Geometric Cohomology and Homology of Stratified Objects

by

Robert Mark Goresky

B.Sc., University of British Columbia, 1971

Thesis

Submitted in partial fulfillment of the requirements for the

Degree of Doctor of Philosophy at Brown University

June, 1976

This dissertation by Robert Mark Goresky
is accepted in its present form by the Department of
Mathematics as satisfying the
dissertation requirement for the degree of Doctor of Philosophy.

Date . . . .

Recommended to the Graduate Council

Date . . . .

Date . . . .

Approved by the Graduate Council

Date . . . .

Acknowledgements

I am very happy to thank my friend and advisor, Robert MacPherson, for his patience, encouragement, and good advice; and for teaching me the joys of geometry.

I am grateful to my wife, Shelagh, without whose concern, encouragement, and support, I would never have made it.

Table of Contents

INTRODUCTION

In this thesis, a geometric description is given of the homology and cohomology of a large class of spaces called stratified objects. Rene Thom announced the notion of a stratified object in a paper of great importance [17] and since then his ideas have been applied with success to problems in algebraic geometry, the theory of stability of mappings, and algebraic topology. The class of stratified objects contains all (real or complex) algebraic varieties, analytic varieties, semi-analytic and subanalytic sets. A stratified object is (roughly) a topological space which is put together from smooth manifolds (called strata) similar to the manner in which a cell complex is put together from cells. Chapter 0 contains a quick review of the structure of stratified objects.

A compact k-dimensional stratified object W^k is called a geometric cycle if an orientation of the k-dimensional stratum is specified in such a way that the induced orientations cancel on the $k - 1$ dimensional strata. It is shown in Chapter 2 that a geometric cycle W^k carries a fundamental homology class $[W] \in H_k(W)$, which is a well-known fact when W has only one stratum.

The techniques of chapters 2 and 3 then show that if V is an n-dimensional stratified object, then for any homology class $\alpha \in H_k(V)$ there is a k-dimensional sub-stratified object $W \subset V$ which is a geometric cycle and for which $i_*[W] = \alpha$, where $i: W \to V$ is the inclusion. W is said to represent α.

Two geometric cycles W and W' in a stratified object V are said to be cobordant if there is an oriented stratified object $W'' \subset V \times [0,1]$ whose oriented boundary is $\partial W'' = W \times \{1\} - W' \times \{0\}$. The cobordism W'' can be interpreted as a "deformation" from W to W' inside V. If W is cobordant to W' in this manner then they represent the same homology class.

In Chapter 4 a similar interpretation of the cohomology of V is made in terms of "geometric cocycles" $W \subset V$ which are embedded substratified objects satisfying a certain local geometric condition (called the π-fibre condition) near the singularities of V. (A substratified object W of a stratified object V is said to satisfy the π-fibre condition if, for each stratum X of V, we have $W \cap T_X = \pi_X^{-1}(W \cap X)$ where $\pi_X: T_X \to X$ is the "tubular neighborhood" of X in V guaranteed by the stratified structure of V).

Assuming the π-fibre condition is satisfied for a substratified object $W^k \subset V^n$, W has a costratification, each "costratum" of W being a certain union of strata of W, and each costratum of W has a normal bundle and tubular neighborhood in V. The π-fibre subobject W is called a geometric cocycle if an orientation of the costratum of W with codimension

$n - k$ is specified and if the orientations induced on the normal bundles of the costrata of codimension $n - k + 1$ cancel. If $W^k \subset V^n$ is a geometric cocycle then the Thom class of the normal bundle of this "top costratum" has a unique lift to a cohomology class $\beta \in H^{n-k}(V)$ and we say W <u>represents</u> β .

It is then shown that, for any cohomology class $\beta \in H^{n-k}(V)$ there is a geometric cocycle $W^k \subset V$ which represents β , and that two geometric cocycles in V represent the same cohomology class if and only if there is a π-fibre cobordism between them. Intuitively this means that one can be deformed into the other by a <u>deformation</u> which obeys the π-fibre condition.

In Chapter 5 it is shown that if $f: V \to V'$ is a continuous map between two stratified objects and if $W' \subset V'$ is a geometric cocycle then f is homotopic to a map $f': V \to V'$ for which $f'^{-1}(W')$ is a geometric cocycle in V representing the pullback (by f) of the cohomology class represented by W' .

In Chapter 6 the cup and cap products are interpreted as follows: If W_1 is a geometric cocycle in a stratified object V and if $W_2 \subset V$ is a geometric cycle (respectively geometric cocycle) then W_1 is cobordant to a geometric cocycle $W_1' \cap V$ which is tranverse to W_2 and $W_1' \cap W_2$ is a geometric cycle (respectively cocycle) representing the cap (respectively cup) product of the classes determined by W_1 and W_2 .

The technical difficulties arise because there are two types of stratified objects ("Whitney" and "Thom-Mather") and the notions of transversality are best suited to Whitney stratified objects while the ideas of algebraic topology are most naturally suited to Thom-Mather stratified objects. The appendix is concerned with a method of freely translating from one type of stratified object to the other.

The idea of a geometric cocycle is not new. In 1947, Whitney [20] essentially understood the geometry of cohomology and the fact that the cup and cap product could be interpreted as intersection of geometric cocycles in "general position." Whitney did not have the technicalities of stratified objects and transversality needed to make his proofs rigorous. The algebraic approach to cohomology became prevalent.

In [5], McCrory made a study of the Zeeman spectral sequence and among other things, gained a similar understanding of which cycles in a P.L. space come from cohomology. (The study of the Zeeman spectral sequence and the failure of Poincare Duality was posed as a problem by Dennis Sullivan in [14]). About this time, Sullivan remarked that homology is studied using "tangential geometry" of a geometric cycle while cohomology is the study of "normal geometry." Rourke and Sanderson [12] have a similar understanding of cohomology in the P.L. category.

The history of geometric homology theories is more complex. Poincare's original treatment of homology in Analysis Situs was an attempt at such a theory although it lacked rigor.

Thom [16] in 1954 considered the problem of which homology classes in a manifold can be represented by an embedded submanifold, giving examples of homology classes without such a representation. The question of representing homology classes by mapped in manifolds gave rise to the bordism groups, investigated by Thom, Atiyah, Milnor, Conner and Floyd, etc.

In 1971, Sullivan [15], mentioning Thom's counterexample for motivation, gave a "minimal list of singularities" necessary to represent homology, i.e., a small list of spaces so that any homology class in a space X could be represented by a map of one of the spaces on the list into X. Sullivan also studied the question of representing a homology class in a manifold by an embedded goemetric cycle with only "join-like singularities." A more rigorous treatment of Sullivan's theory is found in Baas [1] together with a detailed study of the effect of certain restrictions on the list of allowed singularities. Both Sullivan and Baas use stratified objects as a natural setting for this theory. Rourke and Sanderson [12] give a definitive treatment of Sullivan's theory of "Bordism with singularities" in the P.L. context, proving that any generalized homology theory is isomorphic to a bordism theory with certain singularities.

In this present paper, no attempt is made to restrict the allowed list of singularities and the representing cycles (and cocycles) are always embedded in the space X which may itself be singular. Homology and cohomology are treated as dual notions using the language and techniques of stratified objects systematically. I am grateful to Bob MacPherson for suggesting to me that cohomology might admit such a highly geometric interpretation using the π-fibre condition.

CHAPTER O.

TECHNICAL PRELIMINARIES

In what follows the term "manifold" will mean real paracompact C^∞ Hausdorff manifold with a countable basis. Submanifolds are assumed to take the relative topology. A superscript on a manifold denotes its dimension. A bar over a subspace of a space denotes its closure. All vectorbundles are C^∞ unless otherwise specified.

The notations π_X, T_X, S_X, ρ_X, r_X, C_X^o, C_X, h_X associated with a stratum X in a stratified object will be defined in this chapter and used consistently throughout the paper.

0.1. Whitney Objects

0.1.1. Definition: Suppose X^n and Y^m are smooth submanifolds of $\mathbb{R}^p$ with $X \subset \bar{Y}$. The pair X, Y is said to satisfy Whitney's condition B at $x \in X$ if the following holds:

Suppose $y_i \in Y$ is a sequence of points converging to x. Suppose $x_i \in X$ is a sequence converging to x. Suppose the secant lines $\widehat{x_i y_i}$ converge to some limiting line ℓ (in the Grassmannian of lines in $\mathbb{R}^p$). Suppose the tangent planes $T_{y_i}Y$ converge to a limiting plane τ (in the Grassmannian of m-planes in $\mathbb{R}^p$). Then $\ell \subset \tau$.

The submanifolds X and Y are said to satisfy condition A at x if, under the above hypotheses, $T_xX \subset \tau$.

X and Y satisfy condition A (respectively B) if they do so at every point $x \in X$.

If X and Y are submanifolds of a smooth manifold M then conditions A and B are defined by reference to a local coordinate system on M and if X and Y satisfy condition A or B at the point $x \in X$ with respect to one coordinate system about x then they do so with respect to any coordinate system about x .

Condition B implies condition A and it also implies that $\dim(Y) > \dim(X)$.

A Whitney stratified object W is defined to be a closed subset of a manifold M , with a locally finite decomposition of W into disjoint submanifolds of M (called strata) which satisfy:

(a) The condition of the frontier: If X and Y are strata and $X \cap \overline{Y} \neq \phi$ then $X \subset \overline{Y}$, and we write $X < Y$.

(b) If $X < Y$ then X and Y satisfy Whitney's condition B (and hence also condition A).

Remark: For most purposes, the strata need not be connected and we may therefore assume there is at most one stratum of any given dimension.

The dimension of W is the maximum of the dimensions of the strata of W, and is sometimes designated by a superscript on W.

0.2. Tubular Neighborhoods and Control Data

0.2.1. Definition: A tubular neighborhood T_X of a stratum X of a Whitney object $W \subset M$ is a Riemannian normal bundle $E \to X$ together with a smooth embedding

$$\phi_X: E_\epsilon \to M$$

where

$$E_\epsilon = \{v \in E \mid \|v\|^2 < \epsilon\}$$

for which (1) $\phi(E_\epsilon)$ is an open neighborhood of X

(2) ϕ maps the zero section identically to X.

T_X is used to denote the neighborhood $\phi(E_\epsilon)$ and $\pi_X: T_X \to X$ the projection and $\rho_X: T_X \to \mathbb{R}$ the "distance function" $\rho_X(p) = \|\emptyset^{-1}(p)\|^2$.

0.2.2. Definition: Control data on a Whitney object $W \subset M$ is a family of tubular neighborhoods $\{T_X \mid X$ is a stratum of $W\}$ for which

(a) If X and Y are not comparable then $T_X \cap T_Y = \phi$

(b) If $X < Y$ then $\pi_X \pi_Y = \pi_X$ and $\rho_X \pi_Y = \rho_X$ whenever both sides of the equations are defined.

In this case define

$$S_X(\epsilon) \equiv \{p \in T_X | \ \rho_X(p) = \epsilon\}$$

and

$$T_X(\epsilon) \equiv \{p \in T_X | \ \rho_X(p) < \epsilon\}$$

0.2.3. Theorem: (Thom [17], Mather [9]).

Let $W \subset M$ be a Whitney object and $f: M \to P$ a smooth map between smooth manifolds and suppose that for each stratum X of W, $f|X: X \to P$ is a submersion. Then there exists a system of control data on W for which $f(\pi_X(p)) = f(p)$ for every stratum X and for every $p \in T_X$.

0.3. Abstract Stratified Objects

0.3.1. Definition: An abstract or Thom-Mather stratified object W is a Hausdorff space satisfying the following axioms:

(1) W has a locally finite decomposition into smooth locally closed manifolds, called strata, which satisfies the axiom of the frontier: if X and Y are strata and $X \cap \overline{Y} \neq \phi$ then $X \subset \overline{Y}$, (and we write $X < Y$).

(2) Each stratum X has a fixed neighborhood T_X in W and a continuous retraction $\pi_X : T_X \to X$ and a continuous "distance function" $\rho_X : T_X \to [0, \infty)$ which satisfy:

(a) $X = \{v \in T_X | \rho_X(v) = 0\}$

(b) If $X < Y$ then $(\pi_X, \rho_X) | (T_X \cap Y) : T_X \cap Y \to X \times (0, \infty)$ is a smooth submersion.

(c) If $X < Y$ then $\pi_X \pi_Y = \pi_X$ and $\rho_X \pi_Y = \rho_X$ whenever both sides of the equations are defined.

0.3.2. Remark: By shrinking the neighborhoods and multiplying the distance functions ρ_X by a scale factor $f: X \to \mathbb{R}$, we may assume, and from now on we will assume that

(3) If X and Y are strata, then $T_X \cap T_Y = \phi$ unless $X < Y$ or $Y < X$.

(4) For any stratum X and any $\epsilon > 0$ sufficiently small, $\pi_X | \rho_X^{-1}(\epsilon) : \rho_X^{-1}(\epsilon) \to X$ is proper.

According to the theorem 0.2.3., every Whitney stratified object admits the structure of a Thom-Mather stratified object satisfying property (4) above.

Property (4) guarantees that

$\overline{S_X(\epsilon)}$ = the closure of $\{p \in T_X | \rho_X(p) = \epsilon\}$

is a Thom-Mather stratified object.

For a Whitney stratified object W or an Abstract stratified object W we shall use the following notations and terminology:

bW denotes the union of all strata whose dimension is less than the dimension of W. It is called the singularity set of W.

$W - bW$ is called the <u>nonsingular part</u> or the <u>top stratum</u> of W. (When working in a context where strata must be connected, the term "top stratum" will refer to a component of $W - bW$).

$W^{(p)}$ denotes the p-skeleton of W, i.e., the union of all strata of W with dimension p or less.

If W is an abstract stratified object, or if W is a Whitney object with a system of control data, then we denote

$$T_p = \cup\{T_X \mid X \text{ is a stratum of } W \text{ with dimension } \leq p\}$$

$$T_p(\epsilon) = \cup\{T_X(\epsilon) \mid X \text{ is a stratum of dimension } \leq p\}$$

$$S_X(\epsilon) \equiv \{p \in T_X \mid \rho_X(p) = \epsilon\} \text{ whenever } X \text{ is a stratum of } W$$

<u>Caution</u>: If W is a Thom-Mather stratified object and X is a stratum of W then T_X is a neighborhood of <u>X in W</u>. If W is <u>Whitney</u> then T_X is a neighborhood of X <u>in the ambient manifold</u>.

0.4. Families of Lines

In this section, W denotes a Thom-Mather stratified object with strata X, Y, etc.

0.4.1. Definition: A family of lines on W is a number $\delta > 0$ together with a system of radial projections for each stratum X

$$r_X(\epsilon): T_X - X \to S_X(\epsilon)$$

defined whenever $0 < \epsilon < \delta$, which satisfy the following commutation relations: If $X < Y$ then

(1) $r_X(\epsilon) \circ r_Y(\epsilon') = r_Y(\epsilon') \circ r_X(\epsilon) \in S_X(\epsilon) \cap S_Y(\epsilon')$

(2) $\rho_X \circ r_Y(\epsilon) = \rho_X$

(3) $\rho_Y \circ r_X(\epsilon) = \rho_Y$

(4) $\pi_X \circ r_Y(\epsilon) = \pi_X$

(5) If $0 < \epsilon < \epsilon' < \delta$ then
$r_X(\epsilon') \circ r_X(\epsilon) = r_X(\epsilon')$

(6) $\pi_X \circ r_X(\epsilon) = \pi_X$

(7) $r_X(\epsilon) | T_X(\epsilon) \cap Y: T_X(\epsilon) \cap Y \to S_X(\epsilon) \cap Y$
is smooth.

0.4.2. Remark: It follows from (3) that if $p \in T_X \cap Y$ then $r_X(\epsilon)(p) \in S_X(\epsilon) \cap Y$ so $r_X(\epsilon)$ takes strata to strata.

Therefore the map

$$h_X\colon T_X - X \to S_X(\epsilon) \times (0\ ,\ \infty)$$

$$h_X(p) = (r_X(\epsilon)(p)\ ,\ \rho_X(p))$$

is a homeomorphism, is a diffeomorphism on each stratum of $T_X - X$, and extends to a stratum preserving homeomorphism between the neighborhood $T_X(\epsilon)$ and "half open" mapping cylinder of $S_X(\epsilon) \to X$:

$$h_X\colon T_X(\epsilon) \to (S_X(\epsilon) \times [0\ ,\ \epsilon)) \cup X/(p\ ,\ 0) \sim \pi_X(p)$$

and h_X is compatible with the projection to X.

The curves $h_X^{-1}(p\ ,\ t)$ for fixed p are called X-lines and they foliate $Y \cap T_X(\epsilon)$ for each stratum $Y > X$.

0.4.3. Definition: Suppose W is a Thom-Mather stratified object with a family of lines and X is a stratum of W. The cone operation C_X associates to any subset $Q \subset S_X(\epsilon)$ the set $C_X(Q) \subset T_X(\epsilon)$ consisting of the union of all X-lines which intersect Q. $C_X(Q)$ is homeomorphic to the mapping cylinder of the map

$$\pi_X | Q\colon Q \to \pi_X(Q) \subset X$$

0.4.4. Remark: Property (1) in the definition of a family of lines guarantees that if $Q \subset S_X(\epsilon) \cap S_Y(\epsilon)$ then

$$C_X C_Y(Q) = C_Y C_X(Q)$$

which is an important step in the triangulation procedure.

0.4.5. <u>Definition</u>: The <u>open cone</u> $C_X{}^o(Q)$ is defined to be $C_X(Q) - \pi_X(Q)$.

0.4.6. <u>Construction of a Family of Lines</u>

<u>Proposition</u>: Every Thom-Mather stratified object W admits a family of lines.

<u>Proof</u>: Mather [9] proves that for each stratum X of W and for ϵ sufficiently small, there is a stratum preserving homeomorphism between $T_X(\epsilon)$ and the mapping cylinder of $\pi_X| \ S_X(\epsilon): S_X(\epsilon) \to X$, compatible with the projection to X . The resulting map $T_X - X \to S_X(\epsilon) \times (0\ ,\ 1) \to S_X(\epsilon)$ is a candidate for a family of X-lines since all conditions except (1) are satisfied.

A family of lines on W is constructed by increasing induction on the dimension of the stratum X . The above construction applied to the 0-dimensional stratum begins the induction.

Suppose a family of X_p-lines has been constructed for each stratum X_p of dimension $p \leq i - 1$ so that all conditions (1) - (5) hold. Let X be the stratum of dimension i . Applying the above construction to X defines a radial projection $r_X(\epsilon): T_X - X \to S_X(\epsilon)$ which must be altered slightly so as to satisfy condition (1) with respect to all smaller strata.

Suppose inductively that a radial retraction r_X has been constructed which satisfies condition (1) with respect to all strata Y where $p < \dim Y \leq i - 1$ and which satisfies conditions (2) - (5) with respect to all other strata.

Let $Y < X$ be a stratum of dimension p. Redefine r_X in the region $T_Y(\epsilon') \cap T_X$ by setting

$$r_X'(\epsilon)(p) =$$

$$h_Y^{-1}(r_Y(\tfrac{\epsilon'}{2})r_X(\epsilon)h_Y^{-1}(r_Y(\tfrac{\epsilon'}{2})(p) \, , \, \phi(\rho_Y(p))) \, , \, \rho_Y(p))$$

where

$$h_Y\colon T_Y - Y \to S_Y(\tfrac{\epsilon'}{2}) \times (0 \, , \, \infty)$$

$$h_Y(p) = (r_Y(\tfrac{\epsilon'}{2})(p) \, , \, \rho_Y(p))$$

and where $\phi\colon \mathbb{R} \to \mathbb{R}$ is a smooth nondecreasing function with

$$\phi(x) = \tfrac{\epsilon'}{2} \text{ if } x \leq \tfrac{\epsilon'}{2}$$

and

$$\phi(x) = x \text{ if } x > \tfrac{2}{3}\epsilon'$$

If $\rho_Y(p) \leq \frac{\epsilon'}{2}$ then $r_X'(p)$ can be interpreted as first sliding p along its Y-line to the point

$$p' = r_Y(\tfrac{\epsilon'}{2})(p) \in S_Y(\tfrac{\epsilon'}{2}) \, ,$$

then sliding p' along its X-line to the point

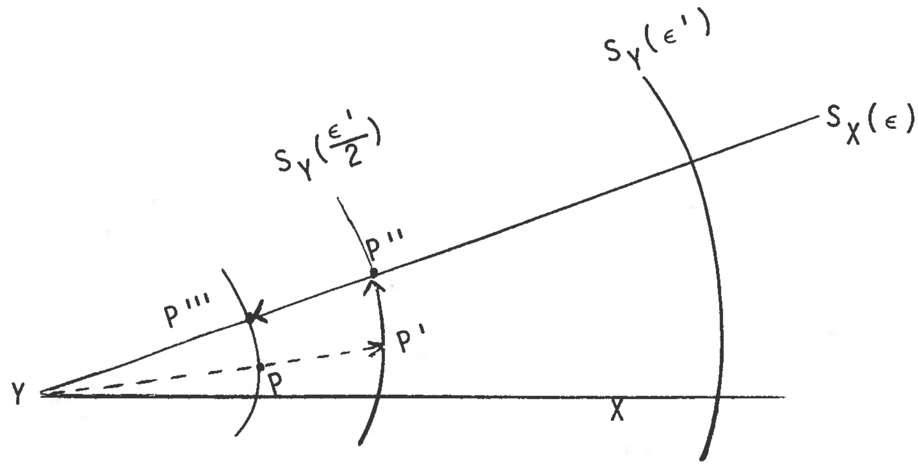
$$p'' = r_X(\epsilon)(p') \in S_X(\epsilon) \cap S_Y(\tfrac{\epsilon'}{2}) \, ,$$

and then sliding p'' back along its Y-line until it reaches a point p''' with the same ρ_Y - value as p.

$$p''' = h_Y^{-1}(r_Y(\tfrac{\epsilon'}{2})(p''), \rho_Y(p))$$

If $\rho_Y(p) \geq \frac{2}{3}\epsilon'$ then $r_X'(\epsilon)(p) = r_X(\epsilon)(p)$ so r_X has not been changed outside $T_Y(\frac{2}{3}\epsilon')$.

Diagram:



It is now clear that $r_X'(\epsilon)$ satisfies the necessary commutation relations with π_Y , ρ_Y , and r_Y in the region $T_X(\epsilon) \cap T_Y(\frac{\epsilon'}{2})$ and that it continues to satisfy conditions (2) - (5) with respect to all other strata. Furthermore, if $Y < Z < X$ then $r_X'r_Z = r_Zr_X'$ because

$$r_X'(\epsilon) \circ r_Z(p)$$

$$= h_Y^{-1}(r_Y(\tfrac{\epsilon'}{2})r_X(\epsilon)h_Y^{-1}(r_Y(\tfrac{\epsilon'}{2})(r_Z(p)),\ \phi(\rho_Y(r_Z(p))),\ \rho_Y r_Z(p))$$

$$= h_Y^{-1}(r_Y(\tfrac{\epsilon'}{2})r_X(\epsilon)h_Y^{-1}(r_Z r_Y(\tfrac{\epsilon'}{2})(p),\ \phi(\rho_Y(p)),\ \rho_Y(p))$$

$$= h_Y^{-1}(r_Z r_Y(\tfrac{\epsilon'}{2})r_X(\epsilon)h_Y^{-1}(r_Y(\tfrac{\epsilon'}{2})(p),\ \phi(\rho_Y(p)),\ \rho_Y(p))$$

$$= r_Z \circ r_X'(\epsilon)(p)$$

This concludes the nested induction and it follows that X-lines may be constructed compatible with all Y-lines for $\dim Y < \dim X$, which completes the induction. QED.

Remark: This procedure is essentially described in Hendricks [3] although his construction is not well defined.

0.5. Regular Neighborhoods

0.5.1. Proposition: Suppose W^n is a Thom-Mather stratified object and $Q^m \subset W^n$ is a closed union of strata. Let $\epsilon > 0$ be so small that $T_X(\epsilon) \subset T_X$ is defined for each stratum X of Q^m. Let

$$U' \equiv \cup\{T_X(\tfrac{\epsilon}{2}) \mid X \text{ is a stratum of } Q\}$$

Then the inclusion $Q \to U'$ is a homotopy equivalence. (U' is called a <u>regular neighborhood</u> of Q in W.)

<u>Proof</u>: We shall construct a homotopy inverse $H': U' \to Q$, called a <u>canonical retraction</u>:

Choose a family of lines on W subordinate to the Thom-Mather data. Let $U \equiv \cup\{T_X(\epsilon) \mid X$ is a stratum of $Q\}$ and define a smooth function

$$f: \mathbb{R} \to [0, 1]$$

so that

$$f(x) = 0 \quad \text{whenever} \quad x \le \frac{1}{2}$$

$$f(x) = 1 \quad \text{whenever} \quad x \ge \frac{3}{4}$$

$$f'(x) > 0 \quad \text{whenever} \quad x \in (\frac{1}{2}, \frac{3}{4})$$

Define $\phi_m: \mathbb{R} \to [0, 1]$ by $\phi_m(x) = f(x-m)$ for $m = 0, 1, 2, \ldots$.

For the stratum X^i of dimension i, define

$$H_i: U \times \mathbb{R} \to U$$

by

$$H_i(p, t) = p \quad \text{if} \quad p \notin T_X(\epsilon)$$

$$H_i(p, t) = h_X^{-1}(r_X(\epsilon)(p), \rho_X(p)[(1-\phi_i(t))f(\rho_X(p)/\epsilon) + \phi_i(t)]) \quad \text{if} \quad p \in T_X(\epsilon)$$

(where h_X is defined as in section 0.4.6.).

Notice that if $\rho_X(p) < \frac{\epsilon}{2}$ then

$$H_i(p\ ,\ t) = \pi_X(p) \quad \text{for} \quad t \leq i$$

$$H_i(p\ ,\ t) = p \quad \text{for} \quad t \geq i + 1$$

H_i is continuous and if $Y > X$ then $H_i | Y \times \mathbb{R}$ is smooth on the subset $H_i^{-1}(Y) \cap (Y \times \mathbb{R})$.

Define

$$H: U \times [0\ ,\ m+1] \to Q$$

by

$$H(p\ ,\ t) \equiv H_0(H_1(H_2(\ldots(H_m(p\ ,\ t)\ ,\ t)\ ,\ t)\ ,\ \ldots\ ,\ t)$$

and let

$$H': U' \to Q$$

be defined by

$$H'(p) = H(p\ ,\ 0)\ .$$

Then $H(p\ ,\ m+1) = p$ and $H(p\ ,\ 0) \in Q$ whenever $p \in U'$. H' is a homotopy equivalence with homotopy inverse given by the inclusion $Q \to U'$, and the homotopy is easily constructed using H.

Terminology: H' is called a canonical retraction and it is a weak deformation retraction.

0.5.2. Corollary: Suppose W is a Whitney stratified object with a system of control data and $W \subset M$ for some manifold M. For $\epsilon > 0$ sufficiently small, define

$$U \equiv \cup\{T_X(\epsilon) | \ X \text{ is a stratum of } W\}$$

Then the inclusion $W \to U$ is a homotopy equivalence.

Proof: The control data on W gives M the structure of a Thom-Mather stratified object for which W is a closed union of strata. The proposition applies immediately.

Note: If W is a Whitney stratified object in a manifold M and if $\epsilon > 0$ is chosen so small that the collection of submanifolds $S_X(\epsilon)$ are mutually tranverse (for all strata X of W) then the set

$$\overline{U} = \cup\{\overline{T_X(\epsilon)} | \ X \text{ is a stratum of } W\}$$

is a Whitney stratified object in M whose top stratum U is a regular neighborhood of W. It then follows from Verona [19] (theorem 1.4) that U is diffeomorphic to the interior of a smooth manifold with boundary.

