U}$ – smooth



we want a family ~~φ_f, φ_g~~

$\tilde{f}, \tilde{g} : U \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ (more generally $f, g, \tilde{f}, \tilde{g}$ ar rational).

such that:

a. $\tilde{f}, \tilde{g}|_{U \times 0} = f, g$

b. $\tilde{f}, \tilde{g}|_{U \times 1}$ have divisors which are smooth and intersect transversally

c. $\operatorname{Supp} D(\tilde{f}) \cap \operatorname{Supp} D(\tilde{g})$ is finite over $\mathbb{A}^1$!

We are given U together with an etale map $U \xrightarrow{p} \mathbb{A}^2$. Let $\bar{U} \xrightarrow{\bar{p}} \mathbb{P}^2$ be the normal com. of p. ~~Let further~~ For any divisor D at ∞ i.e. with the support in $U_\infty = \mathbb{P}^2 - \bar{U}$ the space of rat. funct. $\mathcal{L}(D)$ with poles at most in D is a linear space and we may think of f, g as of elements in it.

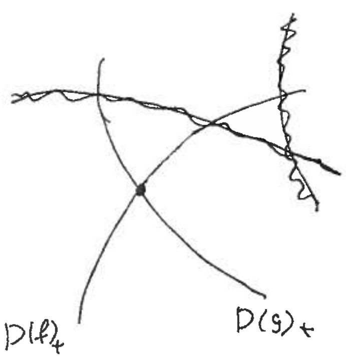
~~What We need to consider~~

Q.1: what is a map $\mathbb{A}^1 \longrightarrow \mathcal{L}(D)^{\times 2}$ such that the cov. family $(\hat{f}, \bar{g})$ has property C?

it is certainly suf. for c. to hold to require tha

$$\mathrm{Supp}(D(\hat{f})_+) \cap \mathrm{Supp}(D(\bar{g})_+) \cap U_\infty = \emptyset.$$

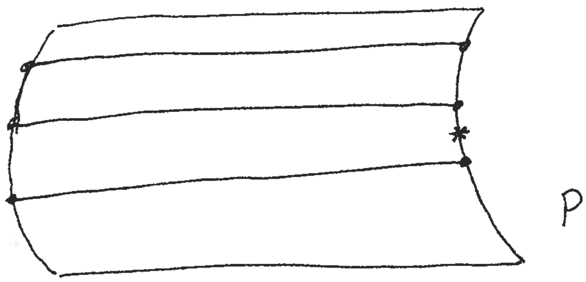
$\boxed{\text{NB we only consider ~~regular~~ } f, g \text{ regular on } U \text{ so far.}}$



Fix f. ~~We have a finite set of points~~

Consider the ir. comp. of $D(f)_+$ which ~~lie in~~ do not lie in U_∞. Call then $P_1, \dots P_n$. Consider the finite set $\{p_\alpha\}$ of points $\{p_\alpha\} = (\cup P_i) \cap U_\infty$

Consider now a family $\tilde{g}$ of g's such that the non-infinite components of $D_+(g_t)$ do not ~~touch~~ pass through $\{p_\alpha\}$. Is it true then that $(\bar{f} = f, \bar{g})$ satisfies c?



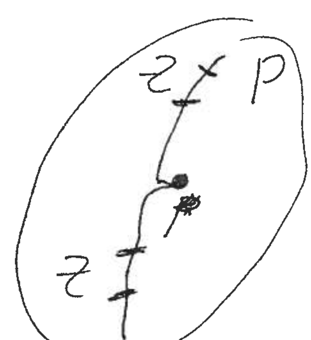
We may fix ~~th comp~~ $P =$ the union of comp. of $D(f)_+$ which contain our point x.

Let further $Z \hookrightarrow P$ be the closed subset which consists of points where P intersects U_∞ or any other comp. of $D(f)$.