CHAPTER 1.

TRANSVERSALITY

1.1. The Whitney Topology:

The Whitney topology will be used for all function spaces unless otherwise noted.

1.1.1. Definition: Let M and N be smooth manifolds and $C^\infty(M, N)$ the space of smooth functions from M to N. The Whitney (C^1) topology on this space is defined as follows:

For each open set $U \subset J^1(M, N)$, (the space of 1-jets of functions from M to N), let

$$M(U) \equiv \{f \in C^\infty(M, N) \mid J^1 f(M) \subset U\} .$$

Then the family of sets $M(U)$ for U an open subset of $J'(M, N)$ defines a basis for the C^1 topology.

The Whitney topology has the following properties, as proven in Mather [7]:

(1) If M, N, and P are manifolds and $C_p^\infty(M, N)$ denotes the proper smooth maps from M to N, then the composition operation

$$C_p^\infty(M, N) \times C^\infty(N, P) \to C^\infty(M, P)$$

is continuous.

(2) The identification $C^\infty(M, N) \times C^\infty(M, P) \to C^\infty(M, N \times P)$ is a homeomorphism.

(3) The diffeomorphisms of a manifold M form an open subset Diff (M) of the space $C^\infty(M, M)$ and the inversion map Diff $(M) \to$ Diff (M) is a homeomorphism.

1.2. Supertransversality

1.2.1. Definition: Suppose M and N are manifolds, K is a closed subset of N, $W_1 \subset M$ and $W_2 \subset N$ are closed Whitney stratified objects, and $f: M \to N$ is a smooth map. Then f is said to take W_1 transversally to W_2 on K if the following holds:

For every stratum X of W_1 and every stratum Y of W_2 and every $p \in X$ for which $f(p) \in Y \cap K$,

$$df(p)(T_pX) + T_{f(p)}Y \supset T_{f(p)}N .$$

The following theorem states that the set of such f form an open and dense subset of $C^\infty(M, N)$.

1.2.2. Transversality Theorem

Let M, N, K, W_1, W_2 and f be as above, and suppose f takes W_1 transversally to W_2 on K. Then

(1) There is a neighborhood U of f in $C^\infty(M, N)$ so that if $g \in U$ then g takes W_1 transversally to W_2 on K.

(2) For any neighborhood U' of f in $C^\infty(M, N)$ there exists a function $f' \in U'$ which takes W_1 transversally to W_2 <u>on N</u>, and satisfies $f'|f^{-1}(K) = f|f^{-1}(K)$. Furthermore f' may be chosen to be homotopic to f by a smooth homotopy which is constant on K.

The proof appears in section 1.3. following a discussion of supertransversality.

1.2.3. <u>Definition</u>: Suppose M and N are manifolds and let $A^a \subset M$ and $B^b \subset N$ be submanifolds. Let $G_a(TM) \to M$ be the Grassmann bundle of a-planes in TM whose fibre over a point $p \in M$ is the Grassmannian $G_a(T_pM)$ of a-planes in T_pM. Let $p: J^1(M, N) \to M \times N$ be the bundle of 1-jets of maps from M to N and let $E \to M \times N$ be the fibre product in the diagram

$$\begin{array}{ccc} E & \cdots\cdots\to & G_a(TM) \times G_b(TN) \\ \vdots & & \downarrow \\ J^1(M, N) & \xrightarrow{p} & M \times N \end{array}$$

Thus, a point in E can be thought of as a collection $(p, q, \tau_1, \tau_2, L)$ where $(p, q) \in M \times N$, $\tau_1 \in G_a(T_pM)$, $\tau_2 \in G_b(T_qN)$ and $L \in \mathrm{Hom}(T_pM, T_qN)$. Define the <u>canonical set</u> $C \subset E$ to be

$$C = \{(p, q, \tau_1, \tau_2, L) \in E \mid L(\tau_1) + \tau_2 \neq T_qN\}.$$

C is the closed set representing those linear maps L which fail to take τ_1 transversally to τ_2.

Define

$$A' \equiv \{(p, q, \tau_1, \tau_2, L) \in E \mid p \in A \text{ and } \tau_1 = T_pA\}$$

and

$$B' = \{(p, q, \tau_1, \tau_2, L) \in E \mid q \in B \text{ and } \tau_2 = T_qB\}$$

and let T be the projection of $\overline{A'} \cap \overline{B'} \cap C$ to $J^1(M, N)$. Then T is a closed subset of $J^1(M, N)$.

Definition: A smooth function $f: M \to N$ is said to take A supertransversally to B if $J^1f(M) \cap T = \phi$.

This means that f takes A transversally to B, and whenever $p \in \overline{A} \subset M$ and $\tau_1 \subset T_pM$ is a limit of tangent planes of A, and whenever $q \in \overline{B} \subset N$ and $\tau_2 \subset T_qN$ is a limit of tangent planes of B, then $df(p)$ takes τ_1 transversally to τ_2.

1.2.4. Remarks:

(1) If A and B are closed submanifolds of M and N respectively, then f takes A transversally to B if and only if f takes A supertransversally to B.

(2) The set of functions which take A supertransversally to B is open in the Whitney C^1 topology.

(3) If K is a closed subset of N then f is said to take A supertransversally to B on K if

$$J^1 f(M) \cap T \cap p^{-1}(M \times K) = \phi$$

and the set of such f is open in the Whitney C^1 topology.

(4) If $W_1 \subset M$ and $W_2 \subset N$ are closed Whitney stratified objects then f is said to take W_1 supertransversally to W_2 on K if f takes each stratum of W_1 supertransversally to each stratum of W_2 on K.

1.3. <u>Proof of the Transversality Theorem</u>

1.3.1. <u>Lemma</u>: Suppose $W_1 \subset M$ and $W_2 \subset N$ are closed Whitney objects contained in manifolds M and N, and suppose $f: M \to N$ is a smooth map which takes W_1 transversally to W_2 on a closed subset $K \subset N$. Then f takes W_1 supertransversally to W_2 on K.

<u>Proof</u>: Let X_1 be a stratum of W_1 and let X_2 be a stratum of W_2. Suppose $p \in \overline{X}_1 \subset M$ and $f(p) \in \overline{X}_2 \cap K \subset N$ and suppose sequences of points $a_i \in X_1$ converging to p and $b_i \in X_2$ converging to $f(p)$ are given, for which the tangent planes $T_{a_i}X_1$ converge to a plane τ_1 and $T_{b_i}X_2$ converge to τ_2. We must show that

$$df(p)(\tau_1) + \tau_2 = T_{f(p)}N .$$

Since $p \in \overline{X}_1$ it lies in some stratum $Y_1 \leq X_1$ of W_1 and $f(p)$ lies in some stratum $Y_2 \leq X_2$ of W_2. Furthermore, f takes W_1 transversally to W_2 on K, so

$$df(p)(T_pY_1) + T_{f(p)}Y_2 = T_{f(p)}N .$$

However, $\tau_1 \supseteq T_pY_1$ and $\tau_2 \supseteq T_{f(p)}Y_2$ so the above equation is verified immediately.

1.3.2. Lemma: Suppose $A_1 < B_1$ are strata of a Whitney object $W_1 \subset M$ and that $A_2 < B_2$ are strata of a Whitney object $W_2 \subset N$, and suppose $f: M \to N$ is a smooth map which takes A_1 transversally to A_2. Then there exist open neighborhoods T_i of A_i so that f takes $B_1 \cap T_1$ transversally to A_2, and f takes A_1 transversally to $B_2 \cap T_2$.

Proof: The proof is an immediate consequence of Whitney's condition A.

1.3.3. Lemma: Suppose M and N are manifolds, $T \subset M$, and $f: M \to N$ and $g: M \to N$ are smooth functions which agree on T. If f and g are sufficiently close in the Whitney C^0 topology (and therefore if they are sufficiently close in the Whitney C' topology) then there is a smooth homotopy H between them, $H: M \times [0, 1] \to N$, for which $H(x, t) = f(x) = g(x)$ whenever $x \in T$ and for all $t \in [0, 1]$.

Proof: We follow Mather's proof [7].

Let U be a neighborhood of the diagonal in $N \times N$. Define a family of geodesics to be a smooth map

$$\gamma: U \times [0, 1] \to N$$

so that

(a) for any $(x_1 y) \in U$, $\gamma(x_1 y, 0) = x$

(b) for any $(x_1 y) \in U$, $\gamma(x_1 y, 1) = y$

(c) for any $x \in U$ and any $t \in [0, 1]$,

$\gamma(x, x, t) = x$.

(The existence of families of geodesics is proven in Mather [7])

Then if f and g are so close that $(f(x), g(x)) \in U$ for all $x \in M$, the desired homotopy is $H(x, t) = \gamma(f(x), g(x), t)$. QED.

1.3.4. Proposition: Suppose $f \in C^\infty(M, N)$, A is a closed submanifold of M, and B is any submanifold of N. Let $K \subset N$ be a closed subset and U be a C^1 neighborhood of f in $C^\infty(M, N)$. Suppose f takes A transversally to B on K. Let $K' = f^{-1}(K)$

Then there exists a function $g \in U$ so that $g| K' = f| K'$ and g takes A transversally to B.

Proof: The inclusion map $i: A \to M$ induces a restriction map $i^*: C^\infty(M, N) \to C^\infty(A, N)$ by $i^*(h) = h \circ i$. We claim that i^* is a continuous open surjection. It is continuous since i is proper and it is a surjection since every smooth function on A can be extended to a smooth function on M. Let T be an open set in $C^\infty(M, N)$. It will be shown that every $h' = i^*(h) \in i^*(T)$ has a neighborhood contained in $i^*(T)$. First construct a continuous extension map $E: C^\infty(A, N) \to C^\infty(M, N)$ so that $i^* \circ E = \text{identity}$ and $E(h') = h$ as follows: By Mather [6] there is a continuous linear extension map $F: C^\infty(A, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ and hence also $F^d: C^\infty(A, \mathbb{R}^d) \to C^\infty(M, \mathbb{R}^d)$. Choose a smooth closed embedding $N \to \mathbb{R}^d$ for some d and a smooth retraction r from a neighborhood of N in $\mathbb{R}^d$ to N.

Choose any smooth extension of r to a map $r: \mathbb{R}^d \to N$, and define $E: C^\infty(A, N) \to C^\infty(M, N)$ by

$$E(g) \equiv r_*(h+F^d(g-h'))$$

where $r_*: C^\infty(M, \mathbb{R}^d) \to C^\infty(M, N)$ is the continuous map

$$r_*(k) = r \circ k .$$

Then E is continuous, $E(h') = r_*(h+F^d(0)) = h$, and $i^* \circ E(g) = g$.

Since E is continuous, $E^{-1}(T)$ is an open neighborhood of h' in $C^\infty(A, N)$ and if $g \in E^{-1}(T)$ then $E(g) \in T$, so $g = (i^* \circ E)(g) \in i^*(T)$ which proves $E^{-1}(T) \subset i^*(T)$, so $i^*(T)$ is open.

Now suppose $f \in C^\infty(M, N)$ and U is a neighborhood of f. Then $i^*(U)$ is a neighborhood of $f|A$ so there is a function $g' \in i^*(U) \subset C^\infty(A, N)$ which is transverse to B and satisfies $g'|A \cap K' = f|A \cap K'$, by the usual transversality theorem. (See, for example, Guillemin and Golubitsky [2]). However $g' = i^*(g'')$ for some $g'' \in U$ and therefore g'' takes A transversally to B and $g''|A \cap K' = f|A \cap K'$.

It only remains to adjust g'' so as to agree with f on all of K assuming it was sufficiently close to f to begin with. This is accomplished as follows.

Let L be an open neighborhood of K' so that f takes A transversally to B on $\bar{L}$ (transversality is an open condition on the domain) and let $\rho: M \to \mathbb{R}$ be a smooth function with $0 \le \rho(x) \le 1$ and

$$x \in K' \Rightarrow \rho(x) = 0$$

$$x \notin L \Rightarrow \rho(x) = 1 .$$

Choose a family of geodesics $\gamma: Q \times [0, 1] \to N$ on N, where Q is a neighborhood of the diagonal in $N \times N$. (See Lemma 1.3.3.).

Then the map

$$\theta: C^{\infty}(M, N) \times C^{\infty}(M, N) \to C^{\infty}(M, N)$$

$$\theta(g_1, g_2)(p) = \gamma(g_1(p), g_2(p), \rho(p))$$

is continuous on the open neighborhood of the diagonal on which it is defined; $\theta(f, f) = f$ and so if $T' \subset T$ is an open neighborhood of f consisting of functions which take A transversally to B at each point $x \in \overline{L} \cap A$, then $\theta^{-1}(T')$ is a neighborhood of (f, f) in $C^{\infty}(M, N) \times C(M, N)$. Choose a (smaller) neighborhood of T'' of f so that $T'' \times T'' \subset \theta^{-1}(T')$ and choose the above g'' to lie in T''.

The result is that $g \equiv \theta(f, g'') \in T' \subset T$ and so

(a) if $x \in K'$ then $\rho(x) = 0$ so $g(x) = f(x)$

(b) if $x \in A - \overline{L}$ then $\rho(x) = 1$ so $g(x) = g''(x)$ so $dg(x)(T_xA) + T_{g(x)}B = T_{g(x)}N$

(c) if $x \in A \cap \overline{L}$ then $dg(x)(T_xA) + T_{g(x)}B = T_{g(x)}N$ since $g \in T'$.

Therefore g takes A transversally to B, and g agrees with f on K'. . QED.

1.3.5. Proof of the Transversality Theorem

Suppose $W_1 \subset M$ and $W_2 \subset N$ are Whitney objects and $f \in C^\infty(M, N)$ takes W_1 transversally to W_2 on the closed subset $K \subset N$. Then by Lemma 1.3.1., f takes W_1 supertransversally to W_2 on K and hence every sufficiently nearby function g does so also. This proves part (1) of the transversality theorem.

To prove part (2), let $K' = f^{-1}(K)$ and suppose U is a given neighborhood of f. We must find $g \in U$ so that $g| K' = f| K'$ and so that g takes W_1 transversally to W_2.

Suppose inductively that a function $h \in U$ has been found so that $h| K' = f| K'$ and h takes each stratum X of the $i - 1$ skeleton $W_1^{(i-1)}$ transversally to W_2. We must find a function $h' \in U$ so that $h'| K' = f| K'$ and h' takes each stratum of $W_1^{(i)}$ transversally to W_2. (The case $i = 0$ is trivial). Let R be an open subset of $J'(M, N)$ so that

$$\{f \in C^\infty(M, N) | J'(f)(M) \subset R\} \subset U .$$

By Lemma 1.3.2. there is a neighborhood T_X of each stratum X of $W_1^{(i-1)}$ so that f takes $Y \cap T_X$ transversally to W_2, where Y is the i-dimensional stratum of W_1. Let T be the union of these neighborhoods and T' be a slightly smaller neighborhood of $W_1^{(i-1)}$ so that $T' \subset \overline{T'} \subset T$. Then $Y \cap (M-\overline{T'})$ is closed in $M - \overline{T'}$ and so by the preceding proposition, there is a smooth function $h': M - \overline{T'} \to N$ so that:

(a) $J^1 h'(M-\overline{T'}) \subset R$

(b) $h' | (T-\overline{T'}) = h| (T-\overline{T'})$

(c) h' takes $Y \cap (M-\overline{T'})$ transversally to W_2

(d) $h' | (K' \cap \overline{T'}) = h| (K'-\overline{T'})$.

(h' is found by repeated application of the proposition to the strata of W_2 in any order).

h' extends to a smooth function $g: M \to N$ by setting $g| T' = h| T'$ and g therefore satisfies:

(a) $g \in U$

(b) g takes $W_1^{(i)}$ transversally to W_2

(c) $g| K' = h| K' = f \;\; K'$.

This completes the inductive step and we conclude part (2) of the transversality theorem.

Finally, it follows from Lemma 1.3.3. that if g is chosen sufficiently close to f then it is homotopic to f by a smooth homotopy which agrees with f on K'.

1.4. Corollary

If W_1 and W_2 are closed Whitney stratified objects in a manifold M, which are transverse on a closed subset $K \subset M$, then there exists a diffeomorphism $\phi: M \to N$, arbitrarily close to the identity in the Whitney C^1 topology, so that $\phi(W_1)$ is transverse to W_2 and $\phi(x) = x$ if $x \in K$.

Furthermore, ϕ may be chosen so as to be smoothly isotopic to the identity I by an isotopy $\phi: M \times [0, 1] \to M$ for which $\phi(x, t) = x$ whenever $x \in K$ and $t \in [0, 1]$.

Proof: Let U be an open neighborhood of the diagonal in $M \times M$ and fix a family of geodesics $\gamma: U \times [0, 1] \to M$ (see section 1.3.). By shrinking U slightly we may assume γ extends to a smooth map $\gamma: M \times M \times [0, 1] \to M$.

Define

$$\gamma_*: C^\infty(M\times M , M\times M) \times [0 , 1] \to C^\infty(M\times M , M)$$

by

$$\gamma_*(F , t) \equiv \gamma_t \circ F$$

where

$$\gamma_t(p , q) \equiv \gamma(p , q , t)$$

Define

$$\Delta^*: C^\infty(M\times M , M) \to C^\infty(M , M)$$

by

$$\Delta^*(f)(x) \equiv f(x , x)$$

Define

$$j_*: C^\infty(M , M) \times C^\infty(M , M) \to C^\infty(M\times M , M\times M)$$

by

$$j_*(f , g)(x , y) \equiv (f(x) , g(y)) .$$

Then the composition

$$\theta \equiv \Delta^* \circ \gamma_* \circ (j_* \times I): C^\infty(M,M) \times C^\infty(M,M) \times [0 , 1] \to C^\infty(M,M)$$

is continuous and

$$\theta(I , I , t)(x) = \gamma_t(x , x) = x \quad \text{for all} \quad t \in [0 , 1] .$$

Therefore there is a neighborhood T of $\{I\} \times \{I\} \times [0 , 1]$ in $C^\infty(M , M) \times C^\infty(M , M) \times [0 , 1]$ so that if $(\phi , \psi , t) \in T$ then $\theta(\phi , \psi , t)$ is a diffeomorphism.

Since $[0, 1]$ is compact, this neighborhood may be chosen to be of the form $T' \times T' \times [0, 1]$ for some neighborhood T' of the identity in $C^{\infty}(M, M)$. This insures that if ϕ and ψ are elements of T' then $\theta(\phi, \psi, t)$ is a homotopy from ϕ to ψ which is a diffeomorphism at each time t. Furthermore, if $p \in M$ and $\phi(p) = \psi(p)$ then for any $t \in [0, 1]$,

$$\theta(\phi, \psi, t)(p) = \phi(p) = \psi(p) .$$

To prove the corollary, then, by the transversality theorem there is a function $\phi \in T$ which takes W_1 transversally to W_2 and restricts to the identity on K. Therefore $\Phi: M \times [0, 1] \to M$ defined by $\Phi(p, t) = \theta(\phi, I, t)(p)$ is an isotopy from ϕ to I which exhibits the desired properties. QED.

1.5.1. <u>Proposition</u>

Suppose $f: M \to N$ is a smooth map between smooth manifolds and suppose $W_1 \subset M$ and $W_2 \subset N$ are Whitney stratified objects. Suppose f takes W_1 transversally to W_2. Then $W \cap f^{-1}(W_2) = (f|W_1)^{-1}(W_2)$ is a Whitney stratified object.

Proof:

The proposition is first proven for the special case $M = N$ and f = the identity. Then $W_1 \cap W_2$ is stratified with one stratum $X_1 \cap X_2$ for each stratum X_i of W_i. To check Whitney's condition B, suppose $X_1 \cap X_2 < Y_1 \cap Y_2$ are two strata of $W_1 \cap W_2$, and suppose a sequence of points $p_i \in Y_1 \cap Y_2$ converges to a point $q \in X_1 \cap X_2$. Suppose a sequence of points $q_i \in X_1 \cap X_2$ also converges to q, that the tangent planes $T_{p_i}(Y_1 \cap Y_2)$ converge to a plane τ, and that the secant lines $\widehat{p_i q_i}$ converge to a line ℓ. We claim $\ell \subset \tau$.

By choosing subsequences if necessary (since the Grassmannian is compact) we may suppose $T_{p_i}Y_1$ converge to some τ_1 and $T_{p_i}Y_2$ converge to some τ_2 and therefore $\ell \subset \tau_1$ and $\ell \subset \tau_2$ by condition B. Thus, $\ell \subset \tau_1 \cap \tau_2$ and it remains to show $\tau = \tau_1 \cap \tau_2$. Certainly $\tau_1 \cap \tau_2 \subset \tau$ so they need only have the same dimension, which will be guaranteed if τ_1 is transverse to τ_2. However, by condition A, $\tau_1 \supseteq T_qX_1$ and $\tau_2 \supseteq T_qX_2$, and X_1 and X_2 are transverse, so τ_1 and τ_2 are also. Note that this proof allows for the possibility $X_1 = Y_1$ or $X_2 = Y_2$.

Local finiteness of $W_1 \cap W_2$ follows from local finiteness of each, however the strata of $W_1 \cap W_2$ need not be connected.

To prove the axiom of the frontier, suppose $X_1 \cap X_2$ and $Y_1 \cap Y_2$ are strata of $W_1 \cap W_2$ with X_i and Y_i strata of W_i . Suppose $p \in (X_1 \cap X_2) \cap \overline{(Y_1 \cap Y_2)}$. Then $p \in X_1 \cap \overline{Y_1}$ so $X_1 < Y_1$ and similarly $X_2 < Y_2$. Furthermore $X_1 \cap Y_2 \neq \emptyset$ because X_1 is transverse to X_2 and contains a normal slice through X_2 which intersects Y_2 . By Mather [9] we have $X_1 \cap X_2 \subset \overline{X_1 \; Y_2}$ and similarly $X_1 \cap Y_2 \subset \overline{Y_1 \cap Y_2}$, so $X_1 \cap X_2 \subset \overline{Y_1 \cap Y_2}$. This completes the proof in the particular case.

<u>General Case</u>

If f takes W_1 transversally to W_2 then there is a neighborhood U of W_1 in M so that $f|\,U$ takes U transversally to W_2 . Therefore the graph of $f|\,U$ is transverse to $M \times W_2$ in $M \times N$, for if $p \in X$, a stratum of W_2 , and if $(u,v) \in T_{(p\ ,\ f(p))}(M \times N)$ then there exists $u_1 \in T_pM$ and $v_1 \in T_{f(p)}X$ so that $v = df(p)(u_1) + v_1$

so

$$(u_1 , v) = (u-u_1 , v_1) + (u_1 , df(p)(u_1))$$

$$\in T_{(p , f(p))}(M\times W_1) + T_{(P , f(p))}(\text{graph } f| U) .$$

It follows that graph $(f) \cap (U\times W_2)$ is a Whitney object in $M \times N$. The projection $pr: M \times N \to M$ takes graph $(f| U)$ diffeomorphically to U and hence $pr(\text{graph } (f| U)\cap(M\times W_2)) = f^{-1}(W_2)$ is a Whitney stratified object in M.

Finally, it must be shown that W_1 is transverse to $f^{-1}(W_2)$ which will imply that $W_1 \cap f^{-1}(W_2)$ is a Whitney object. Suppose X_1 is a stratum of W_1, $f^{-1}(X_2)$ is a stratum of $f^{-1}(W_2)$ and $p \in X_1 \cap f^{-1}(X_2)$. Then for any $v \in T_pM$ there is a $w_1 \in T_pX_1$ and $w_2 \in T_{f(p)}X_2$ so that

$$df(p)(v) = df(p)(w_1) + w_2$$

Therefore

$$v - w \in df(p)^{-1}(T_{f(p)}X_2) = T_p(f^{-1}(X_2))$$

so

$$v = (v-w_1) + w_1 \in T_p(f^{-1}(X_2)) + T_pX_1$$

as desired.

1.5.2. <u>Remark</u>: A collection of submanifolds $S_1, \ldots, S_k$ of a manifold M are said to be <u>mutually transverse</u> if the intersection of any subcollection $S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_p}$ is transverse to the intersection of any other disjoint subcollection $S_{j_1} \cap S_{j_2} \cap \ldots \cap S_{j_q}$.

If W is a Thom-Mather stratified object then there is an $\epsilon > 0$ so that for every collection of strata $X_1, \ldots, X_k$ and for any stratum $Y > X_i (1 \le i \le k)$ the subsets

$$Y \cap S_{X_1}(\epsilon), Y \cap S_{X_2}(\epsilon), \ldots, Y \cap S_{X_k}(\epsilon)$$

are mutually transverse in Y. This follows easily from the fact that (for ϵ sufficiently small) the map

$$f: Y \cap T_{X_1}(\epsilon) \cap T_{X_2}(\epsilon) \cap \ldots \cap T_{X_k}(\epsilon) \to \mathbb{R}^k$$

defined by

$$f(p) = (\rho_{X_1}(p), \rho_{X_2}(p), \ldots, \rho_{X_k}(p))$$

is a submersion.

A similar remark applies to any Whitney stratified object with a system of control data.

CHAPTER 2.

HOMOLOGY OF STRATIFIED OBJECTS

2.1. Orientations

2.1.1. Definition: The set of ordered bases of an n-dimensional real vector space V can be given a topology by its inclusion in $V^n = V \times V \times \ldots \times V$.

An orientation of a real vector space V is an equivalence class of ordered bases of V, two being equivalent of there is a continuous one-parameter family of ordered bases connecting them.

If $0 \to A \to B \to C \to 0$ is an exact sequence of oriented vector spaces, their orientations are said to be compatible with the exact sequence if the following holds:

Let D be a subspace of B complementary to the image of A, and choose an ordered basis (and hence an orientation) of D so that the orientation of A followed by that of D coincides with the given orientation of B. Then $D \to C$ defines an orientation on C which must coincide with the given orientation of C if the orientations are to be compatible.

An orientation of any two of A, B or C determines a unique compatible orientation on the third

An orientation of a manifold is a locally constant orientation of each tangent space.

2.1.2. <u>Definition</u>: A homological orientation of a real n-dimensional vector space V is an element of $H_n(V, V-\{0\})$.

2.1.3. <u>Proposition</u>: If V^n is a real vector space then there is one to one correspondence between orientations of V and homological orientations of V.

<u>Proof</u>:

Let

$$\Delta^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 + x_2 + \ldots + x_{n+1} = 1$$

$$\text{and } 0 \leq x_i \leq 1 \text{ for all } i\}$$

Δ^n is the standard n-simplex. Choose a linear embedding $f: \Delta^n \to V$ so that $0 \in V$ is in the interior of $f(\Delta^n)$. Then any ordering of the vertices $v_1, \ldots, v_{n+1}$ of Δ^n defines both an orientation of V and a homological orientation of V as follows: The vectors $\{(f(v_2)-f(v_1)), (f(v_3)-f(v_2)), \ldots, f(v_{n+1}) - f(v_n)\}$ are an ordered basis for V thus giving an orientation of V. On the other hand, f is a singular n simplex and since $f(\partial\Delta^n) \subset V - \{0\}$, f is a cycle whose homology class generates $H_n(V, V-\{0\})$. Similarly, every orientation or homological orientation of V corresponds to an equivalence class of orderings of the vertices of Δ^n, two being considered equivalent if they differ by an even permutation.

2.2. Abstract Geometric Chains and Cycles

In this section we will define a geometric cycle to be a stratified object with an oriented top dimensional stratum for which the induced orientations on the strata of codimension 1 cancel. A geometric cycle lying inside another stratified object will represent a homology class in that stratified object.

2.2.1. Definition: An orientation of a compact Thom-Mather stratified object W^n is a specification of an orientation and multiplicity (possibly 0) to each connected component $X_i (1 \leq i \leq p)$ of the stratum X of W with $\dim(X) = n$. However, a given orientation and multiplicity is identified with the opposite orientation and negative multiplicity, and nonorientable strata are permitted provided they are assigned the multiplicity 0.

An orientation of W^n determines an element of the homology group $H_n(W, bW)$ (where bW is the singularity subset of W). This element, called a homological orientation, is defined as follows:

According to 0.5., there is a regular neighborhood U in W, of the singularity set bW, so that bW is a (weak) deformation retract of U and so

$$\bigoplus_{i=1}^{p} H_n(X_i , X_i \cap U) \cong H_n(X , X \cap U) \cong H_n(W , U) \cong H_n(W , bW)$$

Since $X_i - (X_i \cap U)$ is compact, each orientation-with-multiplicity on X_i determines an element of $H_n(X_i\ , X_i \cap U)$ as in Milnor [10] (p. 273). The desired homological orientation is the sum of these elements. (It can also be shown that every element of $H_n(W\ , bW)$ corresponds to an orientation of W.)

2.2.2. The Boundary Map

An orientation of W^n as above induces an orientation on bW, described in this section. Recall from Mather [9] that for any pair of strata $Y < X$ of W, the map

$$(\pi_{YX}\ , \rho_{YX}) : T_Y \cap X \to Y \times (0\ , \epsilon)$$

is a smooth submission. In particular for $Y^{n-1} < X^n$, the set $\pi_{YX}{}^{-1}(p)$ consists of finitely many smooth curves in X, oriented by the tangent vector fields

$$-\text{grad}\ (\rho_Y |\ \pi_{YX}{}^{-1}(p))$$

which "point from X to Y". If ℓ denotes the 1-dimensional subbundle of $TX|\ (X \cap T_Y)$ consisting of all tangents to the curves $\pi_{YX}{}^{-1}(p)$ (for $p \in Y$) then there is an exact sequence

$$0 \to \ell \to TX|\ (X \cap T_Y) \underset{d\vec{\pi}_Y}{\to} \pi_{YX}{}^{*}(TY) \to 0\ .$$

An orientation with multiplicity of TX determines a unique orientation with multiplicity of $\pi_{YX}{}^*(TY)$ compatible with this exact sequence.

For each contractible open subset $U \subset Y$ of Y and each connected component Q of $\pi_{YX}{}^{-1}(U) \subset X$ there is a unique orientation with multiplicity of $TY|\,U$ which pulls back to the above orientation with multiplicity of $\pi_{YX}{}^*(TY)|\,Q$, (since the orientation of $\pi_{YX}{}^*(TY)$ is constant along each curve in $\pi_{YX}{}^{-1}(p)$ for $p \in U$.) Summing over all such components Q , we obtain an orientation with multiplicity for $TY|\,U$. These orientations are locally compatible and we have thus defined an induced orientation with multiplicity on Y and therefore an orientation of bW . (If some component of Y is not in the closure of an n-dimensional stratum, a multiplicity 0 is assigned to that component.) The oriented stratified object bW is called the boundary of W .

<u>Remark</u>: Suppose σ^n is the standard n-simplex and an ordering of the vertices $\{v_0 , v_1 , \ldots , v_n\}$ of σ is specified, and suppose $\tau^{n-1} < \sigma$ is the face determined by the vertices $\{v_0 , v_1 , \ldots , \hat{v}_i , \ldots , v_n\}$. The (usual) induced orientation of τ is defined to be $(-1)^i$ times the orientation given by this ordering of the vertices of τ . This induced orientation coincides with the "boundary map" described above when σ is considered a stratified object with τ a codimension one stratum.

2.2.4. <u>Definition</u>: A <u>geometric cycle</u> W^n is an oriented compact Thom-Mather stratified object such that the boundary of W is 0 (i.e., all induced multiplicities are 0) .

2.2.5. <u>Proposition</u>: Let W^n be a geometric cycle. Then there is a unique class $[W] \in H_n(W)$, called the fundamental class of W , which maps to the homological orientation of W in $H_n(W , bW)$.

<u>Proof</u>: The proposition will be verified once it is shown that the operation of computing the boundary of W coincides with the map denoted "θ" in the following diagram with exact row and column:

$$\begin{array}{cccccccl}
 & & & & \text{orientation of } W & & \downarrow & \\
 & & & & \downarrow & & H_{n-1}(bbW) = 0 & \\
 & & & & & & \downarrow & \\
\to H_n(bW) & \to & H_n(W) & \to & H_n(W,bW) & \to & H_{n-1}(bW) & \to \\
\| & & & & & \searrow^{\theta} & \downarrow & \\
0 & & & & & & H_{n-1}(bW, bbW) & \leftarrow \text{orientation of } bW \\
 & & & & & & \downarrow &
\end{array}$$

To compute ∂W it is necessary to use theorem 3.6. which constructs an explicit piecewise linear cellulation of W with the property that, for some $\epsilon < 0$, $\overline{S_Y(\epsilon)}$ is a closed subcomplex and if τ is a cell in $S_Y(\epsilon) \cap X$ then $\pi_Y(\tau)$ is a cell in Y and $C_Y(\tau)$ is a cell in $T_Y(\epsilon)$, and every cell in Y and in $T_Y(\epsilon)$ is of this type.

It is necessary to show that the following diagram commutes, and only the top square is nontrivial.

orientations of W	geometric boundary $\longrightarrow$	orientations of bW
$\downarrow$		$\downarrow$
locally compatible orientations of each n-simplex in W	simplicial boundary $\longrightarrow$	locally compatible orientations of each n - 1 simplex in bW
$\downarrow$		$\downarrow$
simplicial-homological orientations of W	homological boundary = θ $\longrightarrow$	homological orientations of bW in simplicial theory
$\downarrow$		$\downarrow$
orientations of W in singular homology theory	θ $\longrightarrow$	homological orientations of bW in singular theory

Therefore an orientation with multiplicity for each connected component of X induces an orientation with multiplicity for each n-cell in X and therefore defines a cellular n-chain C, in W. The boundary of C contains no contribution from the $n-1$ cells in X (since X is a manifold) so C is a relative cycle in (W, bW) which in fact defines the homological orientation of W: $[C] \in H_n(W, bW)$.

If τ is an $n-1$ cell in Y then τ is a face of only those n-cells $\sigma_1, \ldots, \sigma_p$ which are connected components of $\pi_{YX}^{-1}(\tau)$. Therefore τ appears in C with the coefficient

$$\sum_{i=1}^{p} m_i \epsilon_i$$

where m_i is the multiplicity of σ_i and $\epsilon_i = \pm 1$ is the boundary coefficient relating τ and σ_i. We conclude that $\partial C \in H_{n-1}(bW)$ is a cellular $n-1$ chain supported on $\overline{Y}$ and ∂C defines the induced orientation with multiplicity of Y. Therefore the image of ∂C in $H_{n-1}(bW, bbW)$ is the induced orientation of bW. (The sign of $\theta(C)$ can be seen to agree with the sign defined by ∂W by checking both definitions on the standard n-simplex).

2.3. Sub-Stratified Objects

2.3.1. Definition: Suppose $f: M_1 \to M_2$ is a smooth map between manifolds and suppose $W_1 \subset M_1$ and $W_2 \subset M_2$ are Whitney stratified objects. Then $(f| W_1)$ is called a stratified map provided $f(W_1) \subset W_2$ and f takes each stratum X of W_1 submersively to a stratum $f(X)$ of W_2. Such a stratified map f is said to satisfy condition D if the following holds:

Condition D: For every pair of strata $X_1 < Y_1$ of W_1 and any sequence $p_i \in Y_1$ converging to some point $p \in X_1$, if $T_{p_i}Y_1$ converge to a plane τ_1 and if $T_{f(Pi)}f(Y_1)$ converge to a plane τ_2 then

$$df(p)(\tau_1) \supset \tau_2$$

2.3.2. Definition: Suppose V and W are Thom-Mather stratified objects, $W \subset V$, and V is supplied with a family of lines. Then W is called a sub-stratified object of V (and we say W follows the lines of V) if the following holds:

1. Each stratum of W is contained in one stratum of V.
2. For each stratum X of V, $W \cap X$ satisfies the Whitney conditions.

3. If X is a stratum of V and if $\epsilon > 0$ is sufficiently small then

$$W \cap (T_X(\epsilon)-X) = C_X{}^o(W \cap S_X(\epsilon))$$

4. If $X < Y$ are strata of V and $\epsilon > 0$ is sufficiently small then

$$\pi_X|\ (W \cap Y \cap S_X(\epsilon)):\ W \cap Y \cap S_X(\epsilon) \to W \cap X$$

is a stratified map which satisfies condition D.

2.3.3. Remark: Condition D is used in section 6.4. to guarantee that certain intersections of sub-stratified objects will be sub-stratified objects. It can be weakened considerably and perhaps omitted completely although this would necessitate considerably more technical analysis when intersections of sub-stratified objects are considered.

2.3.4. Definition: The reduction of an oriented stratified object W^n is the oriented stratified object consisting of the closure of the union of all components of the n-dimensional strata which have been assigned a nonzero multiplicity.

A geometric chain W^n in a Thom-Mather stratified object with a family of lines, V , is an equivalence class of compact oriented sub-stratified objects, two being considered equivalent if they have the same reduction.

A <u>geometric cycle</u> W^n in V is a geometric chain whose boundary is 0. A geometric cycle $W^n \subset V$ is said to represent the homology class $i_*[W] \in H_n(V)$ where i is the inclusion map $W \to V$.

Two geometric cycles $W_o{}^n$ and $W_1{}^n$ in V are called <u>cobordant</u> if there is a geometric chain $W^{n+1} \subset V \times [0, 1]$ so that for some $\epsilon > 0$,

(a) $W \cap V \times [0, \epsilon) = W_o \times [0, \epsilon)$

(b) $W \cap V \times (1-\epsilon, 1] = W_1 \times (1-\epsilon, 1]$

(c) $\partial W = W_1 \times \{1\} - W_o \times \{0\}$

2.3.5. <u>Remark</u>: The first two conditions will guarantee that W follows a certain family of lines near the edges of $V \times [0, 1]$ even though we have not yet defined the family of lines on $V \times [0, 1]$. Notice also that the conditions are written modulo reduction, so it is possible for W to have codimension 1 strata, but the induced orientations must cancel on any codimension 1 strata of W which are contained in $V \times (0, 1)$.

2.4. <u>Homology Theorem</u>

2.4.1. <u>Theorem</u>: Suppose V^n is a Thom-Mather stratified with a family of lines. Then for any homology class

$\alpha \in H_k(V)$ there is a geometric cycle $W^k \subset V$ which represents α . If W_1 and W_2 are cobordant geometric cycles in V then they represent the same homology class.

Remark: Although it has not been proven, it is almost certainly true that if W_1 and W_2 are geometric cycles in V and if they represent the same homology class, then they are cobordant.

2.4.2. Corollary: Every homology class in a smooth manifold can be represented by an embedded Whitney stratified object.

Proof of Theorem 2.4.1.: It will be shown in section 3.6. that for any Thom-Mather stratified object with a family of lines, V , there is a regular piecewise linear cell complex K and a homeomorphism $f: K \to V$ which is locally a smooth cellulation of each stratum and therefore if σ is a face of a cell τ in K then $f(\sigma) < f(\tau)$ satisfies condition B provided $f(\sigma)$ and $f(\tau)$ lie in the same stratum of V . Furthermore, each cell $f(\sigma)$ follows the lines of V and any closed union of cells forms a sub-stratified object of V as defined in section 2.3.

Therefore any cellular chain in this cellulation is also a geometric chain and the map

$$\text{cellular chains} \longrightarrow \text{geometric chains}$$

is compatible with the boundary map. Therefore there is a commutative diagram

$$\begin{array}{ccc}
\text{cellular cycles} & \longrightarrow & \text{geometric cycles in } V \\
\downarrow & & \downarrow \\
\text{cellular homology of } V & \xrightarrow{\cong} & \text{singular homology of } V
\end{array}$$

which proves representability.

Suppose W_1 and W_2 are geometric cycles in V and suppose $W \subset V \times [0\,,\,1]$ is a cobordism between them. Then the vertical composition in the following commutative diagram is 0 by exactness, which shows $i_*([W_1] - [W_2]) = 0$.

$$\begin{array}{ccccc}
[W] \in H_{n+1}(W\,,\,bW) & & & & \\
\downarrow \qquad \downarrow \partial & & & & \\
[V_1] - [V_2] \in H_n(bW) & & \overset{i_*}{\searrow} & & \\
\downarrow & & & & \\
H_n(W) & \longrightarrow & H_n(V \times [0\,,\,1]) & \xrightarrow{\cong} & H_n(V)
\end{array}$$

Q.E.D.

2.5. Relative Homology in a Manifold with Boundary

In this section we specialize to a manifold with boundary considered as a stratified object with 2 strata. Control data consists of a collaring of the boundary.

2.5.1. Definition: A relative geometric cycle W^k in a manifold M with boundary bM (possibly empty) is a compact geometric chain in M for which the induced orientations cancel on all codimension 1 strata of W except those contained in bM.

Using a relative version of the argument in section 2.3., it is easily seen that W represents a unique homology class in $H_k(M, bM)$.

2.5.2. Definition: Two such relative cycles $W_1{}^k$ and $W_2{}^k$ are cobordant in (M, bM) if there is a geometric chain $W^{k+1} \subset M \times [0, 1]$ whose boundary is $W_2 \times \{1\} - W_1 \times \{0\} + Q$ where $Q \subset bM \times [0,1]$.

2.5.3. Theorem: Every homology class $\alpha \in H_k(M, bM)$ can be represented by a relative geometric cycle $W^k \subset (M, bM)$ and two such cycles represent the same homology class if and only if they are cobordant in (M, bM).

Proof: Representability follows immediately from the preceding theorem 2.4.1. If two relative cycles are cobordant then they represent the same relative homology class by a relative version of 2.4. It remains to show that if two relative cycles W_1 and W_2 in $(M, \partial M)$ represent the same homology class then they are cobordant. This is a difficult procedure which depends on the technical results of Chapter 3.

According to corollary 3.8.2., there is a cobordism $S_1 \subset M \times [0, \frac{1}{4}]$ between $W_1 \times \{0\}$ and a geometric cycle $W_1' \subset M \times \{\frac{1}{4}\}$ and there is a Whitney cellulation of $M \times \{\frac{1}{4}\}$ so that each stratum of W_1' is a cell of this cellulation. Similarly there is a cobordism $S_2 \subset M \times [\frac{3}{4}, 1]$ between $W_2 \times \{1\}$ and a geometric cycle $W_2' \subset M \times \{\frac{3}{4}\}$ which is stratified by cells of a Whitney cellulation of $M \times \{\frac{3}{4}\}$.

By Corollary 3.6.2., these two cellulations can be refined and extended to a Whitney cellulation of $M \times [\frac{1}{4}, \frac{3}{4}]$ for which any closed subcomplex is a sub-stratified object of $M \times [\frac{1}{4}, \frac{3}{4}]$.

Let $W_1'' \subset M \times \{\frac{1}{4}\}$ and $W_2'' \subset M \times \{\frac{3}{4}\}$ be the induced refinements of W_1' and W_2'. By standard cellular homology theory arguments, W_1'' and W_2'' are both cellular cycles representing the same homology classes as W_1' and W_2' respectively, which in turn represent the same homology class in $H_k(M) \cong H_k(M \times [\frac{1}{4}, \frac{3}{4}])$, by assumption.

Therefore, there is a cellular homology $S_3 \subset M \times [\frac{1}{4}, \frac{3}{4}]$ between W_1'' and W_2'' and so $S_1 \cup S_3 \cup S_2 \subset M \times [0, 1]$ is the desired cobordism between W_1 and W_2. Q.E.D.

CHAPTER 3.

RADIAL STRATIFIED OBJECTS

In this chapter the technical lemmas needed for theorem 2.6. are proven. Intuitively, a radial stratified object is a Whitney object in some manifold, with a fixed system of control data for which the rays in the vectorbundle structure of each tubular neighborhood form a family of lines on the stratified object.

Radial objects are important because they can be cellulated in such a way that <u>every</u> cellular incidence satisfies the Whitney conditions.

The first goal is to prove that every Whitney object is cobordant to a radial object. The two-stratum version of this theorem is proven in section 3.1.2. and then the whole theorem is proven (by induction as usual) in section 3.3.1.

The cellulation theorem is then proven and finally in section 3.7.5. it is shown that the cellulation of a radial object is Whitney. This fact depends on the strong control over the cells near the boundary of a stratum.

Putting these results together in Corollary 3.8.2., we conclude that every Whitney stratified object (in a manifold) is cobordant to one which admits a Whitney cellulation.

3.1.1. Definition

Suppose $W \subset M$ is a Whitney stratified object with a fixed system of control data. Recall from Chapter 0. that if X is a stratum of W then T_X is the image of a fixed smooth embedding $\phi: E_\epsilon \to M$ where E_ϵ is a neighborhood of the zero section of a Riemannian normal bundle for X, $E \to X$.

Define a smooth radial retraction map

$$r_X(\epsilon): (T_X - X) \to S_X(\epsilon) \quad \text{by}$$

$$r_X(\epsilon)(p) = \phi\left(\frac{\epsilon\phi^{-1}(p)}{||\phi^{-1}(p)||^2}\right)$$

A stratum $Y > X$ is said to be radial with respect to X if there is an $\epsilon > 0$ so that $Y \cap T_X(\epsilon)$ $= r_X(\epsilon)^{-1}(Y \cap S_X(\epsilon))$.

A tubular neighborhood T_Y of such a stratum Y is said to be radial with respect to X if the radial retractions r_X and r_Y satisfy the conditions for a family of lines, i.e., if there exist $\epsilon > 0$ and $\epsilon' > 0$ so that on $T_X(\epsilon) \cap T_Y(\epsilon')$ we have

$$r_Y(\epsilon')r_X(\epsilon) = r_X(\epsilon)r_Y(\epsilon')$$

$$\rho_Y r_X(\epsilon) = \rho_Y$$

$$\rho_X r_Y(\epsilon') = \rho_X$$

$$\pi_X r_Y(\epsilon') = \pi_X$$

The stratified object W with a fixed system of control data is radial if every stratum and every tubular neighborhood is radial with respect to every other stratum.

3.1.2. Proposition

Suppose $\pi: E \to X$ is a smooth Riemannian vectorbundle with distance function $\eta(v) = \sqrt{\langle v,v\rangle}$ for $v \in E$. Suppose $W \subset E$ is a Whitney stratified object which is closed in some neighborhood of the 0-section and that X = the 0-section is a stratum of W. Let $\rho(v) = \eta(v)^2$.

By adjusting the metric (i.e., multiplying by a factor which may depend on $x \in X$), we may assume that

(1) W is closed in $\eta^{-1}[0,5)$

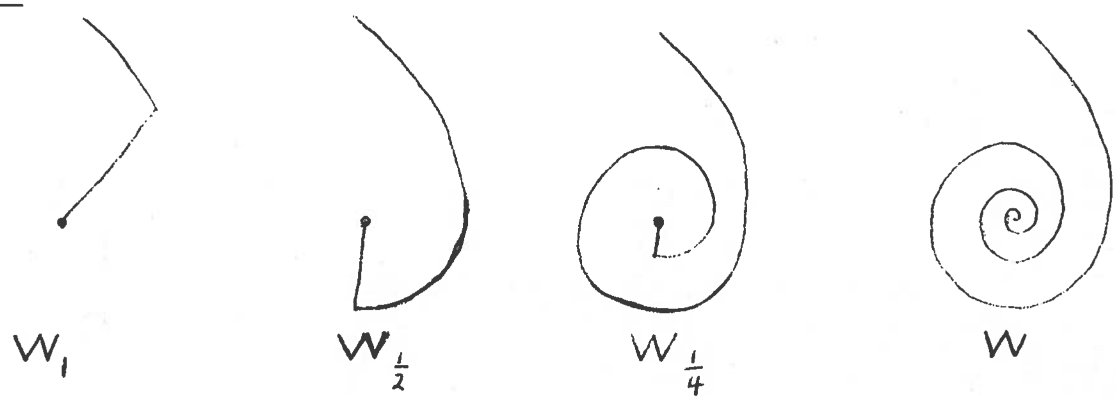
(2) If $0 < \delta \leq 2$ then W is transverse to $\eta^{-1}(\delta)$

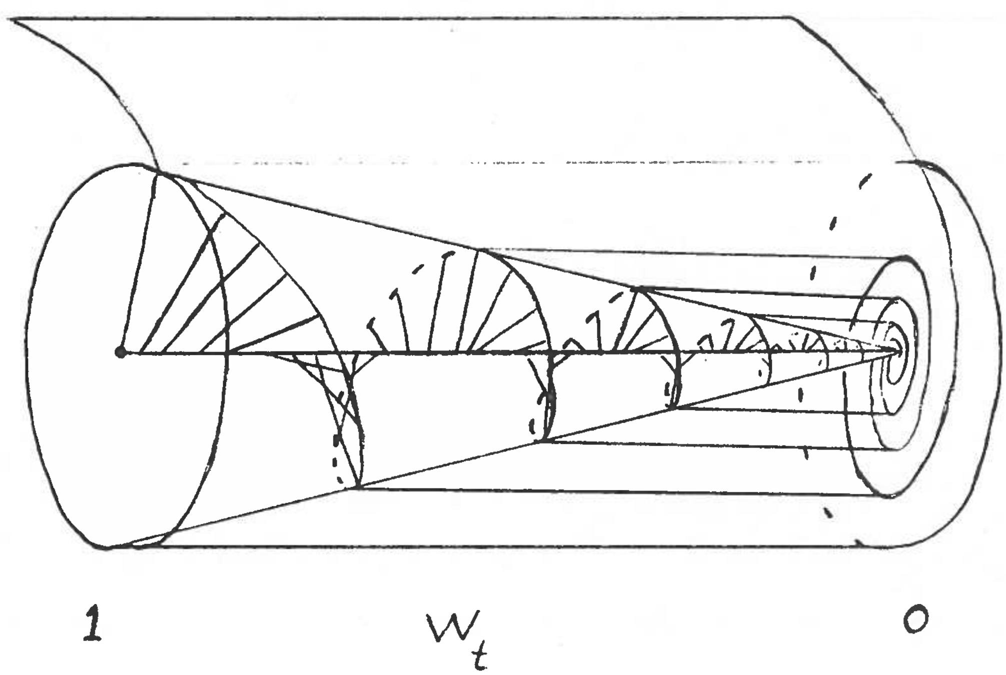
(3) If Y is a stratum of W then either $Y \cap \eta^{-1}(0,3] = \emptyset$ or else Y is comparable to X, so $X < Y$ and $(\pi,\eta)|(Y\cap\eta^{-1}(0,2)): Y \cap \eta^{-1}(0,2) \to X \times \mathbb{R}$ is a proper submersion.

Then there is a Whitney cobordism in $E \times [-\frac{1}{2}, 1\frac{1}{2}]$ between $W \times \{-\frac{1}{2}\}$ and a stratified object $W' \times \{1\frac{1}{2}\}$ where every stratum of W' is radial with respect to X, and W' coincides with W outside $\eta^{-1}[0,2)$.

Proof: The idea is to construct W' by replacing W with the mapping cylinder of $W \cap \eta^{-1}(1) \to X$ near X. The cobordism between W' and W is achieved using the 1-parameter family of Whitney objects W_t where $W_t \cap \eta^{-1}[0,t)$ is the mapping cylinder of $W \cap \eta^{-1}(t) \to X$ and $W_t \cap \eta^{-1}[t,\infty) = W \cap \eta^{-1}[t,\infty)$, for $t \in [0,1]$. Then the cobordism is made constant at the ends of the interval $[-\frac{1}{2},1\frac{1}{2}]$ by adding constant cobordisms $E \times [-\frac{1}{2},0]$ and in $E \times [1,1\frac{1}{2}]$. Two technical problems arise: the "kink" which occurs at $W_t \cap \eta^{-1}(t)$ must be smoothed out, and the Whitney conditions must be verified.

Diagram:





Details of Proof

Choose a smooth function $f: \mathbb{R} \to \mathbb{R}$ which vanishes outside $(0,1)$ and satisfies $0 \le f(x) \le f(\frac{1}{2}) = 1$. Let

$$A \equiv \int_0^1 f(x)dx$$

and choose $\epsilon < \min(\frac{A}{2}, \frac{1}{4})$ and define

$$h(x) \equiv 1 - \frac{1}{\epsilon A}\int_0^{x-1} f(\frac{y}{\epsilon})dy$$

Notice that if $x \le 1$ then $h(x) = 1$ and if $x \ge 1 + \epsilon$ then $h(x) = 0$. Furthermore, $dh/dx \le \frac{1}{\epsilon A}$ for all $x \in \mathbb{R}$.

Define $g: (0,2] \to [0,\infty)$ by

$$g(s) = \frac{sh(s)-h(s)-s+2}{sh(s)-h(s)+1}$$

and check that g is smooth and invertible.

Define $\phi: (0,\infty) \to [0,1]$ by

$$\phi(\Delta) = h(g^{-1}(\Delta))$$

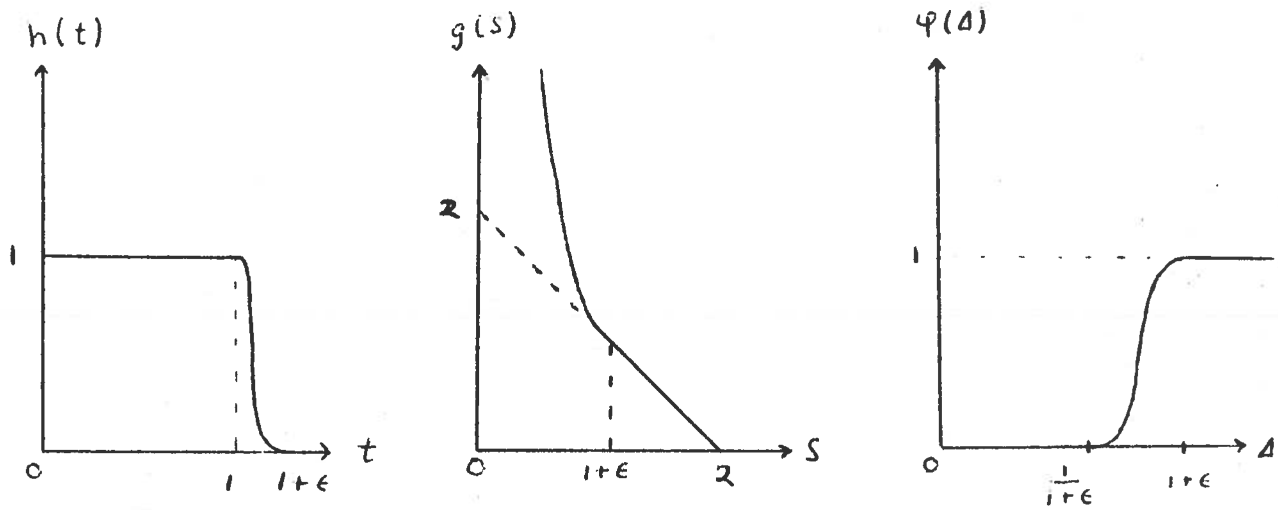
and verify that

$$(1)\quad h(m(\Delta)) = \phi(\Delta) \quad \text{for all} \quad \Delta \in (0,\infty)$$

$$(2)\quad h(s) = \phi(g(s)) \quad \text{for all} \quad s \in (0,\infty)$$

where $$m(\Delta) = 1 + \frac{1-\Delta}{1+\phi(\Delta)(\Delta-1)}$$

Diagram:



Using the vectorbundle structure of E, define:

$F: E \times [0,1] \to E$ by

$$F(p,t) = \begin{cases} (1+\phi(\frac{t}{\eta(p)})(\frac{t}{\eta(p)}-1))p & \text{if } \eta(p) \neq 0 \\ p & \text{if } \eta(p) = 0 \end{cases}$$

<u>Note</u>: F is not continuous at (p,t) if $\eta(p) = 0$.