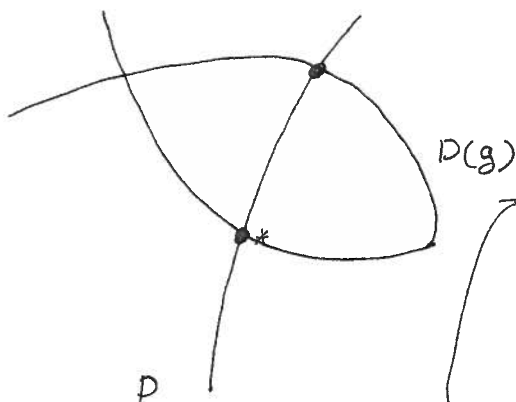
Let us now choose $\tilde{g}$ such that

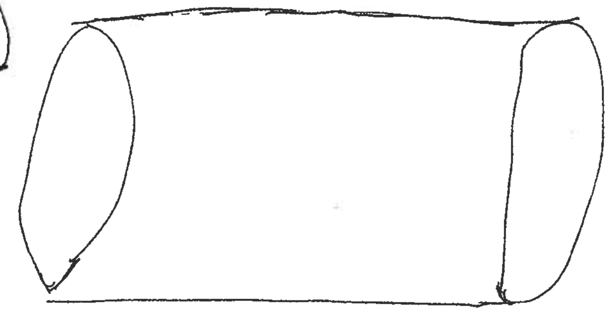
$|D(\tilde{g})| \cap P - Z$ is finite over $\mathbb{A}^1$. ~~This intersection will the support a framed~~ The image of this intersection in $\mathbb{A}^2 \times \mathbb{A}^1$ will be a closed subset ~~supporting~~ with a frame.

Now we have a (very bad) curve



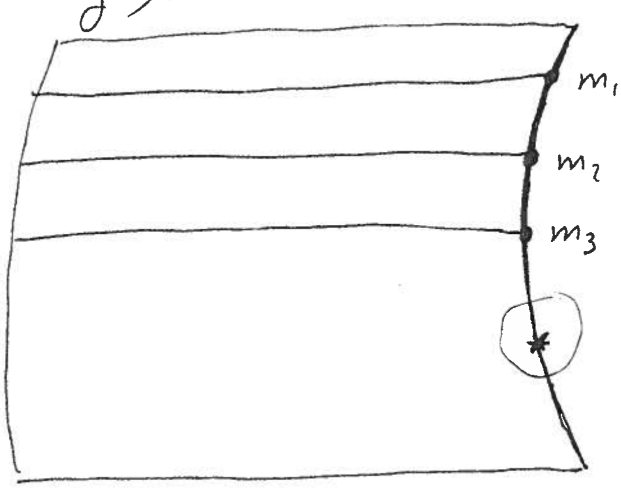
with a point P, have g - a ~~rational (or~~ reg. function with divisor in p.

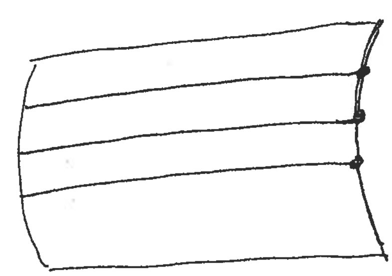




$$\tilde{g} : \overline{U} \times \mathbb{A}^1$$

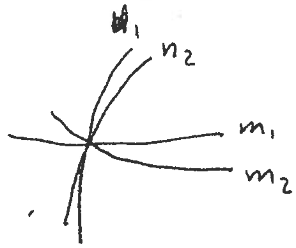
we get a problem if originally $D(f) \cap D(g) \cap U_\infty =$ Ara. Instead of requesting that $\tilde{g}_t$ is regular in points $Z \subset P$ we may request that $\tilde{g}|_{P \times \mathbb{A}^1}$ has fixed orders in $Z \times \mathbb{A}^1$ (the same as the original g).





Specialise again to f, g regular.

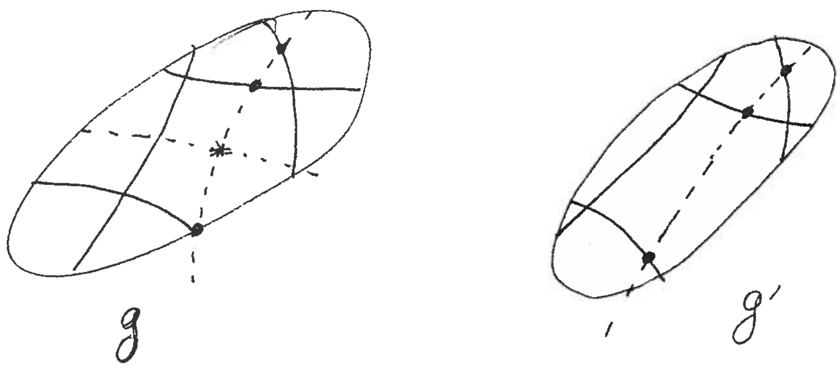
Our ~~or~~ point of origin ~~has seen~~ is of the form

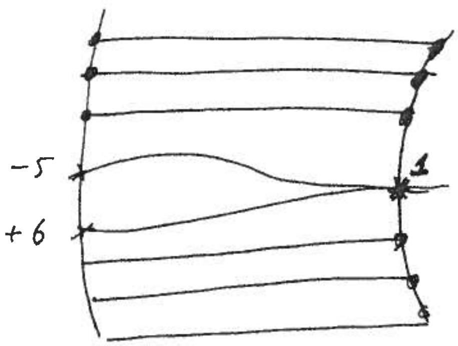


where n_i are the mult. of the comp. of $D(f)$ and m_j are the mult. of the comp. of $D(g)$.

When g is restricted to P ($\hookrightarrow |D(f)|_+$) we get a rat. function with all poles necessarily in $Z \cap P$ and at least d zeroes in $* = P$. We may assume safely that the rest of zeros is in Z as well. Since we want to keep at least all zeroes and poles in Z we have a finite dimension of the space of functions g.

Can we help by passing to a higher etale cover? Aha. That means starting with f, g on $U \to \mathbb{A}^2$ but using $V \to U \to \mathbb{A}^2$ to move them.





∀ ~~q~~ q of cod. = 1 in our set ~~q~~ $g/g'(q) = 1$.

$$(1-t)g + tg' = g \cdot \left((1-t) + t\,\frac{g'}{g}\right)$$

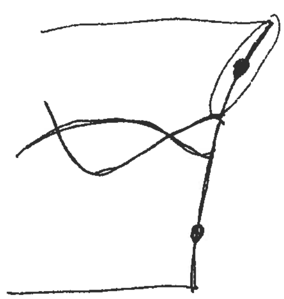
we may try to find g' such that g/g' is regular at ∞ and $= 1$ in Z.

we just need to find g' such that

$\frac{g}{g'}$ is regular and $= 1$ in a neighborhood of Z. ~~Then~~ Consider the $\tilde{g} = \cancel{g'}\ (1-t)g + tg'$

$$D(\tilde{g}) = D(g) + D(\tilde{g}/g) = D(g) + D\left((1-t) + t\frac{g'}{g}\right)$$

the later is $\equiv 1$ in a neigh. of $Z\times\mathbb{A}^1$ Hence there exists an open neigh. V of $Z\times\mathbb{A}^1$ such that $D(\tilde{g}) \cap (V \cancel{\times \mathbb{A}^1}) = D(g)\cap(V\cancel{\times\mathbb{A}^1}) \cancel{= Z\times\mathbb{A}^1}$



~~Hence $(P - Z \cap D(\tilde{g})) =$~~

~~$\neq$~~

$\Rightarrow$ ~~$D(\tilde{g}) = (Z\times\mathbb{A}^1) \amalg W$~~ where

W

$D(\tilde{g}) = W \cup$ ~~a subset of~~ a subset of $D(g)\times\mathbb{A}^1$

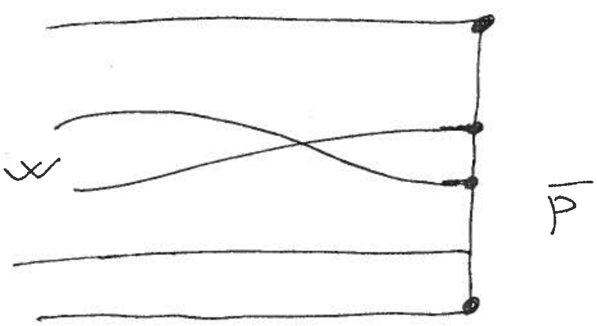
$(W \cap Z\times\mathbb{A}^1 = \emptyset)$

$$D(\tilde{g}) \cap P = \underbrace{(W \cap P^{\times\mathbb{A}^1})}_{\text{finite}} \cup (P^{\times\mathbb{A}^1} \cap \text{ a subset of } D(g)\times\mathbb{A}^1)$$

Better. Intersecting with $\bar{P}$ we get: that

$\cancel{|D(\bar{g})| \cap \bar{P}}$

there exists V an open neigh. of $Z \times A'$ in $\bar{P} \times A'$ such that $(|D(\bar{g})| \cap \bar{P}) \cap V = (|D(g)| \cap \bar{P}) \cap V$



$D(\bar{g}) \cap P$

(+).