If $\delta > 0$ and $t \in (0,1)$ then F maps each sphere bundle $\eta^{-1}(\delta) \times \{t\}$ isomorphically to a sphere bundle $\eta^{-1}(\delta')$, and if $\eta(p) > 1 + \epsilon$ then $F(p,t) = p$. Thus, $F|(E-X) \times (0,1)$ is transverse to W so

$$F^{-1}(W) \cap (E-X)$$

is a Whitney object in $E-X$. $F^{-1}(W)$ will be the desired portion of the cobordism lying between times $t = 0$ and $t = 1$, consisting of the one-parameter family of stratified objects

$$W_t = F_t^{-1}(W)$$

where $F_t(p) = F(p,t)$. In fact, $F_o(p) = p$ so $W_o = W$ and if $\eta(p) > (1+\epsilon)t$ then $F(p,t) = p$, so W_t coincides with W in the region $\eta^{-1}(t+t\epsilon,\infty)$. Furthermore W_1 is radial near X because $F(p,1) = \frac{p}{\eta(p)}$ whenever $\eta(p) < \frac{1}{(1+\epsilon)}$ so the pre-image of W by F_1 will contain the rays from $W \cap \eta^{-1}(1)$ to X.

The object $F^{-1}(W) \subset E \times [0,1]$ is a candidate for a cobordism between $F_o^{-1}(W) = W$ and $F_1^{-1}(W) = W'$, however it is not trivial near the ends of the interval $[0,1]$. We therefore define an object $V \subset E \times [-\frac{1}{2},1\frac{1}{2}]$ by setting

$$V = W'\times(1,1\tfrac{1}{2}] \cup W\times[-\tfrac{1}{2},0) \cup F^{-1}(W)$$

It only remains to check the Whitney conditions for strata of V which are comparable to X or which lie in $E \times \{0\}$ or $E \times \{1\}$. This requires some analysis and we use another description of the object $F^{-1}(W)$ as the image of $W \times [0,2]$ under the mapping

$$G: E \times [0,2] \to E \times \mathbb{R}$$

defined by

$$G(q,s) \equiv (q(1+sh(s)-h(s)) , \eta(q)(sh(s)-h(s)-s+2)) .$$

Then $G|(E-X) \times [0,2]$ is a smooth nonsingular embedding and it is tedious but straightforward to check that

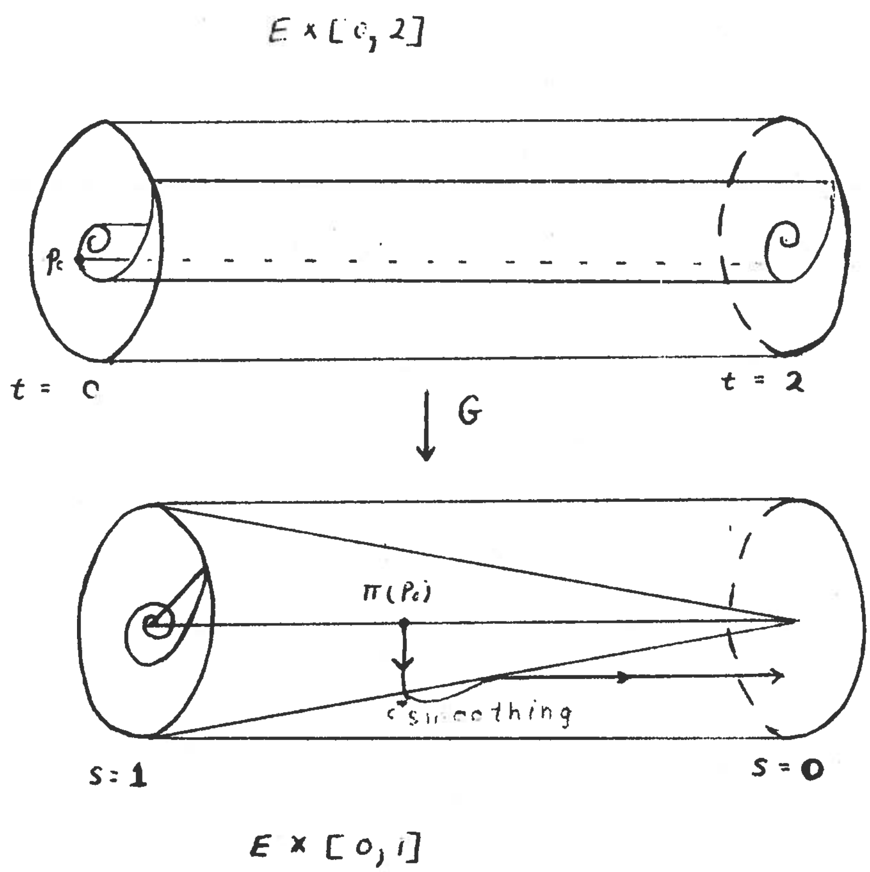
$$F(G(q,s)) = q \quad \text{for any } q \text{ and } s$$

and

$$G(F(p,t)\ ,\ m(\frac{t}{\rho(p)})) = (p,t) \quad \text{for any } p \text{ and } t\ ,$$

so that $F^{-1}(W) = G(W\times[0,2]) \cap E \times [0,1]$.

<u>Diagram</u>: Image of $G(p_0,t)$ for $0 \le t \le 2$.



Suppose $Y > X$ is a stratum of W. Then the cobordism V contains the following strata:

1. $X \times (-\frac{1}{2},0)$

2. $Y \times (-\frac{1}{2},0)$

3. $X = X \times \{0\}$

4. $Y = Y \times \{0\}$

5. $X' = X \times (0,1) = F^{-1}(X) \cap E \times (0,1)$

6. $Y' = F^{-1}(Y) \cap E \times (0,1)$

7. $X'' = X \times \{1\} = F_1^{-1}(X)$

8. $Y'' = F_1^{-1}(Y)$

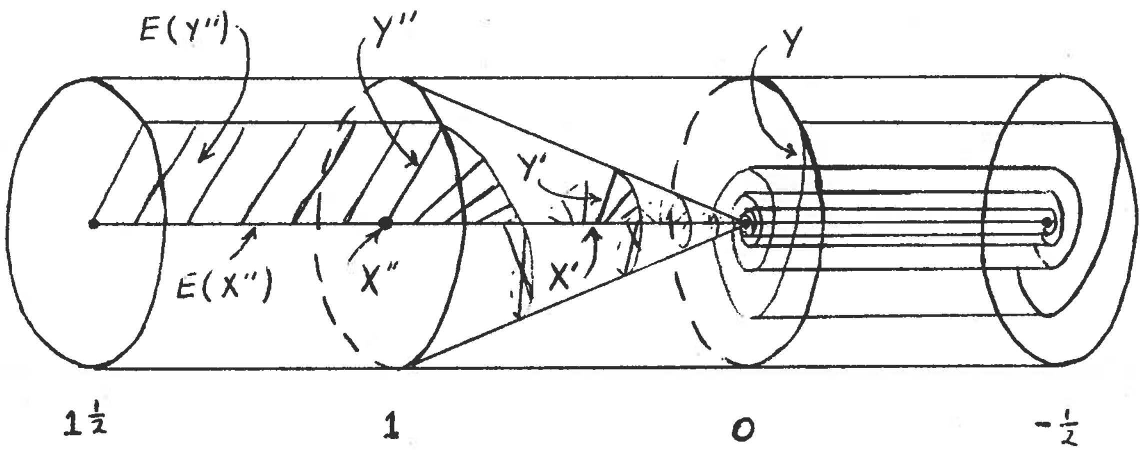
9. $E(X'')$ = trivial extension of X'' inside $E \times (1,1\frac{1}{2})$

10. $E(Y'')$ = trivial extension of Y'' inside $E \times (1,1\frac{1}{2})$.

These strata are arranged with the following incidence relations:

$$\begin{array}{ccccccccc} E(Y'') & > & Y'' & < & Y' & > & Y & < & Y \times (-\frac{1}{2},0) \\ \vee & & \vee & & \vee & & \vee & & \vee \\ E(X'') & > & X'' & < & X' & > & X & < & X \times (-\frac{1}{2},0) \end{array}$$

Diagram:



It is straightforward to check condition B on all of these 17 incidences except:

(a) $X' < Y'$

(b) $X < Y'$

which will now be done separately.

Proof of Condition B for the Incidence $X' < Y'$

Given a sequence of points $G(q_i,s_i) \in Y'$ and a sequence $(x_i,t_i) \in X'$, both converging to some $(x_o,t_o) \in X'$, and assuming the tangent planes $\tau_i = T_{G(q_i,s_i)}Y'$ converge to a plane τ and the secant lines $\ell_i = \overline{(x_i,t_i)\,,\,G(q_i,s_i)}$ converge to a line ℓ, we must show $\ell \subset \tau$. Notice that (q_i,s_i) necessarily converges to a point $(q_o,0)$ where $\pi(q_o) = x_o$.

By choosing a local coordinate system for X near x_o and a Riemannian trivialization of E, we may assume that $E = \mathbb{R}^n$ and $X = \{(x_1,\ldots,x_n) \in E \mid x_{k+1} = x_{k+2} = \ldots = x_n = 0\}$; that $\pi\colon E \to X$ is the usual Euclidean projection and that the Riemannian metric in E coincides with the standard inner product in $\mathbb{R}^{n-k}$ so that $\eta(q) = ||q - \pi(q)||$. For sufficiently large i we have $h(s_i) = 1$ so in these coordinates, G may be expressed by

$$G(q_i,s_i) = (q_i+(q_i-\pi(q_i))(s_i-1)\,,\,\eta(q_i)) \in E \times \mathbb{R}\,.$$

By differentiating,

$$\tau_i = T_{G(q_i,s_i)}Y' = \{(v+(v-d\pi(q_i)(v))(s-1)+(q_i-\pi(q_i))\dot{s} \ , \ d\eta(q_i)(v)) \mid v \in T_{q_i}Y \text{ and } \dot{s} \in \mathbb{R}\}$$

$$\ell_i = \{(\alpha[s_i(q_i-\pi(q_i))+\pi(q_i)-x_i] \ , \ \alpha[\eta(q_i)-t_i]) \mid \alpha \in \mathbb{R}\}.$$

Choose $\alpha_i \in \mathbb{R}$ so that

$$w_i = \alpha_i(s_i(q_i-\pi(q_i))+\pi(q_i)-x_i \ , \ \eta(q_i)-t_i)$$

converges to a vector of finite length (say, 1) and denote the limit by w . We must show $w \in \tau$.

Since w_i has two perpendicular components, each component must converge and furthermore the first component can be written as a sum of the two vectors

$$\gamma_i = \alpha_i s_i(q_i-\pi(q_i))$$

and

$$\theta_i = \alpha_i(\pi(q_i)-x_i) \ .$$

Since γ_i and θ_i become perpendicular as $i \to \infty$ they each converge to some vector γ and θ respectively. Finally, since $(q_i - \pi(q_i)) \to q_0 - \pi(q_0) \neq 0$ the sequence $\alpha_i s_i$ converges to some real number s_0 .

Now, $(\pi,\eta)|Y\colon Y \to X \times \mathbb{R}$ is a submersion so there is a vector $v_0 \in T_{q_0}Y$ so that $d\pi(q_0)(v_0) = \theta \in T_{x_0}X$ and $d\eta(q_0)(v_0) = \lim_{i\to\infty} \alpha_i(\eta(q_i)-t_i)$. Choose $v_i \in T_{q_i}Y$ converging to v_0 . Then for sufficiently large i ,

$$z_i = (v_i+(v_i-d\pi(q_i)(v_i))(s_i-1)+(q_i-\pi(q_i))(s_0),d\eta(q_i)(v_i)) \in \tau_i$$

and $||z_i - w_i||$

$$= ||(s_i(v_i-d\pi(q_i)(v_i))+(d\pi(q_i)(v_i)-\theta_i)+(s_0(q_i-\pi(q_i))-\gamma_i) ,\; d\eta(q_i)(v_i)-\alpha_i(\eta(q_i)-v_i)||$$

converges to

$$||(d\pi(q_0)(v_0)-\theta+s_0(q_0-\pi(q_0))-\gamma,\;\; d\eta(q_0)(v_0)-\lim \alpha_i(\eta(q_i)-t_i)|| = 0$$

Thus the vector $w \in \ell$ is a limit of vectors $z_i \in \tau_i$ and hence $w \in \tau$. Q.E.D.

Condition B for the Incidence $X < Y'$

Suppose a sequence of points $G(q_i,s_i) \in Y'$ and a sequence $(x_i,0) \in X \times \{0\}$ both converge to a point $(x_o,0) \in X \times \{0\}$ and suppose the tangent planes $\tau_i = T_{G(q_i,s_i)}Y'$ converge to a plane τ and the secant lines $\ell_i = \overline{G(q_i,s_i),(x_i,0)}$ converge to a line ℓ. By extracting a subsequence we may suppose the points (q_i,s_i) converge to some limit point (x_o,s_o) where $S_o \in [0,2]$. We may also assume the tangent planes $T_{q_i}Y$ converge to some plane P since the Grassmannian is compact.

Choosing local coordinates for X near x_o and trivializing E as before (in a way which identifies the Riemannian metric on E with the standard inner product in $\mathbb{R}^{n-k}$) we can write

$$G(q,s) = (q+(q-\pi(q))(sh(s)-h(s)) ,$$

$$\eta(q)(sh(s)-h(s)-s+2)) \in \mathbb{R}^n \times [0,1] .$$

If $v \in T_qY$ then

$$dG(q,s)(v,0) = (v+(v-d\pi(q)(v))(sh(s)-h(s)) ,$$
$$d\eta(q)(v)(sh(s)-h(s)-s+2)) .$$

Furthermore,

$$\ell_i = \{\alpha\cdot(q_i-x_i+(q_i-\pi(q_i))(s_ih(s_i)-s_i) ,$$
$$\eta(q_i)(s_ih(s_i)-h(s_i)-s_i+2))|\alpha \in \mathbb{R} .$$

Choose $\alpha_i \in \mathbb{R}$ so that the corresponding vectors w_i above are all unit vectors and hence converge to a unit vector $w \in \mathbb{R}^n \times \mathbb{R}$. Then each of the following sequences of vectors in $\mathbb{R}^n \times \mathbb{R}$ converge because their sum converges and they become mutually perpendicular in the limit:

(1) $(\alpha_i(q_i-x_i),0)$

(2) $(\alpha_i(q_i-\pi(q_i))(s_ih(s_i)-s_i),0)$

(3) $(0,\alpha_i\eta(q_i)(s_ih(s_i)-h(s_i)-s_i+2)$

Notice that $\pi(q_i)$ is the point in X closest to q_i, so

$$|q_i - \pi(q_i)| = \eta(q_i) \le |q_i - x_i| .$$

Therefore $\alpha_i(q_i-x_i)$ does not converge to 0 or else the other two sequences would converge to 0 also.

The vectors $\alpha_i(q_i-x_i)$ lie in the sccant lines $\widehat{q_i x_i}$ which therefore converge and the limit is contained in the plane $P = \lim T_{q_i}Y$, thus vectors $v_i \in T_{q_i}Y$ can be found so that

$$|v_i - \alpha_i(q_i-x_i)| \to 0$$

<u>Claim</u>: The sequence $dG(q_i,s_i)(v_i,0) \in \tau_i$ converges to w. (This will imply $w \in \tau = \lim \tau_i$ which will complete the proof of condition B).

Two local calculations are used. If $q \in \mathbb{R}^n$ and $X = \{x \in \mathbb{R}^n | x_{k+1} = \dots = x_n = 0\}$ then

(1) $d\pi(q)(q-x) = \pi(q) - x$

since $\pi(q_1,\dots,q_n) = (q_1,\dots,q_k,0,\dots,0)$

(2) $d\eta(q)(q-x) = \eta(q)$

since $\eta(q_1,\ldots,q_n) = (q_{k+1}{}^2+\ldots+q_n{}^2)^{\frac{1}{2}}$

The claim is proven by showing the two components of $dG(q_i,s_i)(v_i,0) - w_i$ converge to 0. The first component has length

$$|v_i - \alpha_i(q_i-x_i) + [(v_i-d\pi(q_i)(v_o))-\alpha_i(q_i-\pi(q_o))](s_ih(s_i)-s_i)|$$

$$\le |v_i - \alpha_i(q_i-x_i)| + |s_ih(s_i) - s_i||(v_i-\alpha_i(q_i-x_i) + \alpha_i(\pi(q_i)-x_i) - d\pi(q_i)(v_i)|$$

$$\le |v_i - \alpha_i(q_i-x_i)| + |s_ih(s_i) - s_i|(|v_i-\alpha_i(q_i-x_i)| + |d\pi(q_i)(\alpha_i(q_i-x_i)-v_i)|)$$

which goes to 0.

The second component has length

$$\leq |d\eta(q_i)(v_i) - \alpha_i \eta(q_i)||s_i h(s_i) - h(s_i) - s_i + 2|$$

$$\leq (|d\eta(q_i)(v_i - \alpha_i(q_i - x_i)| + |d\eta(q_i)(\alpha_i(q_i - x_i) - \alpha_i \eta(q_i)|)|s_i h(s_i) - h(s_i) - s_i + 2|$$

$$= (|d\eta(q_i)(v_i - \alpha_i(q_i - x_i))| + |\alpha_i \eta(q_i) - \alpha_i \eta(q_i)|)|s_i h(s_i) - h(s_i) - s_i + 2| \to 0$$

Q.E.D.

3.1.3. <u>Remark</u>: The preceding analysis shows that if $\pi: E \to X$ is a Riemannian vectorbundle and $Q \subset S_X(\epsilon)$ is a Whitney stratified object so that $\pi|T: T \to X$ is a submersion, for each stratum T of Q, then $C_X(Q)$ is also a Whitney stratified object with a stratum $C_X(T)$ for each stratum T of Q, and one more stratum X = the zero section of E.

(The preceding verifies the Whitney conditions for $X < C_X(Y)$. If $Y < Y'$ are strata of Q then the Whitney conditions for $C_X(Y) < C_X(Y')$ are trivial to check.)

This remark will be generalized in section 3.7.

3.2.1. Lemma.

Let W be a closed Whitney object in a manifold M. Suppose a system of control data on the k-skeleton of W has been specified for which the k-skeleton is a radial stratified object. Suppose Y is a $k+1$ —dimensional stratum which is radial with respect to all lower dimensional strata. Then there is a tubular neighborhood T_Y of Y which is radial with respect to all lower dimensional strata.

Proof: The lemma is proven by induction on k with the case $k = -1$ trivial. Suppose it is true whenever $k \leq n - 1$ and we shall prove it for the case $k = n$.

For any stratum X in the n-skeleton of W, and for $\epsilon > 0$ sufficiently small, $W \cap S_X(\epsilon)$ is a Whitney object in the manifold $S_X(\epsilon)$. The $n - 1$ skeleton of $W \cap S_X(\epsilon)$ is radial and the stratum $Y \cap S_X(\epsilon)$ has dimension equal to n and is radial with respect to X. Therefore, the induction hypothesis implies the existence of a tubular neighborhood T_Q of $Q = Y \cap S_X(\epsilon)$ in $S_X(\epsilon)$ satisfying all commutation relations with each stratum in the $n - 1$ skeleton of $W \cap S_X(\epsilon)$.

The cone operation $C_X{}^{\circ}$ can be used to extend T_Q to a tubular neighborhood of $Y \cap T_X(\epsilon)$ in an obvious way by setting

$$\pi_Y(p) = \left(\frac{\rho_X(p)}{\epsilon}\right)\cdot\pi_Q(r_X(p)) \quad \text{if} \quad p \in C_X{}^{\circ}(T_Q)$$

$$\rho_Y(p) \equiv \rho_Q(r_X(p))$$

$$r_Y(p) \equiv \left(\frac{\rho_X(p)}{\epsilon}\right)\cdot r_Q(r_X(p))$$

where scalar multiplication is computed with respect to the vectorbundle structure of T_X .

The commutation relations among T_X and T_Y are easily checked.

The above procedure can be applied with X any stratum in the n-skeleton of W , and in any order and the functions π_Y , ρ_Y and r_Y will be well defined in regions $T_X \cap T_{X'} \cap T_Y$ whenever X and X' are comparable strata in the n-skeleton of W . The reason is that although scalar multiplication in T_X is not the same as scalar multiplication in $T_{X'}$, the two operations do commute.

This procedure gives a radial tubular neighborhood on $Y \cap T_n(\epsilon)$ where

$T_n(\epsilon) = \cup\{T_X(\epsilon) | X$ is a stratum of the n skeleton of $W\}$.

By shrinking ϵ slightly, we can extend this to a tubular neighborhood for all of Y in M , completing the inductive step.

3.3.1. <u>Theorem: Radializing a Whitney Object</u>

Let W be a Whitney stratified object in a manifold M . Then there is a Whitney cobordism between W and a stratified object W' which is radial.

<u>Proof</u>: It will be shown by induction that for each integer k there is a cobordism from W to a Whitney object W_k for which every stratum of W_k is radial with respect to every stratum of the k-skeleton of W_k . (This implies the existence of a fixed system of control data on the k-skeleton of W_k) .

For $k = 0$, the proposition 3.1.2. gives a cobordism from W to W_0 (affecting only a neighborhood of the 0-strata) having the desired properties.

Assuming W_k has been constructed, for each stratum Y of dimension $k + 1$ there is a tubular neighborhood T_Y which is radial with respect to all lower dimensional strata, by lemma 3.2.

Apply proposition 3.1.2. to $W_k \cap T_Y \to Y$ and notice that it is not necessary to adjust the metric on Y except possibly by multiplying it by a single constant, since Y , ρ_Y and W_k are radial with respect to each stratum $X < Y$.

We therefore obtain a cobordism $V \subset M \times [0,1]$ between W_k and an object W_{k+1} which coincides with W_k outside T_Y and which is radial with respect to Y . It is necessary to check that W_{k+1} is still radial with respect to any stratum $X < Y$, that the Whitney conditions hold at such a stratum X , and that V satisfies the Whitney conditions at the strata $X \times (0,1)$ if $X < Y$.

W_{k+1} is radial at $X < Y$ because the whole procedure in section 3.1.2. is radial with respect to X . Explicitly, (referring to the proof of the theorem 3.1.2.), W_{k+1} is defined as

$$W_{k+1} = F_{k-1}^{-1}(W_k)$$

where

$F_{k+1}\colon M \to M$

$$F_{k+1}(p) = \begin{cases} r_Y(f(\rho_Y(p)))(p) & \text{if } p \in T_Y \\ p & \text{if } p \notin T_Y \end{cases}$$

for some function f involving $\phi(\rho_Y(p))^{-\frac{1}{2}}$ etc. However, it is obvious that if $X < Y$ then $r_X(\epsilon) \circ F_1 = F_1 \circ r_X(\epsilon)$ and therefore

$$\begin{aligned} r_X(\epsilon)^{-1}(W_{k+1} \cap S_X(\epsilon)) &= r_X(\epsilon)^{-1} F_{k+1}{}^{-1}(W_k) \cap S_X(\epsilon) \\ &= F_{k+1}{}^{-1} r_X(\epsilon)^{-1}(W_k) \cap S_X(\epsilon) \\ &= F_{k+1}{}^{-1} r_X(\epsilon)^{-1}(W_k \cap S_X(\epsilon)) \\ &= F_{k+1}{}^{-1}(W_k \cap T_X(\epsilon)) \\ &= F_{k+1}{}^{-1}(W_k) \cap T_X(\epsilon) \\ &= W_{k+1} \cap T_X(\epsilon) \end{aligned}$$

showing W_{k+1} is radial with respect to X. (The same comment applies to show that the cobordism V is radial with respect to $X \times (0,1)$).

The Whitney conditions are now easily verified: $W_{k+1} \cap T_X(\epsilon) = C_X(W_{k+1} \cap S_X(\epsilon))$ so the remark following proposition 3.1.2. applies. (Similarly $V \cap T_{X\times(0,1)} = C_{X\times(0,1)}(V \cap S_{X\times(0,1)}))$.

3.3.2. <u>Remark</u>: Notice in fact that $W' = g^{-1}(W)$ where $g: M \to M$ is the (discontinuous) function $g = F_n \circ F_{n-1} \circ \dots \circ F_1 \circ F_0$ if $n = \dim (W)$. Similarly there is a function $G: M\times[0,1] \to M$ so that $G^{-1}(W)$ is the cobordism between

$$W = G_0^{-1}(W) \quad \text{and} \quad W' = G_1^{-1}(W)$$

3.3.3. <u>Corollary</u>: Let M be a manifold with boundary ∂M and a collared neighborhood U of ∂M with collaring $f: U \overset{\cong}{\to} \partial M \times [0,1)$ where $f_{(p)} = (p,0)$ for $p \in \partial M$. Suppose $W \subset M$ is a Whitney stratified object and that W follows the lines of the collaring, i.e., $f(W) = (W \cap \partial M)\times[0,1)$. Then there is a Whitney cobordism between W and a stratified object W' where

(a) the cobordism follows the lines of $M \times [0,1]$

(b) $W' \cap M$ is radial and $W' \cap \partial M$ is radial

Proof: By the above remark there are functions $g: \partial M \to \partial M$ and $G: \partial M \times [0,1] \to \partial M$ so that $g^{-1}(W \cap \partial M)$ is radial and $G^{-1}(W \cap \partial M)$ is a cobordism from

$$G_0^{-1}(W \cap \partial M) = W \cap \partial M$$

to

$$G_1^{-1}(W \cap \partial M) = g^{-1}(W \cap \partial M)$$

Choose a smooth function $\phi: [0,1] \times [0,1] \to [0,1]$ so that

$$0 \le \phi(s,t) \le t$$

$$\phi(s,t) = 0 \quad \text{if} \quad s \ge \tfrac{2}{3}$$

$$\phi(s,t) = t \quad \text{if} \quad s \le \tfrac{1}{2}$$

Let $f_1(p) \in \partial M$ and $f_2(p) \in [0,1)$ be the projections of $f(p) \in \partial M \times [0,1)$ for $p \in U$. Define $H: M \times [0,1] \to M$ by

$$H_{(p,t)} = \begin{cases} p & \text{if} \quad p \notin U \\ G(f_1(p), \phi(f_2(p),t)) & \text{if} \quad p \in U \end{cases}$$

and $H_t(p) = H(p,t)$. Then $H^{-1}(W) \subset M \times [0,1]$ is a Whitney cobordism between $H_0^{-1}(W) = W$ and $H_1^{-1}(W) = W_1$. Notice that $W_t = H_t^{-1}(W)$ is a product in $f(\partial M\times[0,\frac{1}{2}])$ because if $f_2(p) \le \frac{1}{2}$ then $H(p,t) = G(f_1(p),t)$ which depends only on $f_1(p)$. Also, $W_t \cap (M-f(\partial M\times[0,\frac{2}{3}])) = W \cap (M-f(\partial M\times[0,\frac{2}{3}]))$. Finally, $W_1 \cap \partial M$ is radial.

It remains to radialize W_1 in the region away from ∂M . This is accomplished as in theorem 3.3.1. except that $W_1 \cap f(\partial M\times[0,\frac{1}{3}])$ may be left fixed throughout the cobordism.

Q.E.D.

3.4. Cellulations of Manifolds and Stratified Objects

For this section we refer to Hudson [4].

3.4.1. Definition: A convex linear cell $\sigma \subset \mathbb{R}^n$ is a subset defined as the solution to finitely many linear equalities $f_i(x) = 0$ and strict inequalities $g_i(x) < 0$. The closure of such a cell is found by replacing all strict inequalities by simple inequalities $g_i(x) \le 0$. A face of such a cell is defined by replacing some strict inequalities by equalities. Each face is therefore a cell, and the boundary of a cell is a union of faces.