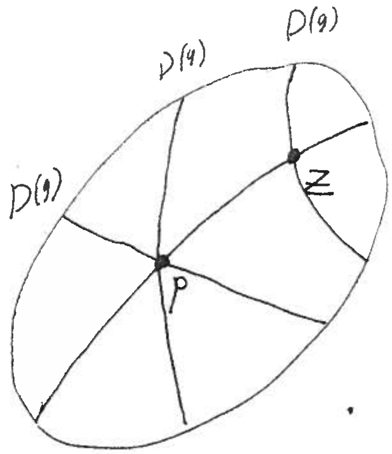
$\cancel{P \in (\bar{P} \cap V)}$

Remains to find g'.

We now have $\bar{U}$, $Z \hookrightarrow \bar{U}$ a collection of points $\bar{P} \hookrightarrow \bar{U}$ a curve $(Z \hookrightarrow \bar{P})$. $p \in \bar{P}$ a point in $\bar{P}$ $p \notin Z$. $\bar{U}$ is normal. We also have a rat. function g. We want a rat. function g' such that:

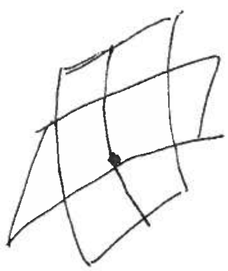
a. g/g' is reg. ~~and~~ in a neigh. of Z and $\equiv 1$ on Z

b. $D(g')$ intersects $P = \bar{P} - Z$ transversally.



(seems easy enough from g.p. arguments)

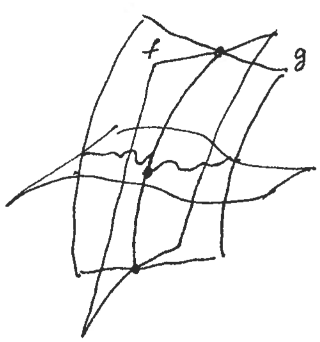
~~What to do if we have f, g~~



f, g, h

g, h fix.

move such that D(f) ∩ D(g) transversal



f, g, h

f, g, h, e

$f \sim g$ (move g s.t.)

move h s.t.

$(f \sim g) \sim h$

move e s.t.

$f_1, \dots f_n$

move f_1 to make it intersect well with

$f_2, \dots f_n$

move f_2 to make it int. well with

$f_1, f_3, \dots f_n$

move f_3 to make it int. well with

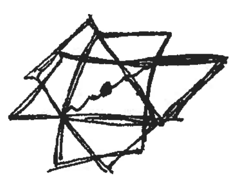
$f_1, f_2, f_4, \dots f_n$

given ~~$f_1, \dots f_n$ ad and g we which are transversal~~ $f_1, \dots f_i, f_{i+1}, \dots f_n$ such that

$\bigcap_{j=1}^{n} |D(f_j)| = \{p\}$ in a neighborhood of p

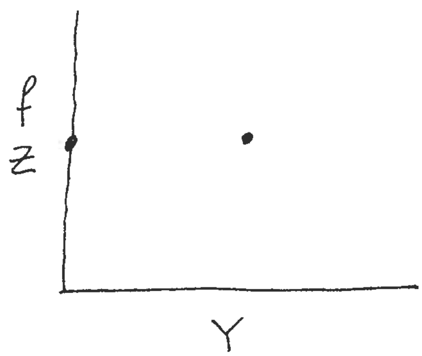
and such that $f_1, \dots f_i$ are transversal to each other in p. we move f_{i+1} creating new intersection points on $\bar{P} = \left(\bigcap_{j \neq i+1} |D(f_j)| \cap \text{small neigh. of } p\right)$

but such that $f_1, \dots f_i, f'_{i+1}$ are now transv. to each other in all these points.



Nov. 16. 2001

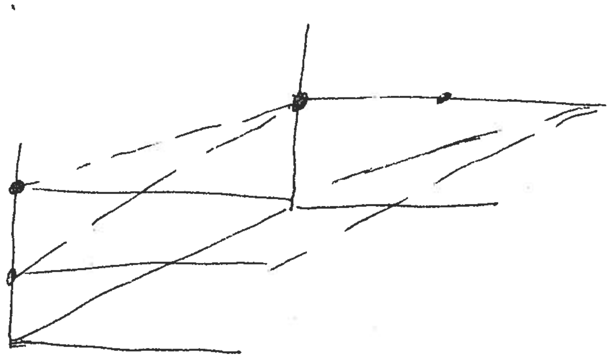
<u>Question:</u> Given $\varphi \in Hom_n(X, Y)$ how can we move it. Say X is local can we make ~~Supp~~ φ - etale over X? Let us first consider the case of $Hom_1(~~Spk~~, Y)$:

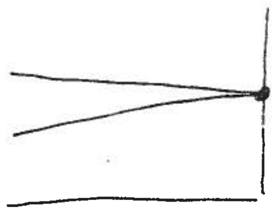


we are given ~~z~~ $z \in \mathbb{A}^1_k$, $f \in k(\mathcal{O}^h_{\mathbb{A}^1, z})^*$

$\varphi : z := Sp.k_z \longrightarrow Y$.

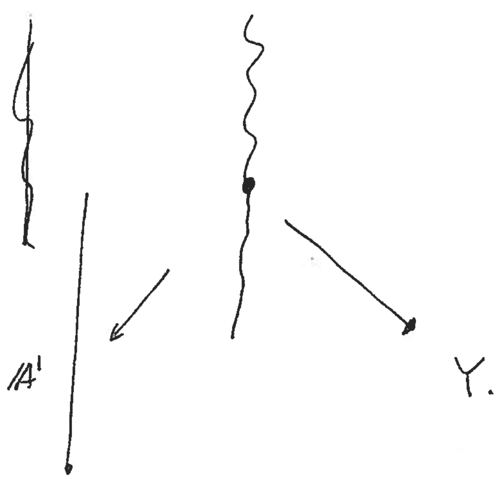
Let's consider the case when f has multiplicity in z other than $1, 0, -1$. Can we move such ~~that~~ as to get several points with simple multiplicities?

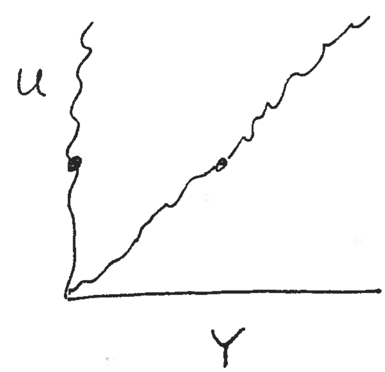


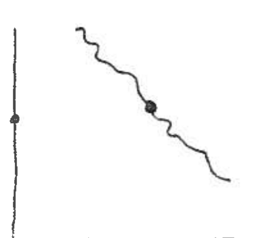


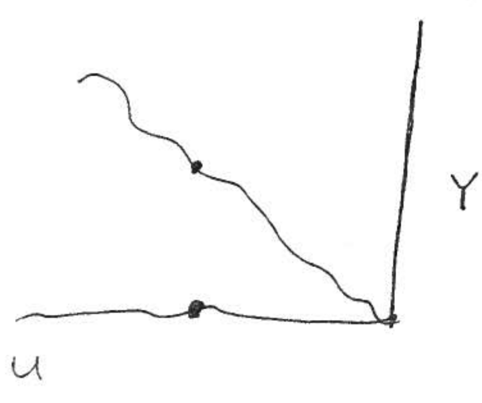
~~Let us consider a slightly different ques~~

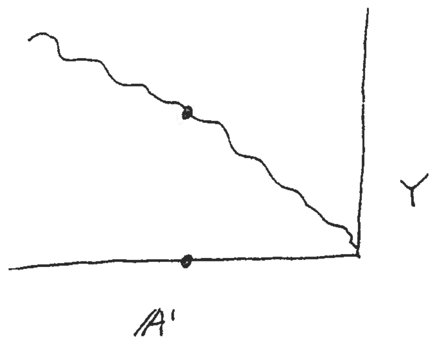
We make most moves along Y in the usual theory of correspondences using representation of Y as a standard triple, i.e. reducing to Y being a curve, and then using description of cor. in terms of ~~paths~~ sections of line bundles.