A <u>closed</u> (convex linear) cell complex $K \subset \mathbb{R}^n$ is a locally finite collection of closed cells so that

(a) If σ is a cell of K then every face of σ is in K

(b) If σ_1 and σ_2 are cells of K then either $\overline{\sigma_1} \cap \overline{\sigma_2} = \emptyset$ or else σ_1 and σ_2 intersect in a common face.

We shall confuse K and $|K|$ = the union of the cells in K.

If $K \subset \mathbb{R}^n$ is a closed cell complex and L is an arbitrary collection of cells in K with $\overline{|L|} = |K|$ then L is called an (arbitrary) cell complex.

A smooth cellulation of a smooth manifold X is a closed cell complex $K \subset \mathbb{R}^n$ together with a homeomorphism $f: K \to X$ so that if σ is a cell of K then $f|\sigma$ extends to a smooth embedding $f: U \to X$ of an open neighborhood U of σ in the plane of σ. If X is not compact then K will in general have infinitely many cells.

A <u>local</u> smooth cellulation of a manifold X is an (arbitrary) cell complex $K \subset \mathbb{R}^n$ together with a homeomorphism $f: K \to X$ so that if σ is a cell of K, lying as an open subset of an affine plane $P \subset \mathbb{R}^n$, and if $x \in \bar{\sigma}$ and $f(x) \in X$ then

(a) the face of σ containing x is in K

(b) $f|\sigma$ extends to a smooth embedding $U \to X$ for some open neighborhood U of x in P.

By a partition of unity argument, it follows that if the faces of σ are in K, i.e., if $f(\bar{\sigma}) \subset X$ then f extends to a smooth embedding of a neighborhood of $\bar{\sigma}$ in P as in the case of a smooth cellulation of X..

A <u>refinement</u> of a local smooth cellulation $f: K \to X$ is a convex linear refinement of $\bar{K}$ together with the induced homeomorphism.

A local smooth cellulation of a smooth fibration $\pi: Y \to X$ is a local smooth cellulation $f: K \to Y$ of Y, and $g: L \to X$ of X for which $g^{-1} \circ \pi \circ f$ is a (linear) cellular vertex map, and in this case we say " π is cellular with respect to the cellulations K and L ".

$$\begin{array}{ccc} K & \xrightarrow{f} & Y \\ {\scriptstyle g^{-1}\pi f}\downarrow & & \downarrow{\scriptstyle \pi} \\ L & \xrightarrow{g} & X \end{array}$$

3.4.2. <u>Remark</u>: Suppose W is a stratified object with a family of lines and $X < Y$ are strata of W. Suppose a local smooth cellulation of the smooth fibration $\pi_X | S_X(\epsilon) \cap Y$: $S_X(\epsilon) \cap Y \to X$ is given as follows:

$$\begin{array}{ccc} K & \xrightarrow{f} & S_X(\epsilon) \cap Y \\ \downarrow & & \downarrow \pi_X \\ L & \xrightarrow{g} & X \end{array}$$

Let J denote the cell complex which is the mapping cylinder of $(g^{-1} \circ \pi_X \circ f)$: $K \to L$. Then there is an obvious homeomorphism $J \to \overline{T_X(\epsilon) \cap Y}$ which is a local smooth cellulation of $T_X(\epsilon) \cap Y$ and is called "the result of <u>coning</u> the cellulation of $S_X(\epsilon) \cap Y$ down to X ". If σ is a cell of K then the corresponding cells of J are denoted

$$\sigma \ , \ C_X{}^o(\sigma) \ , \quad \text{and} \quad \pi_X(\sigma)$$

and the union $C_X{}^o(\sigma) \cup \pi_X(\sigma)$ is denoted $C_X(\sigma)$.

Note: Because connectedness of a stratum is not needed in this chapter, we will assume for the remainder of the chapter that every stratified object is decomposed so as to have at most one stratum of each dimension.

If $d = (d_o, d_1, d_2, \ldots)$ is a sequence of positive numbers and W is a stratified object, denote

$$T_i(d) \equiv \bigcup_{k \leq i} \{T_X(d_k) | \dim X = k\}$$

3.4.3. Definition: Let W be a Thom-Mather stratified object with a family of lines and suppose $d = (d_o, d_1, d_2, \ldots)$ is a sequence of positive numbers for which the link bundles $S_X(d_i)(\dim X = i)$ are mutually transverse (see Chapter 1).

Then a d-cellulation of W is defined inductively to be a closed cell complex $K \subset \mathbb{R}^N$ together with a homeomorphism $f\colon K \to W$ so that for each stratum X^i of W we have:

(1) $f^{-1}(X) \to X$ is a local smooth cellulation of X

(2) $\pi_X\colon S_X(d_i) \to X$ is a (linear) cellular map

(3) $f^{-1}\overline{(S_X(d_i))}$ is a closed subcomplex of K and $f|f^{-1}\overline{(S_X(d_i))}$ is a $(d_1,d_2,\ldots,\hat{d}_i,d_{i+1},\ldots)$ cellulation of $\overline{S_X(d_i)}$.

(4) $T_X(d_i) \cup S_X(d_i)$ is cellulated by coning the cellulation of $S_X(d_i)$ down to X as described in remark 3.4.2.

3.4.4. <u>Definition</u>: Let W^n be a Thom-Mather stratified object with a family of lines and suppose $d = (d_o,d_1,\ldots,d_n)$ and $d' = (d_o',d_1',\ldots,d_n')$ satisfy $0 < d_i' \leq d_i$ for $0 \leq i \leq n$ (written $d' \leq d$). Then a (d',d) cellulation of W is defined to be a d'-cellulation

$$f\colon K \to W$$

of W such that, for each stratum X^i of W , $f^{-1}\overline{(S_X(d_i))}$ is a closed subcomplex of K , $\pi_X|S_X(d_i)\colon S_X(d_i) \to X$ is cellular, and for each cell σ in $f^{-1}(S_X(d_i))$ there are cells

$$f^{-1}h_X{}^{-1}(f(\sigma)\times(d_i',d))$$

$$f^{-1}h_X{}^{-1}(f(\sigma)\times\{d_i'\})$$

$$f^{-1}h_X{}^{-1}(f(\sigma)\times(0,d_i'))$$

in $f^{-1}(T_X(d_i)-X)$, where $h_X\colon T_X(d_i) - X \to S_X(d_i) \times (0,d_i)$ is defined by

$$h_X(p) = (r_X(d_i)(p) , \rho_X(p)) .$$

Such cellulations arise naturally by refinement of a d-cellulation.

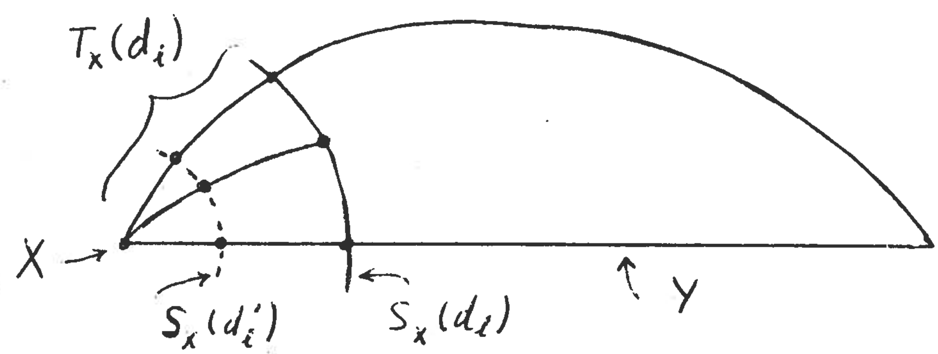
<u>Note:</u> For the rest of this chapter, a cellulation $f\colon K \to W$ of W will generally be identified with the decomposition $f(K)$ of W except when confusion may arise.

3.4.5. <u>Lemma</u>: Suppose W is a k-dimensional Thom-Mather object and a $d = (d_1,d_2,\ldots)$-cellulation of $W^{(p)} \cup \overline{T_i(d)}$ is fixed. (In other words, the restriction of a d-cellulation of W to the closed subcomplex $W^{(p)} \cup \overline{T_i(d))}$. Let $d' \leq d$. Then there is induced a canonical (d',d)-cellulation of $W^{(p)} \cup \overline{T_i(d)}$

which refines the original cellulation and therefore restricts to a d'-cellulation of $W^{(P)} \cup \overline{T_i(d')}$. This is called <u>the canonical d'-refinement</u> of the original cellulation.

<u>Proof</u>: Suppose by induction that a $((d_1',\ldots,d_{\ell-1}',d_\ell,\ldots,d_k)$, $(d_1,\ldots,d_k))$-cellulation of $W^{(p)} \cup \overline{T_i(d)}$ has been found and let $X = X_\ell$ be the ℓ-dimensional stratum. Radial projection from $S_X(d_\ell')$ to $S_X(d_\ell)$ is a caonnical homeomorphism and defines a cellulation of $S_X(d_\ell')$ and hence a refinement of $T_X(d_\ell)$ with cells of three types: those lying in $S_X(d_\ell')$, those inside $T_X(d_\ell')$ and those in $T_X(d_\ell) - T_X(d_\ell')$. (See diagram.)

<u>Diagram</u>:



There is an obvious linear structure on each of the new cells and π_X remains linear and cellular. This subdivision is a $((d_1',\dots,d_\ell',d_{\ell+1},\dots,d_k)$, $(d_1,\dots,d_k)$-cellulation: it must be shown that $S_Y(d_j) \to Y$ is cellular if $\dim Y = j > \ell$, and that $S_Y(d_j') \to Y$ is cellular if $j < \ell$.

If $Y^j > X$ then the new cells in $S_Y(d_j)$ are of the form $\sigma \cap S_X(d'_\ell)$ or $\sigma \cap T_X(d'_\ell)$ or $\sigma \cap T_X(d_\ell) - \overline{T_X(d'_\ell)}$ where σ is an original cell in $S_Y(d_Y)$. However, since $\rho_X\pi_Y = \rho_X$ we have

$$\pi_Y(\sigma \cap S_X(d'_\ell)) = \pi_Y(\sigma) \cap S_X(d'_\ell)$$

$$\pi_Y(\sigma \cap T_X(d'_\ell)) = \pi_Y(\sigma) \cap T_X(d'_\ell)$$

$$\pi_Y(\sigma \cap (T_X(d_\ell) - \overline{T_X(d_\ell')})) = \pi_Y(\sigma) \cap (T_X(d_\ell) - T_X\overline{(d'_\ell)}) ,$$

so π_Y is cellular. It is also clear that $T_Y(d_j)$ is cellulated by cones over cells of $S_Y(d_j)$.

If $Y^j < X$ a very similar argument holds since $\pi_Y = \pi_X\pi_Y$, which completes the induction.

3.4.6. Remark: Suppose K and L are (convex linear) cell complexes and $f: K \to L$ is a (linear) cellular map. Let L' be a subdivision of L. Then the pullback subdivision K' of K is defined to have cells $\sigma' = \sigma \cap f^{-1}(\tau')$ where τ' is a cell of L' and σ is a cell of K. Then $f': K' \to L'$ is cellular.

If $f: K \to L$ is cellular and K' is a subdivision of K then the pushforward subdivision L' if L is defined to have cells $\tau' = \tau \cap f(\sigma')$ where τ is a cell of L and σ' a cell of K'. In this case, if K'' is the pullback of L' then $f: K'' \to L'$ is cellular.

3.5. Relative Cellulation of Fibre Bundles

3.5.1. Lemma: Suppose $\pi: Y \to X$ is a smooth fibration and suppose a smooth cellulation of X is given. Let U and V be open subsets of X with $U \subset \overline{U} \subset V \subset \overline{V} \subset X$ and suppose $\overline{U}$ and $\overline{V}$ are closed subcomplexes. Suppose a smooth cellulation of $\pi^{-1}(\overline{V})$ is given which makes $\pi|\pi^{-1}(\overline{V})$ into a (linear) cellular map. Then there is a smooth cellulation of Y and a refinement of $X - \overline{U}$ with the following properties:

(1) The cellulation of Y agrees with the given one on $\pi^{-1}(\overline{U})$

(2) π is a (linear) cellular map

(3) No cells in $\overline{U}$ or in $\pi^{-1}(\overline{U})$ need be subdivided.

<u>Proof</u>: This follows essentially from Putz [11]. It is necessary to choose an open set U' with $\overline{U} \subset U' \subset \overline{U'} \subset V$ for which there is a smooth cellulation of $Y - \pi^{-1}(\overline{U'})$ and a refinement of $X - U'$ which makes $\pi | Y - \pi^{-1}(\overline{U'})$ cellular.

Putz then proves that the cellulations of $Y - \pi^{-1}(\overline{U'})$ and of $\pi^{-1}(\overline{V})$ can be refined and ϵ-approximated by fibered cellulations which intersect in a subcomplex and therefore fit together to give a cellulation of Y for which π is cellular. His proof also shows that no approximation or subdivision need take place on $\pi^{-1}(\overline{U})$ if ϵ is chosen sufficiently small.

3.6. <u>Theorem</u>: Every Thom-Mather stratified object has a d-cellulation for d sufficiently small.

<u>Proof</u>: Let B_k and $C_{k,i}$ be the propositions stated below. We shall prove these propositions in the following order:

$$\ldots B_k \,;\, C_{k+1,0} \,;\, C_{k+1,1} \,;\, \ldots \,;\, C_{k+1,k} \,;\, B_{k+1} \,;\, C_{k+2,0} \,\ldots$$

<u>Proposition B_k</u>: A stratified object of dimension k has a d-cellulation for d sufficiently small.

<u>Proposition $C_{k,i}$</u>: Suppose W is a k-dimensional Thom-Mather object and suppose $f_1: K_1 \to W^{(k-1)} \cup \overline{T_{i-1}(d)}$ is a d-cellulation of $W^{(k-1)} \cup \overline{T_{i-1}(d)}$ for some $d = (d_1, \ldots, d_{k-1})$. Let $d' < d$ and let X be the stratum of dimension i. Then there is a d'-cellulation $f_2: K_2 \to W^{(k-1)} \cup \overline{T_i(d')}$ which coincides with the canonical d'-refinement of K_1 on $\overline{T_{i-1}(d')}$ and which refines K_1 on $W^{(k-1)}$.

(In other words, if bW has been cellulated along with neighborhoods of all strata of dimension $< i$ then we can extend to a cellulation of the neighborhood of the i-dimensional stratum, provided we first shrink all previous neighborhoods and therefore refine the cellulation of bW .)

<u>Proofs</u>: B_0 is trivial.

$C_{k,0}$ follows by applying B_{k-1} to $S_X(d_o')$ if X is the 0-dimensional stratum, resulting in a d'-cellulation of $S_X(d_o')$ which, by coning to X, gives a d'-cellulation of $T_X(d_o')$.

B_k follows from $C_{k,k-1}$ which gives a d-cellulation of $T_{k-1}(d)$, leaving the interior of the k-dimensional stratum Z to be cellulated. Since $T_{k-1}(d') \subset \overline{T_{k-1}(d')} \subset T_{k-1}(d)$, lemma 3.5. applies and gives a cellulation of Z which agrees with the d'-refinement of $T_{k-1}(d) \cap Z$ on $T_{k-1}(d') \cap Z$. This result is a d'-cellulation of W.

The heart of the proof is $C_{k,i-1} \Rightarrow C_{k,i}$ and we therefore assume the assumptions of proposition $C_{k,i}$ are satisfied. We will first cellulate $S_X(d)$ and then cone to X, cellulating $T_X(d)$.

By the axiom of the frontier, $\overline{S_X(d_i)}$ is a k-1 dimensional stratified object and $L_1 = K_1 \cap f_1^{-1}\overline{(S_X(d_i))} \to S_X(d_i)$ is a $(d_1,\ldots,\hat{d}_i,\ldots,d_{k-1})$ cellulation of

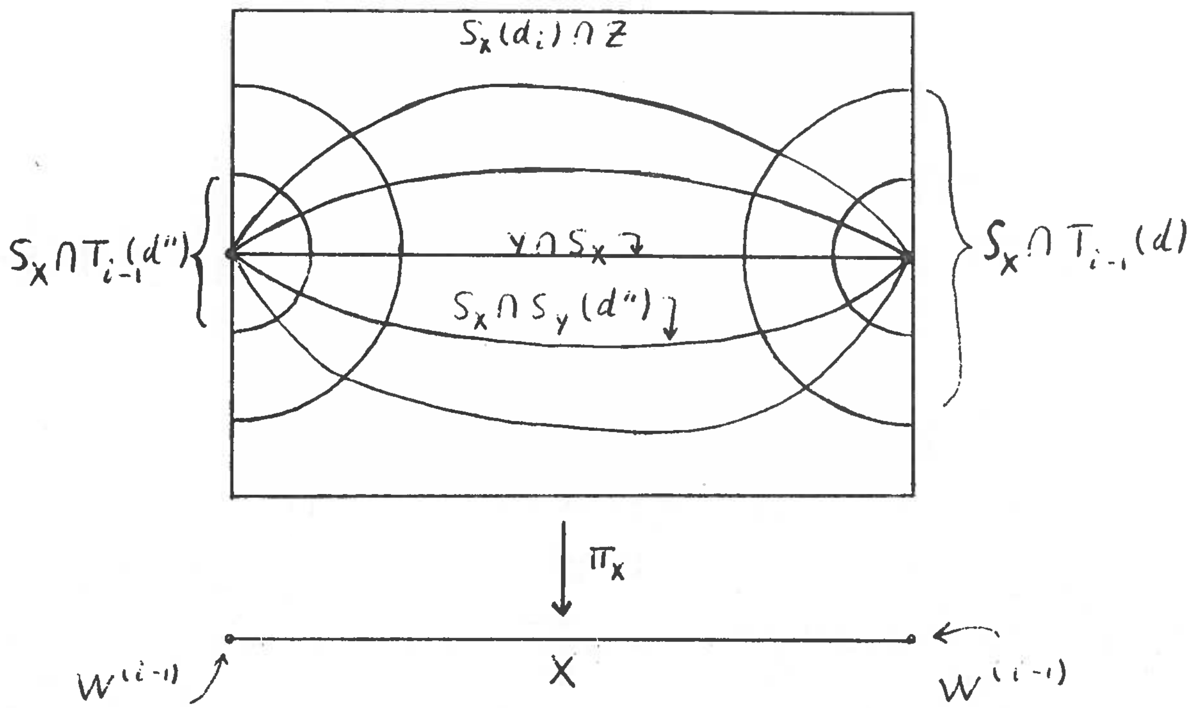
$$(W^{(k-1)} \cup \overline{T_{i-1}(d)}) \cap \overline{S_X(d_i)} .$$

Choose d'' with $d' < d'' < d$ and let K_1'' be the canonical (d'',d)-refinement of $W^{(k-1)} \cup \overline{T_{i-1}(d)}$ and let L_1'' be the corresponding refinement of L_1. By propositions $C_{k-1,i-1}$ through $C_{k-1,k-2}$ there is a d''-cellulation

$$L_2 \to S_X(d_i) \cap \overline{(T_{k-1}(d''))}$$

which coincides with L_1'' on $S_X(d_i) \cap \overline{(T_{i-1}(d''))}$ and which refines L_1 on $S_X(d_i) \cap W^{(k-1)}$.

Diagram: $S_X(d_i)$



Furthermore,

$$\begin{array}{ccc} L_2 & \longrightarrow & S_X(d_i) \cap T_{k-1}(d'') \\ \downarrow & & \downarrow \pi_X \\ K_1'' \cap f^{-1}(X) & \longrightarrow & X \end{array}$$

is a piecewise linear map: If $Y > X$ then π_X can be locally factored

$$S_X(d_i) \cap T_Y(d'') \xrightarrow{\pi_Y} S_X(d_i) \cap Y \xrightarrow{\pi_X} X$$

where π_Y is <u>cellular</u> and $\pi_X|Y$ is piecewise linear (since Y was refined). Notice however that if $Y < X$ then

$$\pi_X\colon S_X(d_i) \cap \overline{T_Y(d_i'')} \to X \cap \overline{T_Y(d'')}$$

is <u>cellular</u> since no refinement of $T_Y(d'')$ was made.

Therefore the pushdown by π_X of L_2 refines $X - T_{i-1}(d'')$ and the pullback by π_X refines L_2 . π_X is cellular with respect to these refinements. In summary we have a $(d_1'', \ldots, d_{i-1}'', d_i, d_{i+1}'', \ldots, d_{k-1}'')$ -cellulation

$$K_3 \to W^{(k-1)} \cup \overline{T_{i-1}(d'')} \cup [S_X(d_i) \cap \overline{T_{k-1}(d'')}]$$

and it remains to cellulate the top stratum $Z \cap S_X(d_i)$ of $S_X(d_i)$.

Since $T_{k-1}(d') \subset \overline{T_{k-1}(d')} \subset T_{k-1}(d'')$ we can apply lemma 3.5.1. to find a refinement of $X - T_{i-1}(d')$ and a cellulation of $Z \cap S_X(di)$ which agrees with K_3 on $Z \cap S_X(d_i) \cap T_{i-1}(d')$, for which $\pi_X | Z \cap S_X(d_i): Z \cap S_X(d_i) \to X$ is cellular. This refinement of $X - T_{i-1}(d')$ again lifts to a further refinement of $S_X \cap \overline{T_{k-1}(d')}$ which does not involve subdividing any cells of $S_X \cap \overline{T_{i-1}(d')}$ and which makes $\pi_X | S_X(d_i): S_X(d_i) \to X$ into a cellular map.

Consequently, coning to X determines a d'-cellulation of $T_X(d_i)$ and therefore a $(d_1',\ldots,d_{i-1}',d_i,d_{i+1}',\ldots,d_{k-1}')$-cellulation of $W^{(k-1)} \cup T_i(d_1',\ldots,d_{i-1}',d_i,d_{i+1}',\ldots,d_{k-1}')$. By shrinking $T_X(d_i)$ we obtain a d'-cellulation of $W^{(k-1)} \cup T_i(d')$ which refines the original cellulation K_1 without subdividing any cell of K_1'' in $T_{i-1}(d')$. Q.E.D.

3.6.2. Corollary: Suppose M is a smooth manifold with collared boundary and that Whitney stratifications of $M \times \{0\}$ and of $M \times \{1\}$ are specified with a system of control data and families of lines. Suppose $f_0: K_0 \to M \times \{0\}$ and $f_1: K_1 \to M \times \{1\}$ are Whitney d_0 and d_1 cellulations of $M \times \{0\}$ and of $M \times \{1\}$ respectively. Then there is a Whitney cellulation of $M \times [0,1]$ which refines K_0 and K_1.

Proof: Stratify $M \times [0,1]$ with strata $(M-\partial M) \times (0,1)$ and $\partial M \times (0,1)$ along with the strata in $M \times \{0\}$ and $M \times \{1\}$. There is a canonical system of control data and family of lines on $M \times [0,1]$ which restricts to the given control data on $M \times \{0\}$ and $M \times \{1\}$ using the method of Proposition 6.1.1. Let $0 < d' < d'' < \min(d_0,d_1)$. Then $\partial M \times [0,1]$ is a Thom-Mather stratified object whose singularity set $\partial M \times \{0\} \cup \partial M \times \{1\}$ has been cellulated, so the inductive step of the preceding theorem applies, and there is a d''-cellulation of $\partial M \times [0,1]$ which refines the given cellulation of $\partial M \times \{0\} \cup \partial M \times \{1\}$. Constructing the canonical (d'',d_0) and (d'',d_1)-refinements of K_0 and K_1 respectively, we arrive at a d''-cellulation of the singularity set

$$\partial M \times (0,1) \cup M \times \{0\} \cup M \times \{1\}$$

of $M \times [0,1]$. Thus the inductive step of theorem 3.6.1. applies again and there is a d''-cellulation of $M \times [0,1]$ refining all previous cellulations. The Whitney conditions are satisfied between any two cells in $\partial M \times (0,1)$ and between any two cells in $(M-\partial M) \times (0,1)$ since these strata are smoothly cellulated. The Whitney conditions are easily verified between cells in $(M-\partial M) \times (0,1)$ and cells in $\partial M \times (0,1)$; between cells in $(M-\partial M) \times (0,1)$ and cells in $(M-\partial M) \times \{0\}$ (or in $(M-\partial M) \times \{1\}$) ; and between cells in $\partial M \times (0,1)$ and cells in $\partial M \times \{0\}$ or $\partial M \times \{1\}$. The difficulty arises in checking the Whitney conditions between cells in $(M-\partial M) \times (0,1)$ and cells in $\partial M \times \{0\} \cup \partial M \times \{1\}$, and this is a special case of a more general problem treated in section 3.7.

3.7. The purpose of this section is to show that a d-cellulation of a radial stratified object satisfies the Whitney conditions at each incidence between two cells.

3.7.1. <u>Lemma</u>: Suppose $\pi: E \to N$ is a smooth Reimannian vector-bundle over a manifold N and let $M = S_N(\epsilon)$ be the ϵ-sphere bundle. Let $W_1 \subset M$ and $W_2 \subset N$ be Whitney objects and suppose $\pi(W_1) \subset W_2$ and that π satisfies condition D (with respect to W_1 and W_2) as defined in section 2.4. Then $\overline{C_N(W_1)}$ can be stratified with strata $C_N(Y) - \pi_N(Y)$, $\pi_N(Y)$, and Y , for each stratum Y of W_1 , and the resulting stratification is Whitney.

<u>Proof</u>: The Whitney conditions need only be checked locally so we may assume M and N are contained in $\mathbb{R}^n$ with

$$N = \{(x_1,\ldots,x_n) \in \mathbb{R}^n | x_{k+1} = \ldots = x_n = 0\}$$

$$M = \{(x_1,\ldots,x_n) \in \mathbb{R}^n | x_{k+1}^2 + \ldots + x_n^2 = \epsilon\}$$

Suppose $X_1 < Y_1 \subset W_1$ and $X_2 \leq Y_2 \subset W_2$ are strata with $\pi_N(X_1) = X_2$ and $\pi_N(Y_1) = Y_2$. The Whitney conditions must be checked for the incidences

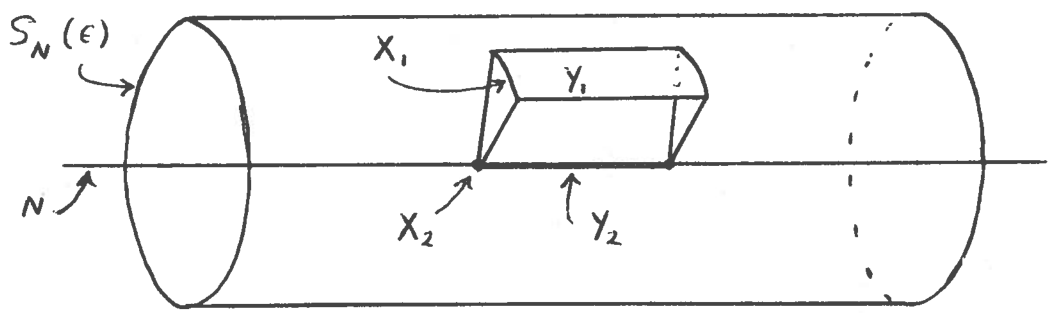
1. $Y_2 < C_N{}^o(Y_1)$

2. $X_2 < C_N{}^o(X_1)$

3. $X_2 < C_N{}^o(Y_1)$

4. $C_N{}^o(X_1) < C_N{}^o(Y_1)$

Diagram:



Incidence (4) is trivial because there is a diffeomorphism of $T_N(\epsilon) - N$ to an open subset of $\mathbb{R}^n$ taking $C_N^{\,o}(X_1)$ to $X_1 \times \mathbb{R}^{n-k}$ and taking $C_N^{\,o}(Y_1)$ to $Y_1 \times \mathbb{R}^{n-k}$ and the Whitney conditions clearly hold for the product of a Whitney object $X_1 \cup Y_1$ with a manifold.

We now calculate $C_N^{\,o}(Y_1)$. Define

$$F: S_N(\epsilon) \times [0,1] \to \mathbb{R}^n$$

by

$$F(p,t) = tp + (1-t)\pi_N(p) .$$

Then $F(Y_1 \times [0,1]) = C_N(Y_1)$ and by differentiation,

$$T_{F(p,t)} C_N^{\,o}(Y_1) = \{tv + (1-t)d\pi_N(p)(v) + s(p-\pi_N(p)) \,|\, v \in T_pY_1 \text{ and } s \in \mathbb{R}\}$$

To prove condition B for incidence (3), suppose a sequence $F(p_i,t_i) \in C_N^{\,o}(Y_1)$ converge to a point $x_o \in X_2$. The p_i have a convergent subsequence which we again denote by $p_i \to p_o \in X_i$. Suppose a sequence $x_i \in X_2$ converges to x_o and that the secant lines

$$\ell_i = \overparen{x_iF(p_i,t_i)} = \{\alpha\cdot(x_i - t_ip_i - (1-t_i)\pi_N(p_i)) \,|\, \alpha \in \mathbb{R}\}$$

converge to some line ℓ, and that the tangent planes

$$\tau_i = T_{F(p_i,t_i)} C_N^{\,o}(Y_1) \quad \text{(as computed above)}$$

converge to some plane τ. We must show $\ell \subset \tau$. (Note that $t_i \to 0$).

Choose $\alpha_i \in \mathbb{R}$ so the vectors

$$w_i = \alpha_i(x_i - \pi_N(p_i)) - \alpha_i t_i(p_i - \pi_N(p_i)) \in \ell_i$$

have unit length and converge to a unit vector $w_0 \in \ell$.

The two components of w_i are perpendicular and hence converge individually. Therefore the secant lines $\widehat{x_i \pi_N(p_i)}$ converge to some line m and by condition B for the incidence $X_2 < Y_2$ we have

$$m \subset \lim_{i\to\infty} T_{\pi_N(p_i)} Y_2$$

(having chosen a subsequence to ensure convergence if necessary). Condition D then implies the existence of a vector $v_0 \in \lim_{i\to\infty} T_{p_i} Y_1$ for which $d\pi_N(p_0)(v_0) = \lim_{i\to\infty} \alpha_i(x_i - \pi_N(p_i)) \in m$. Let $v_i \in T_{p_i} Y_1$ be a sequence of tangent vectors converging to v_0 .

We then conclude that the sequence of vectors

$$t_i v_i + (1-t_i) d\pi_N(p_i)(v_i) + \alpha_i t_i (p_i - \pi_N(p_i)) \in \tau_i$$

converges to the vector $w_0 \in \ell$, so $\ell \subset \tau = \lim \tau_i$ as desired.

The proof for incidences (1) and (2) is similar but easier.

3.7.2. Corollary: Suppose M and N are smooth submanifolds of the same Euclidean space, $\pi: M \to N$ is a smooth submersion, and $W_1 \subset M$ and $W_2 \subset N$ are Whitney objects. Suppose $\pi(W_1) = W_2$ and π satisfies condition D . Then the preceding analysis show the set

$$\overline{C_N(W_1)} = \{tp + (1-t)\pi(p) | p \in W_1, t \in [0,1]\}$$

is a Whitney stratified object provided each line

$$tp + (1-t)\pi(p) \qquad (t \in [0,1], p \in W_1)$$

is not tangent to N when $t = 0$.

3.7.3. Remark: We note at this point that if $\pi_N: X_1 \leq Y_1 \to X_2 \leq Y_2$ satisfies condition D as in the lemma, and if $q_i \in C_N{}^o(Y_1)$ is a sequence converging to a point $x_o \in X_2$ then

$$\lim_{i\to\infty} T_{q_i} C_N{}^o(Y_1) \supset \lim_{i\to\infty} T_{\pi_N(q_i)} Y_2$$

assuming both sides converge.

To see this, suppose $q_i = F(p_i, t_i)$ where $p_i \in Y_1$ and $t_i \in (0,1)$ as before. Suppose all appropriate tangent planes converge and that $p_i \to p_o \in X_1$ by choosing a subsequence if necessary. Let $\eta \in \lim\limits_{i\to\infty} T_{\pi_N(p_i)} Y_2$. Since π_N satisfies condition D, there is a vector $v \in \lim\limits_{i\to\infty} T_{p_i} Y_1$ so that $d\pi_N(p_o)(v) = \eta$. Let $v_i \in T_{p_i} Y_1$ converge to v.

Then the vectors

$$t_i v_i + (1-t_i) d\pi_N(p_i)(v_i) \in T_{q_i} C_N{}^o(Y_2)$$

converge to $\eta = d\pi_N(p_o)(v)$ which proves the remark.

3.7.4. <u>Lemma</u>: Suppose $f\colon Q \to N$ is a smooth fibration and

$$\begin{array}{ccc} K & \xrightarrow{h_1} & Q \\ & & \big\downarrow f \\ L & \xrightarrow{h_2} & N \end{array}$$

are smooth cellulations of Q and N for which $h_2^{-1} \circ f \circ h_1$ is a (linear) cellular map.

Then

(1) If σ is a cell of K and τ is a face of σ then $h_1(\tau) < h_1(\sigma)$ satisfies condition B

(2) f satisfies condition D with respect to the stratifications of Q and N given by the cellulation.

<u>Proof</u>: (1) If $\sigma < \tau$ are linear cells in $\mathbb{R}^m$ then condition B is clearly satisfied. However, h_1 is a smooth cellulation so condition B "pushes forward" to $h_1(\sigma) < h_1(\tau)$ by any smooth extension of $h_1|\tau$ to a neighborhood of τ in the plane of τ.

(2) Suppose $\sigma_1 < \tau_1{}^m$ are cells of K and $\sigma_2 \leq \tau_2{}^n$ are cells of L, with $fh_1(\sigma_1) = h_2(\sigma_2)$ and $fh_1(\tau_1) = h_2(\tau_2)$. We may suppose $\tau_1 \subset \mathbb{R}^m$ and $\tau_2 \subset \mathbb{R}^n$ and there is a linear surjection $L\colon \mathbb{R}^m \to \mathbb{R}^n$ so that

$$f \circ h_1(p) = h_2 \circ L(p) \quad \text{for all } p \in \overline{\tau_1} .$$

Suppose $p_i \in \tau_1$ converge to $p \in \sigma_1$ and all appropriate tangent planes converge. Then (recalling that h_1 has a smooth extension to a neighborhood of $\overline{\tau_1}$) ,

$$\begin{aligned} df(h_1(p))(\lim T_{h_1(p_i)}h_1(\tau_1)) &= df(h_1(p))(\lim dh_1(p_i)(T_{p_i}\tau_1)) \\ &= df(h_1(p))\circ(dh_1(p))(T_p\,\mathbb{R}^m) \\ &= d(f\circ h_1)(p)(T_p\,\mathbb{R}^m) \\ &= d(h_2\circ L)(p)(T_p\,\mathbb{R}^m) \\ &= dh_2(L(p))(T_p\,\mathbb{R}^n) \\ &= \lim dh_2(L(p_i))(T_{p_i}\tau_2) \\ &= \lim T_{f(p_i)}\tau_2 \end{aligned}$$

Q.E.D.

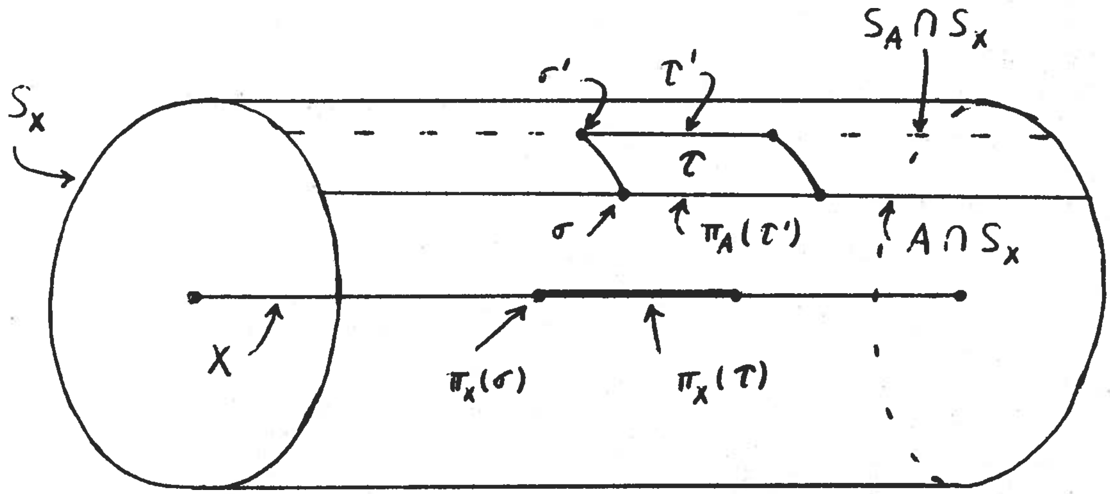
3.7.5. Theorem: A d-cellulation of a radial stratified object is Whitney (i.e., the refined stratification in which each cell is a stratum, satisfies the Whitney conditions).

Proof: It will be shown that if W is a d-cellulated radial stratified object and if X is an (original) stratum of W, then the map π_X: $S_X(d) \to X$ satisfies condition D relative to the refinement given by the cellulation. (Therefore, the cells in $T_X(d)$ obtained by the coning operation satisfy the Whitney conditions. Since each stratum is smoothly cellulated, the cells whose closure lies in a given stratum are also Whitney.)

The above statement is proven by induction on the dimension of W and for the inductive step we assume the condition D statement holds for every radial stratified object of dimension $\leq k - 1$. Let W be a d-cellulated radial object of dimension k and let X be a stratum of W. (We ignore the subscript on d for simplicity in the following.)

Then $S_X(d) \cap W$ is a d-cellulated radial stratified object of dimension less than k. Suppose $\sigma < \tau$ are cells of $S_X(d) \cap W$ and $\pi_X(\sigma) \leq \pi_X(\tau)$ are cells of X. If σ and τ lie in the same stratum of W then condition D follows from the preceding lemma 3.7.4. Therefore, assume σ and τ lie in different strata, say A and B (respectively) where $A < B$. Then $\tau = C_A^o(\tau')$ for some cell $\tau' \subset {}_X(d) \cap S_A(d) \cap B$ and either $\sigma = \pi_A(\tau')$ or $\sigma = \pi_A(\sigma')$ for some cell $\sigma' < \tau'$ in $S_X(d) \cap S_A(d) \cap B$. (See diagram.)

Diagram:



If $\sigma = \pi_A(\tau')$ then $\pi_X(\sigma) = \pi_X\pi_A(\tau') = \pi_X(\tau)$ so there is nothing to check.

We may therefore assume

(1) $\sigma = \pi_A(\sigma') < \pi_A(\tau')$ for cells $\sigma' < \tau'$ of

$$S_X(d) \cap S_A(d) \cap B$$

(2) $\tau = C_A^{\,o}\ (\tau')$

Therefore π_A satisfies condition D with respect to the incidence

$$\pi_A: \sigma' < \tau' \to \sigma < \pi_A(\tau')$$

by induction, and π_X satisfies condition D with respect to the incidence

$$\pi_X: \sigma < \pi_A(\tau') \to \pi_X(\sigma) < \pi_X(\tau)$$

by the preceding lemma.

Now suppose a sequence of points $p_i \in \tau$ converges to a point $p_o \in \sigma$, and suppose the tangent planes $T_{p_i}\tau$ converge, as do $T_{\pi_X(p_i)}\pi_X(\tau)$. According to the remark 3.7.3., we have

$$\lim T_{p_i}\tau \supset \lim T_{\pi_A(p_i)}\pi_A(\tau') \quad \text{(assuming convergence)}$$

since τ is a cone over τ'. Therefore

$$\begin{aligned} d\pi_X(p_o)(\lim T_{p_i}\tau) &\supset d\pi_X(p_o)(\lim T_{\pi_A(p_i)}\pi_A(\tau')) \\ &\supset \lim T_{\pi_X(p_i)}\pi_X(\tau) \end{aligned}$$

as desired.

3.8.

3.8.1. Lemma: Suppose W^k is an abstract geometric cycle and that W' is a refinement of W, i.e., W' is a stratified object and there is a homeomorphism $h: W' \to W$ which embeds each stratum of W' as a smooth submanifold of some stratum of W. Let W' be given the obvious orientation resulting from the embedding of each k-dimensional stratum of W' as an open subset of some k-dimensional stratum of W. Then W' is also a geometric cycle.

Proof: If X is a k-1 dimensional stratum of W' and $h(X) \subset bW$ then the induced orientations on X cancel since $h(X)$ is an open subset of some k-1 dimensional stratum of W where the boundary cancels by hypothesis. If X is a k-1 dimensional stratum of W' and $h(X) \subset W - bW$ then by choosing a local coordinate system about a point $h(p)$ (for $p \in X$) we may assume $W - bW = \mathbb{R}^n$ and $h(X) = \{(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n | x_n = 0\}$ with $\pi_{h(X)}(x_1,\ldots,x_n) = (x_1,\ldots,x_{n-1},0)$ in which case the induced orientations cancel on $h(X)$ as a consequence of:

Sublemma: For any orientation of $\mathbb{R}^n$, the induced orientations on $X = \{(x_1,\ldots,x_n) \in \mathbb{R}^n | x_n = 0\}$ cancel.

Proof: It is only necessary to verify the statement when $\mathbb{R}^n$ is oriented by the ordered basis $(e_1, e_2, \ldots, e_n)$ with $e_i = (0,\ldots,0,1,0,\ldots,0)$. Then X is incident to two strata only,

$$\mathbb{R}^n_+ = \{(x_1,\ldots,x_n) \in \mathbb{R}^n | x_n > 0\}$$

and

$$\mathbb{R}^n_- = \{(x_1,\ldots,x_n) \in \mathbb{R}^n | x_n < 0\}$$

Let $\ell_+(x_1,\ldots,x_n) = (0,0,\ldots,0,-1)$

and

$$\ell_-(x_1,\ldots,x_n) = (0,0,\ldots,0,+1)$$

denote the normal vectorfields "pointing from $\mathbb{R}^n_+$ to X " and "from $\mathbb{R}^n_-$ to X " respectively.

Then the orientation induced on X by the stratum $\mathbb{R}^n_+$ is given by $(-1)^{n+1}$ times the standard ordered basis $(e_1,\ldots,e_{n-1})$ so as to be compatible with the exact sequence on $\mathbb{R}^n_+$

$$0 \to \ell_+ \to T\mathbb{R}^n_+ \to d\pi_X{}^*(TX) \to 0$$

However, the orientation induced on X by the stratum $\mathbb{R}^n_-$ is given by $(-1)^n$ times the ordered basis $(e_1,\ldots,e_{n-1})$ so as to be compatible with the exact sequence on $\mathbb{R}^n_-$

$$0 \to \ell_- \to T\mathbb{R}^n_- \to d\pi_X{}^*(TX) \to 0$$

These orientations are opposite and therefore cancel.

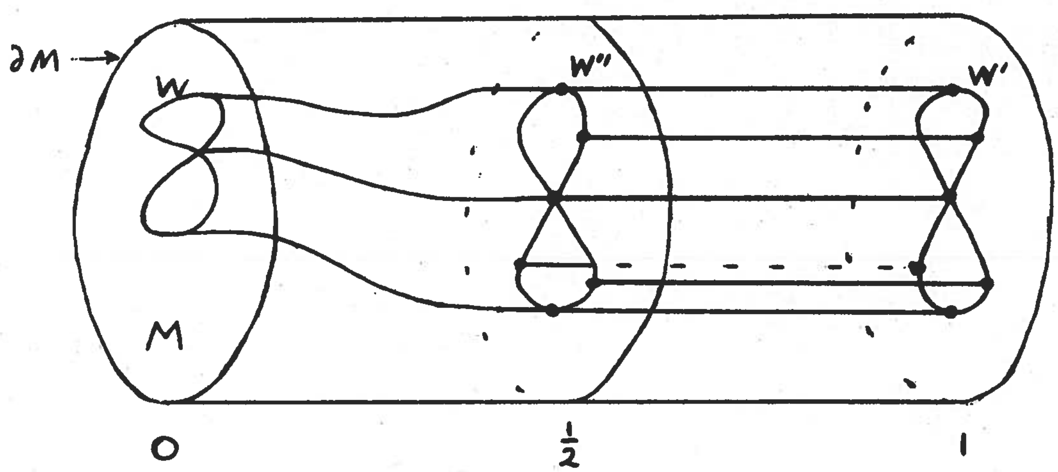
Q.E.D.

3.8.2. Corollary: Let M be a smooth manifold with collared boundary ∂M and suppose W^k is a relative geometric cycle in $(M,\partial M)$. Then there is a cobordism between W and a relative geometric cycle W' in $(M,\partial M)$ and there is a Whitney cellulation $f: K \to M$ of M so that each stratum of W' is a cell of this cellulation (i.e., there is a closed subcomplex $L \subset K$ so that $f(L) = W'$ as Whitney stratified objects).

Proof: By 3.3.1. there is a cobordism $S_1 \subset M \times [0,\frac{1}{2}]$ between W and a radial stratified object $W'' \times \{\frac{1}{2}\} \subset M \times \{\frac{1}{2}\}$ and by 3.6.1. and 3.7.3. there is a Whitney cellulation $f: K \to M$ of M and there is a closed subcomplex $L \subset K$ so that $f|L: L \to W''$ cellulates W''. Extend this cellulation of M trivially across $M \times [\frac{1}{2},1]$ and let $S_2 = f(L) \times [\frac{1}{2},1]$. Then $S_1 \cup S_2 \subset M \times [0,1]$

is the desired cobordism: it is trivial at the ends and $W' = f(L) \times \{1\}$ is a cellulated Whitney object. It is only necessary to check that the induced orientations on k-dimensional strata of $S_1 \cup S_2$ cancel except at the ends. (W' is a cycle by lemma 3.8.1.)

Diagram:



If σ is a k-1 dimensional cell of W' then $\sigma \times [\frac{1}{2},1]$ is a k-dimensional cell of the cobordism $S_1 \cup S_2$, and by lemma 3.8.1. the induced orientations on σ cancel, so the induced orientations on $\sigma \times [\frac{1}{2},1]$ cancel also. If τ is a k-dimensional cell of W' then $\tau \times \{\frac{1}{2}\}$ is a k-dimensional cell of $S_1 \cup S_2$. Suppose τ is contained in a stratum X of W''. Then for some $\epsilon > 0$, τ is incident to the two strata

$$X \times (\tfrac{1}{2}-\epsilon,\tfrac{1}{2}) \quad \text{and} \quad \tau \times (\tfrac{1}{2},1)$$

so the sublemma 3.8.1. applies again and the induced orientations on τ cancel. Q.E.D.

CHAPTER 4.

COHOMOLOGY

In this chapter we prove that, for a Thom-Mather object V, the cohomology group $H^k(V)$ can be interpreted as π-fibre subobjects of V modulo the relation of cobordism.

In 4.1. the definitions for a Thom-Mather object V are set up and the main theorem is stated. In 4.2. the analogous definitions are given in the case V is a Whitney object. Section 4.3. contains the proof of the main theorem which begins by immediately reducing to the case V is a Whitney object.

4.1. Cohomology of a Thom-Mather Stratified Object

4.1.1. Definition: Let V^n be a Thom-Mather stratified object. A substratified object $W^k \subset V$ is said to satisfy the π-fibre condition if, whenever $X < Y$ are strata of V, there is an $\epsilon < 0$ so that

$$W \cap T_{XY}(\epsilon) = \pi_{XY}^{-1}(W \cap X)$$

(where by π_{XY} we mean $\pi_{XY} | T_{XY}(\epsilon)$) .

If A is a stratum of W, X a stratum of V, and $A \subset X$, then the <u>intrinsic codimension</u> of A is defined to be $\dim X - \dim A$. If W is a π-fibre subobject of V then a <u>costratum</u> of W is the union of all strata of W with the same intrinsic codimension. (As usual, one may take components if connected costrata are preferred.) The resulting decomposition of W is called a <u>costratification</u>.

A costratum S of codimension k is denoted ${}^kS$.

Each costratum S of W has a normal vectorbundle E_S whose fibre over a point $p \in S \cap X$ is T_pX/T_pS if X is a stratum of V. Therefore $E_S|(X\ S)$ is the normal bundle to $X \cap S$ in X and these fit together in a locally trivial structure for E_S which satisfies the following:

If U' is a sufficiently small open subset of $X \cap S$ then $E_S|U'$ as trivial and $\pi_{XY}|\pi_{XY}^{-1}(U')$ is trivial for any other stratum $Y > X$. Thus, $d\pi_{XY}$ determines an isomorphism

$$E_S|\pi_{XY}^{-1}(U') \cong \pi_{XY}^{*}(E_S|U')$$

(whereby π_{XY} is meant $\pi_{XY}|T_{XY}(\epsilon)$ for ϵ sufficiently small, as above). But $\pi_{XY}^{*}(E_S|U')$ is trivial and so $E_S|\pi_{XY}^{-1}(U')$ is trivial also.

4.1.2. <u>Tubular Neighborhoods of Costrata</u>: For each costratum S of W there is an embedding $\phi_S \colon E_S \to V$ of the normal bundle of S onto an open neighborhood T_S of S in V, satisfying

(a) $\phi|(E_S|X\cap S)\colon E_S|(X\ S) \to X$ is a tubular neighborhood of $S \cap X$ in X whenever X is a stratum of V,

(b) The following diagram commutes:

$$\begin{array}{ccccc}
\pi_X^*(E_S|X\cap S) & \xrightarrow[d\pi_X]{\cong} & E_S|\pi_X^{-1}(S\cap X) & \xrightarrow{\phi} & T_X \\
\downarrow \pi_X & & & & \downarrow \pi_X \\
E_S|(X\cap S) & & \xrightarrow{\phi} & & X
\end{array}$$

ϕ_S is constructed by induction over $E_S|(S\cap T_k)$ where $T_k = \cup\,\{T_X(\epsilon)\ \dim X \le k$ and X is a stratum of $V\}$ (where ϵ is sufficiently small). For the inductive step, we assume $\phi_S|(E_S|(S\cap T_{k-1}))$ has been constructed and let X^k be a stratum of V. An embedding of $E|(X\cap S\cap T_{k-1})$ has been given, so by the existence theorem for tubular neighborhoods (Mather [9]), after shrinking ϵ slightly, this embedding extends to an embedding of $E_S|(S \cap X)$ as a tubular neighborhood. It is now easy to find an embedding of $E|\pi_X^{-1}(S \cap X)$ for which the above diagram commutes using Mather [9] (proposition 11.3.). This completes the construction of $\phi_S|(E|S \cap T_k)$.

4.2. Coorientations

4.2.1. Definition: A coorientation of a π-fibre subobject W of a Thom-Mather object V is defined to be an orientation with multiplicity of the normal bundle E_{S_i} of each component S_i of the costratum ${}^kS$ with the lowest codimension. (S is called the "top costratum" of W, and $\delta W \equiv W - S$ is called the $k+1$ coskeleton of W). As before, a given orientation with multiplicity is identified with the reverse orientation and negative multiplicity. A multiplicity 0 is attached to all non-orientable normal bundles.

4.2.2. Remark: We recall the Thom Isomorphism Theorem (see Milnor [10]): Suppose E_S is an orientable $\mathbb{R}^k$-vectorbundle over S, and let F_p be the fibre of E_S over a point $p \in S$ and let $i_p \colon F_p \to E_S$ be the inclusion. Then an element $\xi \in H^k(E_S, E_S - S)$ is nonzero if and only if $i_p^*(\xi) \in H^k(F_p, F_p - 0)$

is nonzero for all p . If a locally constant homological orientation $\theta_p \in H_k(F_p, F_p - 0)$ is chosen for all $p \in S$, then there is a unique $\xi \in H^k(E_S, E_S - S)$ so that

$$\langle i_p^*(\xi), \theta_p \rangle = +1 \quad \text{for all } p \in S .$$

4.2.3. Proposition: Let ${}^kW \subset V$ be as above. Then there is a one-to-one correspondence between coorientations of W , and the singular cohomology group $H^k(V - \delta W, V - W)$ of homological coorientations of W .

Proof: Let ${}^kS$ be the top costratum of W and let E_S be the normal bundle of S . Choose a Riemannian metric on E_S and a tubular neighborhood $\phi: E_S \to T_S$ of S , and define, for each $p \in S$,

$$T_S(\epsilon) \equiv \{x \in T_S | <\phi^{-1}(x), \phi^{-1}(x)> < \epsilon\}$$

$$D_S(P) \equiv \{X \in \pi_S^{-1}(P) | <\phi^{-1}(x), \phi^{-1}(x)> \leq \epsilon\}$$

$$S_S(P) \equiv \{x \in \pi_S^{-1}(P) | <\phi^{-1}(x), \phi^{-1}(x)> = \epsilon\}$$

and let $i_p: S_S(P) \to D_S(P)$ be the inclusion. Then according to the Thom isomorphism theorem, for any locally constant choice of orientation

$$\theta_p \in H_k(D_S(p), D_S(p)-p) \cong H_k(D_S(p), S_S(p))$$

and multiplicity m, there is a unique $\xi \in H^k(T_S(\epsilon), T_S(\epsilon)-S)$ so that

$$\langle i_p^*(\xi), \theta_p \rangle = + m \quad \text{for all} \quad p \in S \ .$$

However,

$$H^k(T_S, T_S - S) \approx H^k(T_S, T_S - W) \approx H^k(V - \delta W, V - W)$$

by excision of $V - T_S - \delta W$. (One checks that $\overline{V-T_S-\delta W} \cap (V-\delta W) \subset \text{interior}\ (V-W)$ to validate the excision.)

4.2.4. <u>Definition</u>: A cooriented π-fibre subobject W of the Thom-Mather stratified object V is called an (intrinsic) geometric cochain in V. The <u>reduction</u> of a geometric cochain W is the geometric cochain consisting of the closure of the union of all (components of the) top costrata which have been assigned a nonzero multiplicity. We shall generally identify two geometric cochains if they have the same reduction.

4.3. Coboundary

4.3.1. Definition: Let ${}^{k}W$ be a π-fibre subobject of the Thom-Mather stratified object V. A coorientation of W defines a coorientation on δW as described in this section. It will be helpful to think of V as a smooth manifold in this definition and then to notice that all statements are compatible with the local π-fibre structure of W in V.

Suppose ${}^{k+1}R < {}^{k}S$ are costrata of W and let $f: E_R \to R$ be the normal bundle of R in V with tubular neighborhood $\phi_R: E_R \to T_R$ and $\pi_R = f \circ \phi_R^{-1}$. Then $f^*(E_R)$ can be identified with tangent bundle to the fibres of E_R, so

$$(\phi_R^{-1} \circ f)^*(E_R) = \pi_R^*(E_R) \to T_R$$

is the bundle of tangents to the fibres $\pi_R^{-1}(p) \subset T_R$ for $p \in R$. Let $-\ell \to S \cap T_R$ be the tangents to the curves $\pi_R^{-1}(p) \cap S$ oriented by the vectors $(\text{grad } \rho_R)(s)$ for $s \in \pi_R^{-1}(p) \cap S$. Note that this is the reverse of the orientation considered in section 2.2.