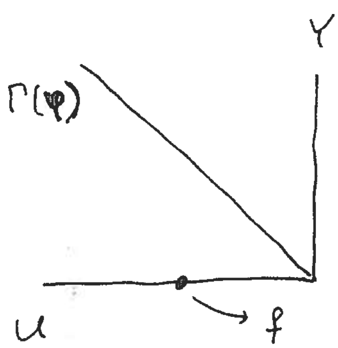




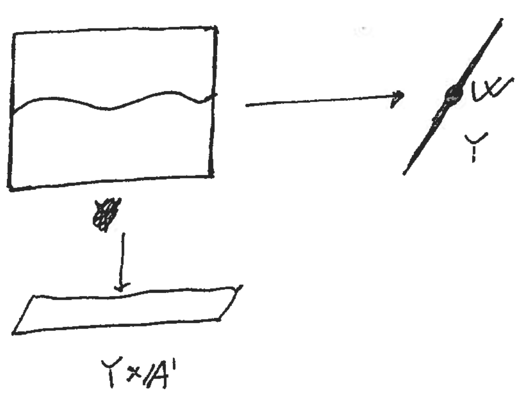


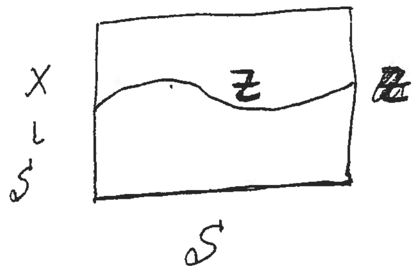


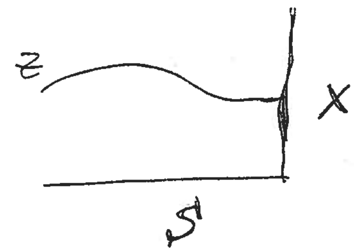
Start with



- we can always move f just as before!







- try to find $X \xrightarrow{p} \mathbb{A}^1_S$ such that p is etale ~~about~~ around Z. and bijective on Z

Then
$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ \mathbb{A}^1_S & & X \end{array}$$
is almost a framed cor.

what remains is to choose a function on X defining Z. (enough to take the canaronical one on $\mathbb{A}^1_S$!).

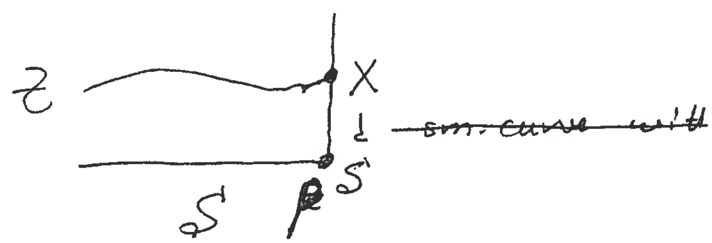
That means that at least if S is henselian (and probably for any S - semilocal) a cor. like that has a lifting to a framed cor. from S to X.

Consider now again the problem of finding for $p \in U \hookrightarrow X$ a $V \ni p$ and a map $U \dashrightarrow V$ s.t.

$$\begin{tikzcd} U \arrow[r, dashed] \arrow[d] & V \arrow[dl] \\ X & \end{tikzcd}$$

commutes.

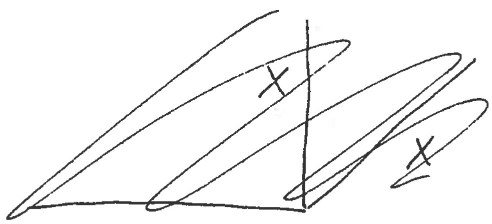
Using the stand. triple lemma can ~~we~~ reduce to:

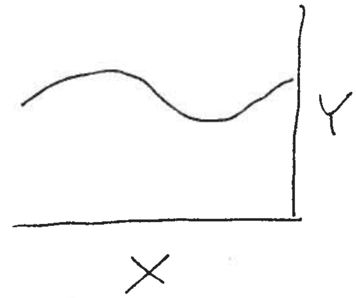


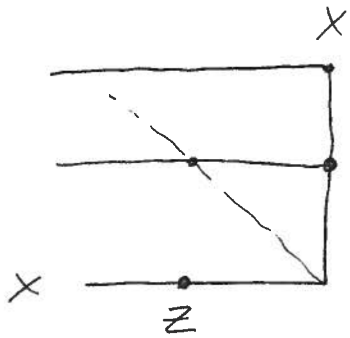
~~such that~~ ~~...~~ $((X,Z) \sim (X, X-U)$ after shrinkage$)$. ~~Find $X \to \mathbb{A}^1_S$ etale and bijective on Z and take (say) the canonical f for Z in $\mathbb{A}^1_S$.~~

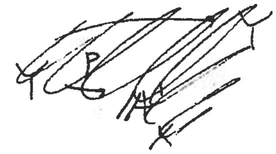
$$\cancel{\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ \mathbb{A}^1_S & & X \end{tikzcd}}$$

after that just move Z as usual ~~so that~~









$f: \quad \Delta = f^{-1}(0)$

$X \xrightarrow{p} \mathbb{A}^1$ - etale.
and finite around Δ (may need to shrink the base X).