Then there is an exact sequence of vectorbundles over $S \cap T_R$,

$$0 \to -\ell \to \pi_R^*(E_R)|(S \cap T_R) \to E_S|(S \cap T_R) \to 0$$

so an orientation with multiplicity of E_S induces an orientation with multiplicity of $\pi_R^*(E_R)|(S \cap T_R)$ compatible with this exact sequence. The orientations and multiplicities of $\pi_R^*(E_R)|(S \cap T_R)$ are constant on each curve in $\pi_R^{-1}(p) \cap S$ (for $p \in R$) and therefore by summing over local connected components of $T_R \cap S$ as in section 2.2. we obtain an orientation with multiplicity for E_R and thus a coorientation of δW .

4.3.2. <u>Proposition</u>: Let $W \subset V$ be as in the above definition. Then the coboundary coincides with the composition

$$\begin{array}{llll} & & \downarrow & \\ \text{coorientations of } W \cong H^k(V-\delta W, V-W) & \to & H^{k+1}(V-\delta W) & \to \\ & \overset{\delta}{\searrow} & \downarrow & \\ & & H^{k+1}(V-\delta\delta W, V-\delta W) & \\ & & \| & \\ & & \text{coorientations of } \delta W & \end{array}$$

<u>Proof</u>: Let ${}^{k+1}R < {}^{k}S$ be costrata of W. Choose tubular neighborhoods T_S of S, T_R of R and define E_S, $D_S(p)$, $D_R(p)$, $S_S(p)$ and $S_R(p)$ as in proposition 4.2.

For each $p \in S$ let α_S: geometric coorientations of $W \to H_R(D_S(p), S_S(p))$ be defined by restricting the orientation of E_S to the fibre over $p \in S$. Let

$$\beta_S: \text{ geometric coorientations of } W \to H^k(V-\delta W, V-W)$$

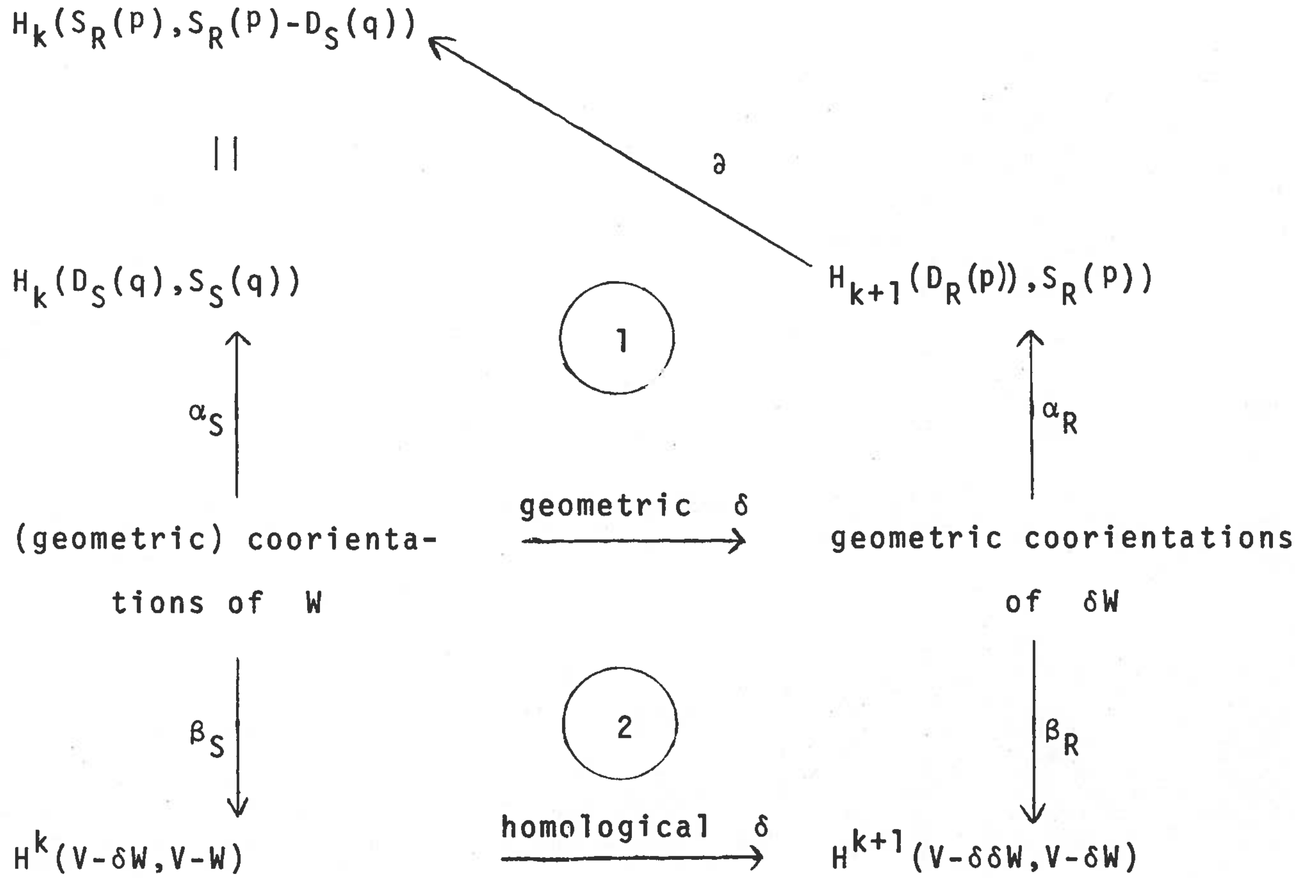
be the map defined in proposition 4.2.3.

Recall that β_S is an isomorphism and $\beta_S(\xi)$ is the unique solution to the equation

$$\langle i_p^*(\beta_S(\xi)), \alpha_S(\xi)\rangle = + m \quad \text{for all } p \in S$$

where m is the multiplicity of S.

Consider the following diagram, where $p \in R$ and $q \in S \cap S_R(p)$:



If the loop labelled "1" commutes then square "2 commutes also because $\beta_R(\delta\xi)$ is the unique solution to the equation

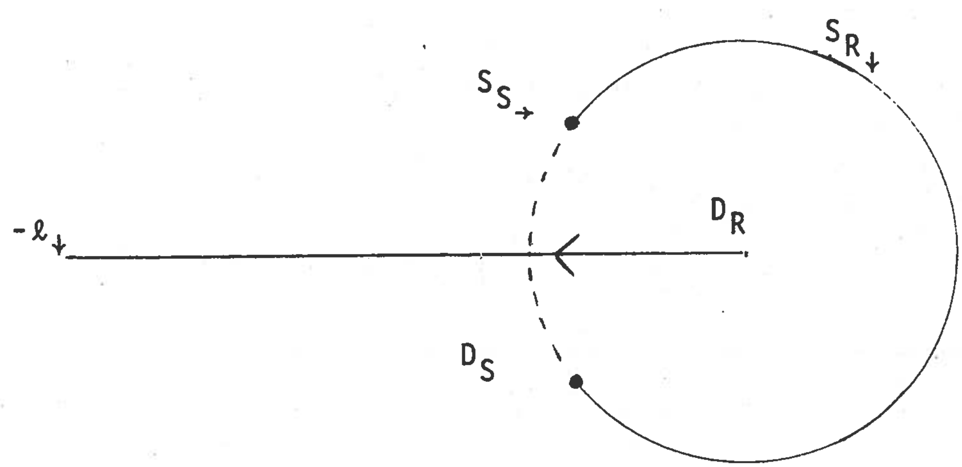
$$\langle i_p{}^* \beta_R(\delta\xi) \; , \; \alpha_R(\delta\xi)\rangle = + \; m$$

which is also satisfied by $\delta\beta_S(\xi)$ since

$$\langle i_p{}^*\delta\beta_S(\xi)\ ,\ \alpha_R(\delta\xi)\rangle = \langle \delta i_p{}^*\beta_S(\xi)\ ,\ \alpha_R(\delta\xi)\rangle = \langle i_p{}^*\beta_S(\xi)\ ,\ \partial\alpha_R(\delta\xi)\rangle$$

$$= \langle i_p{}^*\beta_S(\xi)\ ,\ \alpha_S(\xi)\rangle = +\ m$$

Therefore the proposition is proven if it can be shown that diagram "1" commutes . We examine a normal slice of W through a point $p \in R$:



D_R is a stratified object with two strata, $D_R - S_R$ and S_R . The line $-\ell$ points "from the larger stratum to the smaller stratum" and therefore according to the proof of proposition 2.2.5., the (homological) orientation induced on S_R by a (homological) orientation of D_R is the one for which:

the orientation of $-\ell$ followed by the orientation of S_R equals the orientation of D_R .

However, this is exactly the prescription given above, for finding a coorientation of R given a coorientation of S .

This completes the proof of the proposition.

4.3.3. <u>Definition</u>: If W is a cooriented π-fibre subobject of V and if $\delta W = 0$ then W is called an (intrinsic) <u>geometric cocycle</u>.

4.3.4. <u>Proposition</u>: If ${}^kW$ is a geometric cocycle in V then the homological coorientation of W has a unique lift to $H^k(V,V-W)$ and W therefore determines a cohomology class $\{W\} \in H^k(V)$, and we say W <u>represents</u> $\{W\}$.

<u>Proof</u>: Consider the following diagram:

$$
\begin{array}{ccccccccc}
 & & & & & & & & \downarrow \\
 & & & & & & & & H^{k+1}(V,V-\delta\delta W) = 0 \\
 & & & & & & & & \downarrow \\
\to & H^k(V,V-\delta W) & \to & H^k(V,V-W) & \to & H^k(V-\delta W,V-W) & \to & & H^{k+1}(V,V-\delta W) \quad \to \\
 & \| & & \downarrow & & & \overset{\delta}{\searrow} & & \downarrow \\
 & 0 & & H^k(V) & & & & & H^{k+1}(V-\delta\delta W,V-\delta W) \\
 & & & & & & & & \downarrow
\end{array}
$$

The vertical and horizontal rows are long exact cohomology sequences for a triple. The existence and uniqueness of $\{W\} \in H^k(V,V-W) \to H^k(V)$ depends on the vanishing of the two groups indicated, which follows from:

4.3.5. Proposition: Suppose ${}^kW$ is a π-fibre subobject of the Thom-Mather stratified object V^n (meaning the top co-stratum of V has intrinsic codimension k). If $p < k$ then $H^p(V,V-W) = 0$.

Proof: We proved by decreasing induction on k , the codimension of W .

Case $k = n$: In this case, W consists of a discrete set of points which therefore lie in the nonsingular part of V . Thus, a tubular neighborhood T of W in V consists of a collection of n-discs, so

$$H^p(V,V-W) \cong H^p(T,T-W) \cong H^p(T,\partial T) \cong h^p \text{ (wedge of } n\text{-spheres)} = 0$$

$$\text{if } p < n$$

(The h^p denotes reduced cohomology).

Inductive Step: Assume the proposit on is true whenever $k = \text{cod } W \geq t$ and let W be a π-fibre subobject of V with $k = \text{cod } W = t - 1$. Then $\text{cod } \delta W = t$ so $H^q(V, V-\delta W) = 0$ if $q < t$ by induction. Let $p < k = t - 1$ and consider the long exact cohomology sequence

$$\begin{array}{ccccccccc} \to & H^p(V,V-\delta W) & \to & H^p(V,V-W) & \to & H^p(V-\delta W,V-W) & \to & H^{p+1}(V,V-W) & \to \\ & \| & & & & & & \| & \\ & 0 & & & & & & 0 & \end{array}$$

If S is the top costratum of W, E_S the normal bundle to S and T_S a tubular neighborhood of S in V, then

$$H^p(V,V-W) \cong H^p(V-\delta W,V-W) \cong H^p(T_S,T_S-S) \cong H^p(E_S,E_S-S) = 0$$

Provided $p < t - 1 =$ fibre dimension of E_S.

Q.E.D.

4.3.6. <u>Remark</u>: If W_0 is a π-fibre subobject of V and $\delta W_0 = 0$ then $\delta(W_0 \times [0,1]) = 0$ if $W_0 \times [0,1]$ is considered a π-fibre subobject of $V \times [0,1]$. However, if $W_0 \times [0,1]$ is considered a subobject of $V \times \mathbb{R}$ then $\delta(W_0 \times [0,1]) = W_0 \times \{1\} - W_0 \times \{0\}$. Since $W_0 \times [0,1]$ should be a cobordism between $W_0 \times \{0\}$ and $W_0 \times \{1\}$ the definition of cobordism is a little subtle.

4.3.7. <u>Definition</u>: Two intrinsic cocycles W_1 and W_2 in the Thom-Mather object V are called <u>cobordant</u> if there is a π-fibre cochain $W \subset V \times \mathbb{R}$ (with the product Thom-Mather structure) so that for some $\epsilon < 0$,

(a) $W \cap V \times (-\infty, \epsilon) = W_1 \times (-\infty, \epsilon)$

(b) $W \cap V \times (1-\epsilon, \infty) = W_2 \times (1-\epsilon, \infty)$

(c) $\delta W = 0$ (Modulo reduction.)

According to the above remark, condition (c) may be replaced by

(c') $\delta(W \cap (V \times [0,1])) = W_2 \times \{1\} - W_1 \times \{0\}$ when $W \cap (V \times [0,1])$ is considered a subobject of $V \times \mathbb{R}$.

The main theorem in this chapter states that the set of codimension k geometric cochains on V, modulo the equivalence relation of cobordism, can be identified with the cohomology group $H^k(V)$. The theorem is proven by immediately reducing to the case where V is a Whitney object in some manifold. Therefore the above discussion must first be repeated for Whitney objects.

4.4. Cohomology of a Whitney Stratified Object

4.4.1. Definition: Suppose V^n is a Whitney stratified object in a manifold M^m and suppose a system of control data on V is fixed. Another Whitney object $W \subset M$ is said to be π-fibre relative to V if, for any stratum X of V there is an $\epsilon > 0$ so that

$$W \cap T_X(\epsilon) = \pi_X^{-1}(W \cap X)$$

(where π_X is assumed to be restricted to $T_X(\epsilon)$). In particular, W is transverse to V and $W \cap V$ is an (intrinsic) π-fibre subobject of the stratified object V with its induced Thom-Mather structure. (More about this later.)

A <u>coorientation</u> of W is an orientation and multiplicity of the normal bundle of the top stratum of W. A <u>germ of a stratified object near V</u> is an equivalence class of closed Whitney stratified objects in M, two being considered equivalent if they coincide in some neighborhood of V.

An <u>extrinsic</u> geometric cochain of codimension k on $V^n \subset M^m$ is a germ near V of a π-fibre coriented Whitney object W of dimension $m - k$. (We shall tend to confuse W and the germ (W) it represents.)

The coboundary $\delta(W)$ of such an extrinsic geometric cochain is defined as follows: Choose a representative of W of the germ. Then bW is a π-fibre Whitney object and the coorientation of W induces one on bW (as in section 4.1.). The <u>reduction</u> of bW (i.e., the closure of the union of all top strata of bW with nonzero multiplicity) is denoted δW and the germ (δW) represented by δW is the desired coboundary.

Notice that if $\delta W \cap V = \emptyset$ then δW does not intersect a neighborhood of V, so $(\delta W) = 0$.

An (extrinsic) geometric cocycle on V is an extrinsic geometric cochain whose coboundary is 0. If W is a representative of an extrinsic geometric cocycle and W is closed in some regular neighborhood U of V then W determines a unique cohomology class $\{W\} \in H^k(U) \simeq H^k(V)$ by an argument almost identical to the one in 4.1.

<u>Definition</u>: A cobordism between two reduced (extrinsic) geometric cocycles W_1^{n-k} and W_2^{n-k} on V is a cooriented Whitney object $W^{n+1-k} \subset M\times\mathbb{R}$ which is π-fibre relative to the product system of control data on $V\times\mathbb{R} \subset M\times\mathbb{R}$ and which, for some $\epsilon > 0$, satisfies the conditions

(a) $W \cap M\times(-\infty,\epsilon) = W_1\times(-\infty,\epsilon)$

(b) $W \cap M\times(1-\epsilon,\infty) = W_2\times(1-\epsilon,\infty)$

(c) $\delta W = 0$

Note that, as in definition 4.3.7., we may replace condition (c) by

(c') $\delta(W\cap[0,1]) = W_2\times\{1\} - W_1\times\{0\}$ where $W \cap [0,1]$ is considered a stratified object in $M\times\mathbb{R}$.

We shall prove that the set of extrinsic geometric cocycles of codimension k on V, modulo the relation of cobordism, can be identified with the cohomology group $H^k(V)$.

4.4.2. Lemma: Suppose V^n is a Whitney object in a manifold M^m and a system of control data on V is fixed. Then with respect to the Thom-Mather structure on V (induced by the control data) there is a canonical one-to-one correspondence between intrinsic and extrinsic geometric cochains, cocycles, and cobordisms on V.

Proof: Suppose $W^{m-k} \subset M^m$ is (a representative of) an extrinsic geometric cochain on V. Then W is transverse to V and $W \cap V$ is a intrinsic π-fibre geometric cochain. Each co-stratum ${}^{i}T$ (with codimension i) of $W \cap V$ is the intersection of a stratum A^{m-i} (with codimension i) of W, with all of V:

$$T = A \cap V$$

If E_A is the normal bundle of A in M, and E_T is the normal bundle of T in V then there is a canonical isomorphism

$$E_A | T \cong E_T$$

(because if X is a stratum of V then $T_pM/T_pA \cong T_pX/T_p(A \cap X)$ whenever $p \in X \cap A$). An orientation of E_A thus determines an orientation of E_T .

Furthermore, $bW \cap V$ is the union of all costrata of $W \cap V$ with codimension $\geq i + 1$ and in fact

$$(\delta_{extrinsic} W) \cap V = \delta_{intrinsic}(W \cap V) \quad .$$

The same remarks apply to cobordisms.

Conversely, suppose $W' \subset V$ is an <u>intrinsic</u> geometric cochain. Choose numbers $\epsilon_X > 0$ for each stratum X of V so the intrinsic π-fibre condition holds for $W' \cap T_X(\epsilon_X)$, i.e.,

$$W' \cap T_X(\epsilon_X) \cap V = \pi_X^{-1}(W' \cap X) \cap T_X(\epsilon_X) \cap V \quad .$$

Let S' be a costratum of W' . Then

$$S \equiv \bigcup \{\pi_X^{-1}(S' \cap X) | X \text{ is a stratum of } V\}$$

is a manifold in M which intersects V transversally in S' .

Let

$$W \equiv \bigcup\{\pi_X^{-1}(W'\cap X) \mid X \text{ is a stratum of } V\} \quad .$$

Then W is stratified with one stratum S for each costratum S' of W'. W is transverse to V, $W \cap V = W'$ and W satisfies the Whitney conditions, for if $R' < S'$ are costrata of W' then $R' \cap X < S' \cap X$ satisfy the Whitney conditions for each stratum X of V, and so $\pi_X^{-1}(R\cap X) < \pi_X^{-1}(S\cap X)$ satisfies the Whitney conditions. This decomposition of W is locally finite, and so by Mather [9] the condition of the frontier holds for W.

Finally, if $E_{S'}$ is the normal bundle of a costratum S' of W' and if X is a stratum of V then

$$E_S|(S\cap T_X) \cong \pi_X^{*}E_{S'}$$

(where E_S is the normal bundle of the stratum S of W associated to the costratum S' of W'). In particular, an orientation of $E_{S'}$ determines an orientation of E_S.

Thus, the germ represented by W is an extrinsic geometric cochain on V. It is clear that the two procedures

extrinsic cochain $W \longrightarrow$ intrinsic cochain $W \cap V$

and

intrinsic cochain $W' \longrightarrow$ extrinsic cochain germ $(\pi^{-1}(W'))$

are inverses to each other, which concludes the lemma.

Note that if W and W' are related in this way then the cohomology classes $\{W\}$ and $\{W'\}$ in $H^k(V)$ (defined in sections 4.2. and 4.1. respectively) coincide.

4.5.1. Theorem: Let V be a Thom-Mather (respectively Whitney) stratified object. Then every cohomology class $\alpha \in H^k(V;Z)$ can be represented by an intrinsic (respectively extrinsic) geometric cocycle of codimension k on V, and two such geometric cocycles are cobordant if and only if they represent the same cohomology class.

Proof: If V is a Thom-Mather stratified object, then it can be embedded as a Whitney object V' in $\mathbb{R}^n$ in such a way that its Thom-Mather structure is induced from a system of control data. (See appendix 1.) It is then possible (by 0.5.) to find a regular neighborhood U of V' in $\mathbb{R}^n$ diffeomorphic to the interior of a smooth manifold (also denoted U) with smooth boundary ∂U . According to the preceding lemma, there is therefore a one-to-one correspondence between intrinsic cochains, cocycles and cobordisms on V and extrinsic cochains, cocycles, and cobordisms on V' .

Similarly, if V is originally a Whitney object embedded in a (possibly non-orientable) manifold then by embedding further into Euclidean space and arguing as in the preceding lemma, we may assume V is contained in an orientable manifold.

Therefore the theorem need only be proven for the case of a <u>Whitney</u> object V in the interior of a smooth <u>oriented</u> manifold U^N with smooth boundary ∂U , for which the embedding $V \to U$ is a homotopy equivalence.

Then $H^k(V) \cong H^k(U) \cong H_{N-k}(U,\partial U)$ by Lefshetz duality. According to theorem 2.6., every relative homology class in $H_{N-k}(U,\partial U)$ can be represented by an $N - k$ dimensional (Whitney) relative geometric <u>cycle</u> $W \subset (U, \partial U)$ and two such geometric cycles are cobordant precisely when they represent the same homology class.

An orientation of the top stratum X of a relative geometric cycle W^{N-k} determines a unique orientation of the normal bundle E_X of X in U compatible with the exact sequence

$$0 \to TX \to TU|X \to E_X \to 0$$

and therefore W can be considered a geometric <u>cochain</u> in U. In fact, W is a cocycle for the following reason:

There are two types of top strata of bW: those in ∂U and those in $U - \partial U$. The induced orientations cancel on the top strata of bW which are contained in $U - \partial U$ since W is a cycle, so the induced coorientations must cancel on such strata also. The induced orientations do not necessarily cancel on those top strata of bW which are contained in ∂U, however these strata have intrinsic codimension k and therefore are <u>not</u> part of δW. We conclude $\delta W = 0$.

There is thus a one-to-one correspondence between relative geometric cycles on $(U,\partial U)$ and geometric cocycles on U ; and the same applies to cobordisms. If $GH_{N-k}(U,\partial U)$ denotes cobordism classes of $N-k$ dimensional relative geometric cycles on $(U,\partial U)$ and $GH^k(U)$ denotes cobordism classes of codimension k geometric cocycles on U , we have a diagram:

$$\begin{array}{ccc} GH_{N-k}(U,\partial U) & \xleftarrow{\ \cong\ } & GH^k(U) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle \text{Thom Class}} \\ H_{N-k}(U,\partial U) & \xleftarrow[\cong]{\ \cap[U]\ } & H^k(U) \cong H^k(V) \end{array}$$

This diagram commutes (a fact which may be checked locally).

We now proceed to "deform" the geometric cochains on U into π-fibre geometric cochains on V .

<u>Claim 1</u>: Every geometric cochain on U is cobordant to a geometric cochain which is transverse to V , and two geometric cochains transverse to V are cobordant if and only if there is a cobordism between them which is transverse to V (meaning transverse to $V\times\mathbb{R}$ in $U\times\mathbb{R}$). Therefore if $TH^k(U)$ denotes <u>transverse</u> cobordism classes of codimension k geometric cocycles <u>transverse</u> to V , then $TH^k(U) \cong GH^k(U)$.

Proof of Claim 1: Let W be a geometric cochain in U. Then W is transverse to V near ∂U since $V \cap \partial U = \emptyset$. Let T be a collar of ∂U which does not intersect V and within which W is a product. Then by applying the corollary of the transversality theorem to the identity: $U - T \to U - T$, we obtain a one-parameter family of diffeomorphisms $\phi_t: U - T \to U - T$ $(t \in [0,1])$ so that ϕ_0 = identity and $\phi_1(W \cap (U-T))$ is transverse to V, and ϕ_t may be chosen to be the identity near the edge of T (for all $t \in [0,1]$). Thus, be setting $\phi_t(p) = p$ for $p \in T$, ϕ_t extends to a diffeomorphism of the manifold with boundary U. Thus, $\phi(W \times [0,1])$ is a cobordism from W to $\phi_1(W)$ which is transverse to V.

Similarly, if W_1 and W_2 are geometric cocycles in U which are transverse to V and which are cobordant, then the cobordism $Q \subset U \times [0,1]$ is transverse to $V \times [0,\epsilon)$ and to $V \times (1-\epsilon,1]$ for some $\epsilon > 0$. By the transversality theorem, then, $Q \cap U \times (0,1)$ is cobordant to a new stratified object $Q' \subset U \times (0,1)$ which is transverse to $V \times (0,1)$ and which coincides with Q on $U \times (0,\frac{\epsilon}{2})$ and on $U \times (1-\frac{\epsilon}{2},1)$. Thus,

$\overline{Q^{T}} \subset U\times[0,1]$ is a cobordism from W_1 to W_2 with the desired properties, completing the proof of claim 1.

We digress for a moment to state a technical tool:

<u>Lemma</u>: Let V be a Whitney stratified object with a fixed system of control data. Suppose V is contained in a manifold U, and let $W \subset U$ be an oriented Whitney object. Suppose W is transverse to V. Then W is cobordant to a Whitney object W' which satisfies the π-fibre condition with respect to V, and the whole cobordism Q may be chosen to be transverse to $V\times(0,1)$ in $U\times(0,1)$. Furthermore, if K is a closed subset of U and if L is an open subset of U, $L \supset K$, and if $W \cap L$ satisfies the π-fibre condition then the cobordism can be chosen to be constant on $W \cap K$, i.e.,

$$Q \cap (K\times[0,1]) = (Q\cap(K\times\{0\}))\times[0,1] = (W\cap K)\times[0,1]$$

It will follow from the proof that in fact

$$W' = \psi^{-1}(W)$$

for a certain smooth map $\psi\colon U \to U$ which is transverse to W.

The proof of this lemma will be postponed for a moment.

Returning to the problem of geometric cohomology, the following claim will complete theorem 4.3.

Claim 2: Every geometric cochain W on U which is transverse to V, is cobordant to a geometric cochain which is π-fibre with respect to V, and the cobordism may be chosen to be transverse to $V\times(0,1)$ in $U\times(0,1)$. If W_1 and W_2 are geometric cochains which are π-fibre with respect to V and if there is a cobordism between W_1 and W_2 which is transverse to $V\times(0,1)$ then there is a cobordism between them which is π-fibre with respect to $V\times(0,1)$. (Recall that the control data on $V\times[0,1]$ is the restriction of the product control data on $V\times\mathbb{R}$). Therefore, if $PH^k(U)$ denotes π-fibre cobordism classes of codimension k π-fibre (with respect to V) geometric cochains on V, we have $PH^k(U) \cong TH^k(U)$.

Proof of Claim 2: If W is a geometric cochain transverse to V, then according to the lemma, W is cobordant to a geometric cochain which is π-fibre with respect to V, and the cobordism may be chosen transverse to $V\times(0,1)$. Similarly, if W_1 and W_2 are π-fibre relative to V and if $Q \subset U\times[0,1]$ is a cobordism between them which is transverse to $V\times(0,1)$ in $U\times(0,1)$, then Q is cobordant to a geometric cochain $Q' \subset U\times[0,1]$ which is π-fibre relative to V, and Q' may be chosen to coincide with Q near the beginning and end of the cobordism. Therefore Q' is the desired π-fibre cobordism.

Claim 3: $PH^k(U) \cong PH^k(U-\partial U)$, where $PH^k(U-\partial U)$ denotes the π-fibre) cobordism classes of geometric cocycles in $U - \partial U$ which are π-fibre relative to V. (The point is that a geometric cocycle in $U - \partial U$ does not necessarily follow the lines of a collaring of ∂U.)