~~too~~ Interpret our morph as:

$$\begin{tikzcd} V \times X \arrow[r, "pr"] \arrow[d] & X \\ \mathbb{A}^1_V & \end{tikzcd}$$

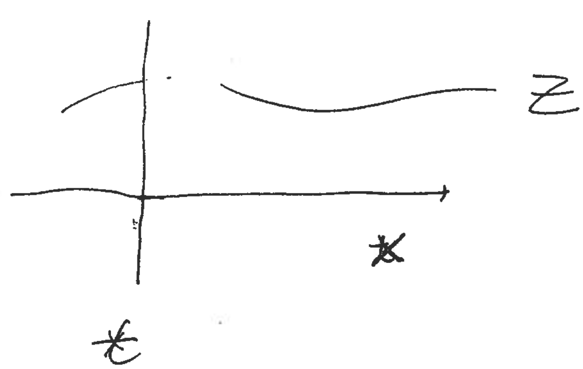
now move $\Gamma \hookrightarrow V \times X$ together with its parameter top (but not p!) to make sure $pr(\Gamma)$ does not touch Z or equiv. that Γ does not touch $pr^{-1}(Z)$.

Let's try to compute $\pi_0 \, Hom_{\mathbb{1}}(Sp.k, Sp.k)$.

We consider collections $x_1, \dots x_n$ where x_i are closed points in $\mathbb{A}^1_k$ + $f_1, \dots f_n$ where f_i is a non-zero rational function on $Spec\, \mathcal{O}^h_{\mathbb{A}^1, x_i}$ ~~which is inver~~ which has a singularity (a pole or a zero) in x_i.

We identify $(\{x_i\}, \{f_i\})$ and $(\{x_i'\}, \{f_i'\})$ if there exists:

- a closed subset $Z \hookrightarrow \mathbb{A}^1_k \times \mathbb{A}^1_k$ finite over the first $\mathbb{A}^1_k$: $(x, t) \longmapsto x$.



- For each point z of Z a rational function f_z on $Spec^k \, \mathcal{O}^h_{\mathbb{A}^2, z}$ with no singularities outside $Spec\, \mathcal{O}^h_{Z,z}$. + compatibility ~~cond~~ cond.

The com condition imply that the images of f_z and $f_{z'}$ in the separable closure of the function coincide. Hence we in fact have f_{z_i} where z_i are the generic points of Z and $f_{z_i} \in \operatorname{Spec} \mathcal{O}^{sh}_{\mathbb{A}^2, z_i}$ with the condition that z_i can be descended to $\operatorname{Spec} \mathcal{O}^{h}_{\mathbb{A}^2, z}$ for any $z \in Z$ and such descents are compatible for z lying in two components simultaneously.

- For a closed ~~in sub~~ subset $Z \hookrightarrow \mathbb{A}^1_X$ which is finite and equidim. over X there exists a unique polynomial f_Z in $\mathcal{O}(X)[t]$ ~~such the wh~~ with the leading coef. 1. ~~Which is~~ such that $Z = f_Z^{-1}(0)$. (Assuming X is normal!). In our case $X = \mathbb{A}^1$.

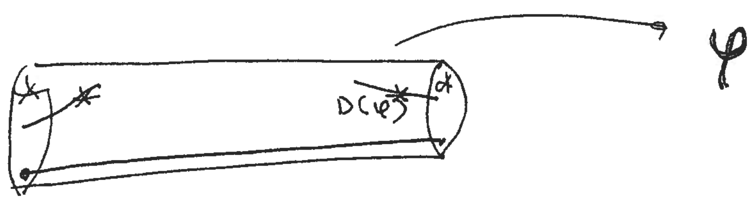
I. ~~Let us try to show that~~ Given a point $z \in \mathbb{A}^1$ and $f \in K(\operatorname{Spec} \mathcal{O}^h_{\mathbb{A}^1, z})$ define $st(f)$ to be $f_z^d \cdot a_f$ where f_z is defined above and a_f is ~~ob~~defined ~~by~~ as follows $f = f_z^d \cdot \varphi$ where $\varphi \in (\mathcal{O}^h_{\mathbb{A}^1, z})^*$.

$a_f := \varphi(z)$. Oops.... Define $st(f)$ as the pair (d, a_f) where $d \in \mathbb{Z}$, $a_f \in (k_z)^*$.

we want to show that (z, f) is equivalent to (z, f') if $st(z,f) = st(z,f')$. We need an element φ in ~~the~~ $\mathcal{O}^h_{\mathbb{A}^1, z}[t]$ such that:

$$\begin{array}{l} \varphi|_{t=0} = f \\ \varphi|_{t=1} = f' \end{array} \quad \text{and} \quad \cancel{\varphi \text{ has no singularities}}$$

$$\mathrm{Supp}(D(\varphi)) \cap (z \times \mathbb{A}^1) = \emptyset.$$



(i.e. φ is invertible in the local ring of any point of the form (z, t). We have

$$\begin{array}{l} f = f_z^d \cdot f_0 \\ f' = f_z^d \cdot f_0' \end{array} \quad \text{where } f_0, f_0' \in \left(\mathcal{O}^h_{\mathbb{A}^1, z}\right)^* \text{ and}$$

$f_0 = f_0'$ in z. Hence $f_0' = f_0 + f_z \cdot h$

and we can define $\varphi = f_0 + t f_z \cdot h$.

~~Conclusion:~~

~~[illegible]~~

$I_n = \{$the subsheaf of $\mathcal{M}^* \wedge \dots \wedge \mathcal{M}^*$ (n times) which consists of $(f_1, \dots f_n)$ such that $\bigcap_{i=1}^{n} \operatorname{Supp} D(f_i) = \emptyset\}$

<u>Example</u>: $I_1 = \mathbb{G}_m$.

For $Z \xhookrightarrow{i} X$ and F on X_{et} define

$$\Phi(F, I_n)(Z) := H^0(Z,\ i^*((j_* j^* I_n / I_n) \wedge F))$$

where $j : X - Z \to X$ is the complimentary open embedding.

<u>L.</u> For $Z_1 \hookrightarrow Z_2$ have a natural map

$$\Phi(F, I_n)(Z_1) \to \Phi(F, I_n)(Z_2)$$

The set $\Phi(F, I_n)(Z)$ is called the set of ~~parametrized~~ framed cycles of depth n on Z with coefficients in F.

For $X \xrightarrow{p} S$ equidimensional of dimension n define $c^{fr}(X/S, F)$ as the set:

$$c^{fr}(X/S, F) = \operatorname*{colim}_{\substack{Z \hookrightarrow X \\ \text{finite, equid.} \searrow\ \downarrow \\ S}} \Phi(F, I_n)(Z).$$

__Lemma__ An ~~section~~ element $a \in \Phi(F, I_n)(Z)$ defines a map in $H_\bullet$ of the form:

$$X/X-Z \longrightarrow \Sigma^n_T(F).$$

__Proof:__ (hint: a section $(f_1, \dots f_n)$ of I_n gives a covering by $U_i = X - \mathrm{Supp}(D(f_i))$ and a map $U_1 \cap \dots \cap U_n \to \mathbb{G}_m^{\wedge n}$).

Set $c_n^{fr}(X,Y) := c^{fr}(\mathbb{A}^n_X/X, (h_Y)_+)$ i.e.

$$c_n^{fr}(X,Y) = \operatorname*{colim}_{\begin{tikzcd}[row sep=small] Z \arrow[r, hook] \arrow[dr, "f.e.q."'] & \mathbb{A}^n_X \arrow[d] \\ & X \end{tikzcd}} \Phi((h_Y)_+, I_n)(Z)$$

Define composition:

$$c_n^{fr}(X,Y) \times c_m^{fr}(Y,Z) \longrightarrow c_{n+m}^{fr}(X,Z)$$

as follows.

Given

$$\begin{tikzcd} Z \arrow[r, hook, "i_Z"] \arrow[dr] & \mathbb{A}^n_X \arrow[d] \\ & X \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, hook, "i_W"] \arrow[dr] & \mathbb{A}^m_Y \arrow[d] \\ & Y \end{tikzcd}$$

consider