Proof: Let T be the collaring of ∂U in U, i.e., T is the image of a smooth embedding

$$\phi\colon \partial U\times[0,1) \to U$$

with $\phi(x,0) = x$ for all $x \in \partial U$ and image $(\phi) = T$ is an open set.

Let $\eta\colon T \to [0,1)$ and $f\colon T \to \partial U$ be the projections for which

$$\phi(f(p),\eta(p)) = p \quad \text{for all} \quad p \in T$$

and let $R = \phi(\partial U\times\{\tfrac{1}{2}\})$.

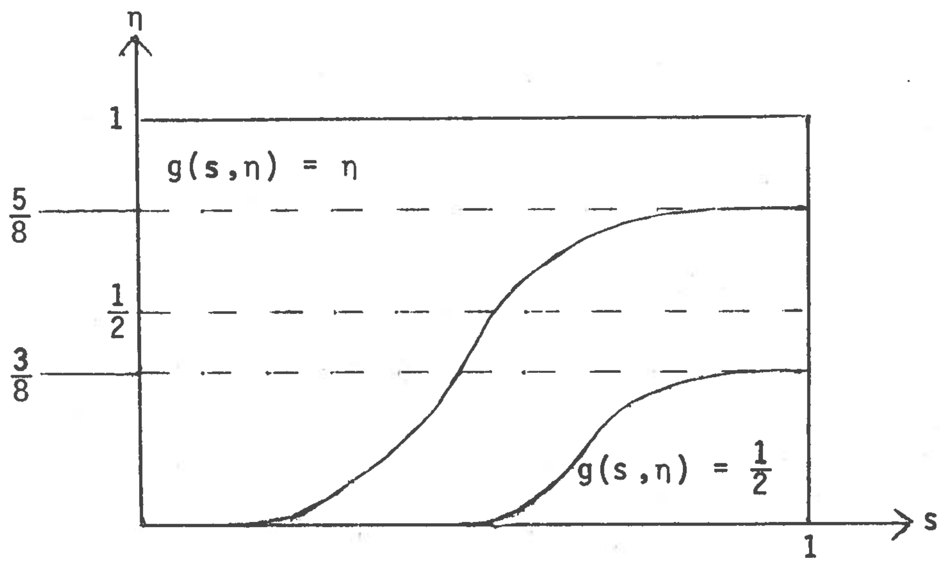
There is an obvious map

$$PH^k(U) \to PH^k(U-\partial U)$$

which we first show is surjective.

Let W be a geometric cocycle in $U - \partial U$. By the transversality theorem, W can be altered slightly in $\phi(\partial U\times(\frac{1}{4},\frac{3}{4}))$ so as to be transverse to $R = \phi(\partial U\times\{\frac{1}{2}\})$ and by the lemma in claim 1, it is cobordant to a geometric cocycle W' which is π-fibre relative to R (meaning that it follows the collaring lines in some neighborhood of R) and we may assmme it remains π-fibre relative to V . It is necessary only to "straighten out W' " in the region $\phi(\partial U\times[0,\frac{1}{2}])$.

Choose a smooth function $g\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ with $\eta \leq g(S,\eta)$ and $\frac{\partial g}{\partial \eta}(S,\eta) \neq 0$ except in the shaded region indicated below, and choose g so as to assume the values indicated:



Define $\psi: U\times[0,1] \to U$ by

$$\psi(x,S) = \begin{cases} \phi(f(x),\ g(s,\eta(x))) & \text{if } x \in T \\ x & \text{if } x \notin T \end{cases}$$

Then ψ is transverse to W' and $\psi^{-1}(W'\times[0,1])$ is a cobordism between $\psi_0^{-1}(W') = W'$ and $\psi_1^{-1}(W')$ which coincides with W' in a neighborhood of V and which follows the lines of the collaring near ∂U . $\psi_1^{-1}(W')$ is the desired geometric cycle in $PH^k(U)$.

An identical argument applies to cobordisms to complete the proof of claim 3.

Claim 4: $PH^k(U-\partial U) \cong GH^k(V)$.

Proof: There is an obvious map $PH^k(U-\partial U) \to GH^k(V)$ since each π-fibre geometric cocycle in U defines a germ near V . This map is surjective as shown in the following argument:

Let W be a representative of a germ of a π-fibre geometric cocycle near V and suppose W obeys the π-fibre condition in the region

$$U' = \cup\{T_X(\epsilon_X) | X \text{ is a stratum of } V\}$$

for some numbers $\epsilon_X > 0$. Let

$$U'' = \cup\{T_X(\tfrac{\epsilon_X}{2}) | X \text{ is a stratum of } V\} .$$

It is possible to find a collaring

$$\phi: \partial U \times [0,1) \to U$$

of ∂U in U so that $\phi(x,0) = x$ for all $x \in \partial U$

and

$$\phi(x,t) \in U'' \quad \text{whenever} \quad x \in \partial U \quad \text{and} \quad t > \frac{1}{2}$$

$$\phi(x,t) \in U' \quad \text{whenever} \quad x \in \partial U \quad \text{and} \quad t > \frac{1}{4}$$

by using the canonical deformation retraction

$$H: U\times[0,1] \to U$$

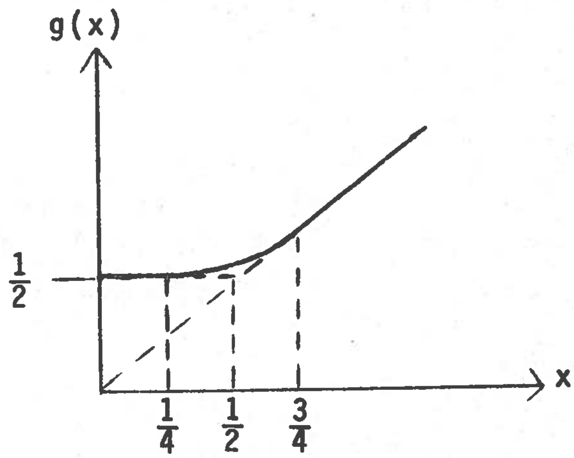
of section 0.5. and noting that $H|((U-V)\times[0,1])$ is smooth and embeds each $(U-V)\times\{t\}$.

ϕ can in fact be chosen so that

$$R \equiv \phi(\partial U\times\{\tfrac{1}{2}\})$$

is transverse to W , by the transeersality theorem. Let $T = \text{image}\ (\phi)$.

Choose a smooth function $g\colon \mathbb{R}\to\mathbb{R}$ with $g(x) = \frac{1}{2}$ for $x \leq \frac{1}{4}$ and $g(x) = x$ for $x \geq \frac{3}{4}$ and $g'(x) > 0$ for $x > \frac{1}{4}$.



Let $\eta: T \to [0,1]$ and $f: T \to \partial U$ be the projections for which

$$\phi(f(p),\eta(p)) = \quad \text{for all } p \in T \ .$$

Define $\psi: U \to U$ by

$$\psi(p) = \begin{cases} \psi(f(p),g(\eta(p))) & \text{if } p \in T \\ p & \text{if } p \notin T \end{cases}$$

Then ψ is transverse to W and $\psi^{-1}(W)$ is the desired geometric cocycle in $U - \partial U$ satisfying $\psi^{-1}(W) \cap U'' = W \cap U''$.

To show $PH^k(U-\partial U) \to GH^k(V)$ is one to one, suppose W_1 and W_2 are geometric cocycles in $U - \partial U$ and suppose there is a germ W of a π-fibre object contained in some neighborhood U' of V and that W is a cobordism between $W_1 \cap U'$ and $W_2 \cap U'$. We must show W_1 and W_2 are cobordant.

By choosing an appropriate collaring of ∂U which "extends into the region U'" as before, and by applying the construction just given together with an argument similar to that in claim 3, we can find a smooth map $\psi: U \to U$ so that

(a) ψ is transverse to W_1 and to W_2

(b) $\psi\times(\text{identity}): U\times[0,1] \to U\times[0,1]$ is transverse to W

(c) $\psi^{-1}(W_1 \cap U')$ is cobordant to W_1

(d) $\psi^{-1}(W_2 \cap U')$ is cobordant to W_2

(e) $(\psi\times(\text{identity}))^{-1}(W\times[0,1])$ is a cobordism between $\psi^{-1}(W_1 \cap U')$ and $\psi^{-1}(W_2 \cap U')$.

Therefore W_1 and W_2 are cobordant, completing claim 4.

In summary, there is a sequence of one-to-one correspondences

$$H^k(V) \simeq H^k(U) \simeq H_{N-k}(U,\partial U) \simeq GH_{N-k}(U,\partial U) \simeq GH^k(U)$$

$$\simeq TH^k(U) \simeq PH^k(U) \simeq PH^k(U-\partial U) \simeq GH^k(V)$$

and it can be checked that the composition coincides with the "representation" map defined in proposition 4.3.4.

This completes theorem 4.5., except for the proof of the lemma.

Proof of Lemma: We shall construct a sequence of cobordisms between Whitney objects $W = W_0, W_1, W_2, \ldots, W_n = W'$ so that for each i there is a smooth mapping $\psi_i: U \to U$ with

(a) $W_{i+1} = \psi_i^{-1}(W_i)$

(b) each cobordism between W_i and W_{i+1} is transverse to $V\times(0,1)$ in $U\times(0,1)$

(c) W_i satisfies the π-fibre condition with respect to the i-skeleton of V

(d) $W \cap K = W_1 \cap K$

(These cobordisms fit together and by reparametizing the time scale, they yield the desired cobordism as a subset of $U\times[0,1]$) .

<u>Construction of the k^{th} Cobordism:</u>

If W_{k-1} is a closed Whitney object in U , and W_{k-1} is transverse to V and satisfies the π-fibre condition over the $k - 1$ skeleton of V , then there is a neighborhood Q of the $k - 1$ skeleton of V for which $W_{k-1} \cap Q$ satisfies the π-fibre condition with respect to $V \cap Q$. To see this, suppose $X < Y$ are strata of V with $\dim (X) \leq k - 1$ and $\dim (Y) \geq k$. There is an $\epsilon_X > 0$ so that in $T_X(\epsilon)$ we have $W_{k-1} \cap T_X(\epsilon_X) = \pi_X^{-1}(W_{k-1}\cap X)$. Let $\pi_Y\colon T_Y \to Y$ be the tubular neighborhood of Y . Then

$$\begin{aligned}\pi_Y^{-1}(W_{k-1}\cap Y\cap T_X(\epsilon_X)) &= \pi_Y^{-1}(\pi_X^{-1}(W_{k-1}\cap X)\cap Y\cap T_X(\epsilon_X))\\ &= \pi_Y^{-1}\pi_X^{-1}(W_{k-1}\cap X) \cap \pi_Y^{-1}(Y) \cap \pi_Y^{-1}(T_X(\epsilon_X))\\ &= \pi_X^{-1}(W_{k-1}\cap X) \cap T_Y \cap \pi_Y^{-1}(T_X(\epsilon_X))\\ &= W_{k-1} \cap T_X(\epsilon_X) \cap T_Y\end{aligned}$$

proving that $W_{k-1} \cap T_X(\epsilon_X)$ is π-fibre with respect to Y .

Therefore, to construct W_k it is only necessary to cobord W_{k-1} near $Y - T_X(\epsilon_X)$ where Y is the union of strata of dimension k . Let R and S be open neighborhoods of the $k-1$ skeleton of V so that $R \subset \overline{R} \subset S \subset \overline{S} \subset Q$. Then $Y - \overline{R}$ is a closed submanifold of $U - \overline{R}$ so there is a locally finite atlas $\{(U_\alpha', \phi_\alpha)\}$ on $U - \overline{R}$ with

(1) U_α' an open subset of U and $\phi_\alpha : U_\alpha' \to \mathbb{R}^N$ a diffeomorphism

(2) If $Y \cap U_\alpha' \neq 0$ then

$$\phi_\alpha(Y \cap U_\alpha') = \{x \in \phi_\alpha(U_\alpha') | x_{k=1} = \ldots = x_N = 0\}$$

(3) ϕ_α trivializes the tubular neighborhood T_Y so the fibres of T_Y become identified with the affine space perpendicular to $\phi_\alpha(Y \cap U_\alpha')$ and π_Y becomes identified with the projection

$$\pi(x_1, x_2, \ldots, x_N) = (x_1, x_2, \ldots, x_k, 0, \ldots, 0) \ .$$

Let $f(x_1, \ldots, x_N) = (x_{k+1}, \ldots, x_N) \in \mathbb{R}^{N-k}$ be the local "projection to the fibre."

For each stratum C of W_{k-1}, the smooth function

$$(f\circ\phi_\alpha)|C\colon C \to \mathbb{R}^{N-k}$$

has surjective differential at each point of $C \cap Y \cap U_\alpha'$ and hence in a neighborhood of $C \cap Y \cap U_\alpha'$. W has locally finitely many strata, so there is an open set $T \subset T_Y$ so that for <u>any</u> stratum C of W_{k-1},

$$(f\circ\phi_\alpha|C)\colon C \to \mathbb{R}^{N-k}$$

is a submersinn on $C \cap T$. Let $\eta\colon Y \to \mathbb{R}$ be a smooth function so that

$$\{x \in T_Y | \rho_Y(x) < \eta(\pi_Y(x))\} \subset T$$

and define

$$T_1 \equiv \{x \in T_Y | \rho_Y(x) < \tfrac{1}{3}\eta(\pi_Y(x))\}$$

$$T_2 \equiv \{x \in T_Y | \tfrac{1}{3}\eta(\pi_Y(x)) \leq \rho_Y(x) < \tfrac{2}{3}\,(\pi_Y(x))$$

$$T_3 \equiv \{x \in T_Y | \tfrac{2}{3}\eta(\pi_Y(x)) < \rho_Y(x) < \eta(\pi_Y(x))\}$$

Choose smooth functions θ_1 and θ_2 so that

$\theta_1: U - R \to [0,1]$ and $\theta_1(p) = 0$ if $p \in T_1$

$\theta_1(p) = 1$ if $p \in T_3$ or if $p \notin T_1 \cup T_2 \cup T_3$

$\theta_2: Y \to [0,1]$ and $\theta_2(p) = 0$ if $p \in R$

$\theta_2(p) = 1$ if $p \in Y - S$

Define $\psi: M \to M$ by

$$\psi(x) = \theta_2(\pi_Y(x))[\theta_1(x)\cdot x + (1-\theta_1(x))\pi_Y(x)] + [1-\theta_2(\pi_Y(x))]x$$

if $x \in T_Y - R$

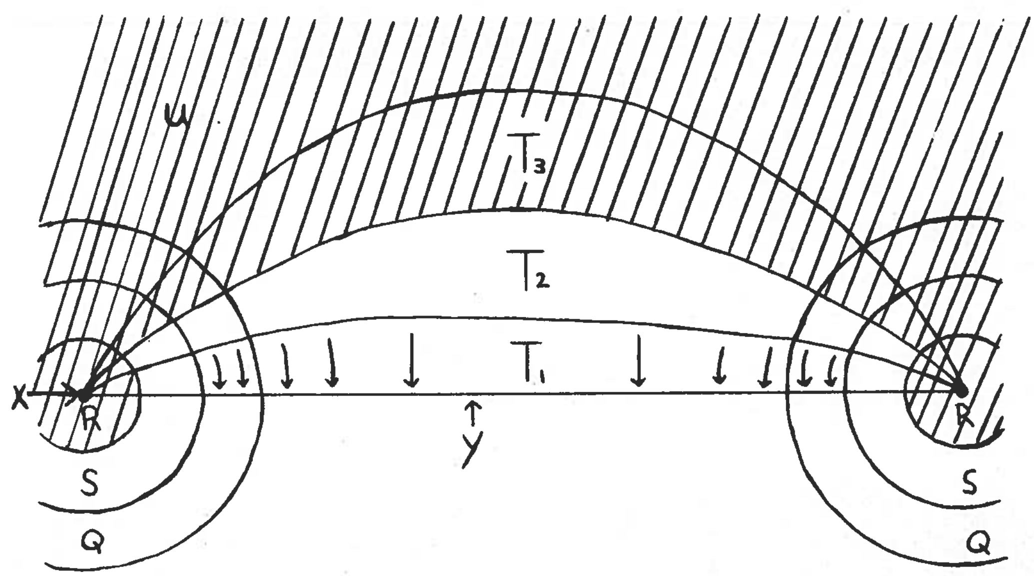
and

$\psi(x) =$ if $x \in R$ or if $x \notin T_Y$.

(See diagram.)

Here addition and scalar multiplication are computed with respect to the vectorbundle structure of T_Y . If $x \in T_1$ and if $\pi_Y(x) \notin S$ then $\psi(x) = \pi(x)$. If $x \in R \cup T_3$ then $\psi(x) = x$. Furthermore, $\psi(\pi_Y(x)) = \pi_Y(\psi(x))$ whenever defined.

Diagram: ψ is the identity in the shaded region and is the projection in the region with arrows.



In any of the charts chosen, the local description of ψ is given by

$$\psi(x_1,\dots,x_N) = (x_1,\dots,x_k,{}^*,\dots,*)$$

so $D\psi(x_1,\dots,x_N)$ has a matrix

$$\left[\begin{array}{c|c} I & * \\ \hline 0 & * \end{array}\right]$$

ψ is therefore transverse to W_{k-1} on T (since the local projection to the fibre is a submersion on each stratum of W_{k-1}). ψ is the identity outside T and is therefore globally transverse to W_{k-1}. Furthermore,

$$W_k \equiv \psi^{-1}(W_{k-1})$$

is the desired object: it coincides with W_{k-1} outside

$T \cap (U-R)$ and is π-fibre with respect to $Y - S$ since

$$\pi_Y^{-1}(W_k \cap Y-S) \cap T_1 = W_k \cap T_1 \cap \pi_Y^{-1}(Y-S) \quad .$$

W_k is also π-fibre with respect to $Y \cap Q$ since $\pi_Y^{-1} \circ \psi^{-1} = \psi^{-1} \circ \pi_Y^{-1}$ and W_k is transverse to V .

W_k and W_{k-1} are cobordant: Let $\theta_3\colon [0,1] \to [0,1]$ be a smooth function with $\theta_3[0,\epsilon] = 0$ and $\theta_3[1-\epsilon,1] = 1$ for small ϵ . Then if we set

$$\Phi\colon U\times[0,1] \to U$$

$$\Phi(p,t) \equiv \begin{cases} \theta_3(t)p + (1-\theta_3(p))\psi(P) & \text{for } p \in T_Y \\ p & \text{for } p \notin T_Y \end{cases}$$

then it is clear that Φ is transverse to W_{k-1} , $\Phi_0 = \psi$ and Φ_1 = the identity, so $\Phi^{-1}(W_{k-1})$ is a cobordism between W_{k-1} and W_k . Since both W_{k-1} and W_k are transverse to V , it is possible to find a new cobordism between W_{k-1} and W_k which is transverse to $V\times(0,1)$ in $U\times(0,1)$. (The cobordisms are constant at the ends of the time interval to permit their inclusions in a long string of cobordisms without necessitating a verification of the Whitney conditions at the ends.)

Remark: There is in general no diffeomorphism of T that will take W_{k-1} to W_k although they are homeomorphic.

Finally notice that if W_{k-1} were originally π-fibre over some open set L containing a closed subset K then W_k can be chosen to coincide with W_{k-1} on K by choosing ψ to be the identity on K — the proof procedes as before with $K \cup R$ in place of R .

This concludes the construction of W_k . It coincides with W_{k-1} near the $k - 1$ skeleton of V so it is π-fibre with respect to all strata in the k-skeleton of V . Q.E.D.

4.6. The Group Structure on Cohomology

4.6.1. Proposition: Suppose W_1 and W_2 are geometric cocycles in the Thom-Mather stratified object V . According to theorem 6.4., W_1 is cobordant to a geometric cocycle $W_1' \subset V$ so that

(a) For each stratum X of V , $W_1' \cap X$ is transverse to $W_2 \cap X$ in X

(b) $W_1 \cap W_2$ is a geometric cocycle in V .

In this case it is easily seen that $W_1 \cup W_2$ is a geometric cocycle in V by an argument similar to that of section 4.1. Furthermore $\{W_1 \cup W_2\} = \{W_1\} + \{W_2\}$ in the cohomology of V , as can be seen by combining section 4.1. with the Mayer-Vietoris sequence for cohomology:

$$\to H^k(V-\delta W_1 , V-W_1) \oplus H^k(V-\delta W_2, V-W_2)$$

$$\to H^k(V-(\delta W_1 \cup \delta W_2), V-(W_1 \cup W_2)) \to$$

as in Spanier [13], page 239.

4.7. <u>Relative Cohomology</u>

4.7.1. <u>Definition</u>: Let V be a connected Thom-Mather stratified object and suppose $A \subset V$ is a closed union of strata. A <u>relative</u>

geometric cochain $W \subset (V,A)$ is a cooriented π-fibre sub-stratified object whose reduction does not intersect A . A relative cobordism between two relative geometric cochains is a cobordism $W' \subset V\times[0,1]$ between the cochains for which the reduction of W' does not intersect $A\times[0,1]$. The set of relative cobordism classes of relative geometric cochains of codimension i is denoted $GH^i(V,A)$.

4.7.2. Proposition: There is a one-to-one correspondence $GH^i(V,A) \cong H^i(V,A)$.

Proof: Let V/A be the space obtained by collapsing A to a point and let $P: V \to V/A$ be the collapsing map. Then V/A admits the structure of a Thom-Mather object. (If U is a regular neighborhood of A in V then the canonical retraction $H: U \to A$ can be used to show that there is a neighborhood of $P(A)$ in V/A which is homeomorphic to the cone over a certain Thom-Mather stratified object.) Therefore a π-fibre geometric cochain in V/A may not intersect $P(A)$ unless it has codimension 0 . There is thus a one-to-one correspondence between geometric cochains, cocycles, and cobordisms on V/A and relative geometric cochains, cocycles, and cobordisms on (V,A) and we have

$$GH^i(V,A) \simeq GH^i(V/A) \simeq H^i(V/A) \simeq H^i(V,A) \quad \text{if} \quad i \geq 1 \quad .$$

For $i = 0$, all groups are 0 . Q.E.D.

4.7.3. Remark: The connecting homeomorphism $GH^i(A) \to GH^{i+1}(V,A)$ can be described as follows: Let W be an intrinsic geometric cocycle in A . Let $W' = \cup\{\pi_X^{-1}(W) | X$ is a stratum of $A\} \subset V$. Then $\delta W' \cap A = \emptyset$ since $W' \cap A = W$ is a geometric cocycle. Therefore $\{\delta W'\} \in GH^{i+1}(V,A)$ is the desired coboundary.

CHAPTER 5

FUNCTORIALITY

In this chapter, the pullback of a cohomology class by a continuous map f will be interpreted geometrically as the primage of a geometric cocycle by a small perturbation of f. A geometric proof that homotopic maps induce the same homomorphism on cohomology will be given.

5.1.1. <u>Proposition</u>. Suppose V_1 and V_2 are Whitney stratified objects (in some manifold) with fixed systems of control data. Let U_i be a regular neighborhood of V_i $(i = 1,2)$ and suppose ${}^kW \subset U_2$ is a (representative of) an extrinsic geometric cocycle of codimension k on W_2. Let $f: U_1 \to U_2$ be a smooth map. Then in any C^1-neighborhood of $f \in C^\infty(U_1,U_2)$ there is a map $g: U_1 \to U_2$ so that

(a) g is homotopic to f;

(b) g takes V_1 transversally to W;

(c) $g^{-1}(W)$ is a geometric cocycle in U_1, which is transverse to V_1 and represents the cohomology class $f^*\{W\} = g^*\{W\}$ in the sense of the theories $TH^k(U_1) = GH^kH(U_1)$ (see Chapter 4).

(d) Any C' sufficiently small perturbation of g satisfies conditions (a), (b), and (c).

We note that, according to the lemma of section 4.3 there is a cobordism between $g^{-1}(W)$ and a geometric cocycle which is π-fibre relative to V_1 and which therefore represents

$f^*\{W\}$ in the sense of the theory $GH^k(V_1)$.

Remark. Such functions $f: U_1 \to U_2$ arise in the following way: if $h: V_1 \to V_2$ is a continuous map then there is a continuous extension to a map $U_1 \to V_2 \subset U_2$ which may then be smoothed to give $f: U_1 \to U_2$. Therefore, there is at least one such f for every homotopy class of maps $V_1 \to V_2$.

Proof of 5.1. The existence of $g: U_1 \to U_2$ which is homotopic to f and takes V_1 transversally to W follows from the transversality theorem (as does property (d)). Each stratum X' of $W' = g^{-1}(W)$ is of the form $g^{-1}(X)$, for some stratum $X \circ fW$, and this correspondence preserves codimension and normal bundles, i.e. if E is the normal bundle to a stratum ${}^kW$ of W and if $X' = g^{-1}(X)$ then $E' = (g| X')^*(E)$ is the normal bundle to ${}^kX'$. Fibre orientations and multiplicities on E determine fibre orientations and multiplicities on E' and the resulting Thom classes S and S' are related by

$$dg^*(S) = S' \in H^k(E',E'-X)$$

where $dg: E' \to E$ is the vectorbundle map induced by g. (This follows from the naturality of the Thom isomorphism theorem.)

The following diagram commutes up to homotopy:

$$\begin{array}{ccc} (T_{X'},T_{X'}-X') & \xrightarrow{g} & (T_X,T_X-X) \\ \uparrow & & \uparrow \\ (E',E'-X') & \xrightarrow{dg} & (E,E-X) \end{array}$$

so the classes S and S' define coorientations R and R' of W and $g^{-1}(W)$ which are related by

$$g^*(R) = R' \in H^k(T_{X'},T_{X'}-X) \cong H^k(U'-\delta W',U'-W')$$

The coboundary of R and R' is calculated locally on the normal slices through the codimension $k+1$ strata of W and W', (see section 4.3) and g gives a natural identification between the normal slices through W and the normal slices through W'. Therefore $\delta g^*(R) = g^*(\delta R)$.

We obtain two copies of the following diagram (a primed and an unprimed version) with commuting g^* maps between them: (as in section 4.3)

$$\begin{array}{ccccccc} & & & & & 0 & \\ & & & & & \downarrow & \\ 0 \to & H^k(U,U-W) & \to & H^k(U-\delta W,U-W) & \to & H^{k+1}(U,U-\delta W) & \to \\ & \downarrow & & & \searrow^{\delta} & \downarrow & \\ & H^k(U) & & & & H^{k+1}(U-\delta\delta W,U-\delta W) & \\ & & & & & \downarrow & \end{array}$$

Consequently, if W is a cocycle then $W' = g^{-1}(W)$ is a cocycle and $\{g^{-1}(W)\} = g^*\{W\}$. Q.E.D.

5.1.2. Remark. It is not true that a generic map $g: U_1 \to U_2$ pulls back π-fibre objects to π-fibre objects because the π-fibre condition is not a "generic condition", although it is in some sense equivalent to "transverse intersection" for the purposes of cohomology. The best theorem to be obtained for generic maps is the above which states that a generic map pulls back transverse intersections to transverse intersections.

Nevertheless, a completely intrinsic description of the cohomology pullback can be obtained by demanding "genericity subject to the π-fibre condition". The next theorem shows that if we choose a specific but C^o-small perturbation of a given map $f: V_1 \to V_2$ then the resulting map will pull back π-fibre objects to π-fibre objects and this property is preserved in some sense under small π-fibre perturbations. However, the theorem does not properly address the question as to what is a generic map between stratified objects.

5.2.1. Theorem. Suppose V_1 and V_2 are Thom-Mather stratified objects and $f: V_1 \to V_2$ is a continuous map. Let ${}^kW \subset V_2$ be a (π-fibre) intrinsic geometric cocycle representing the cohomology class

$$\{W\} \in GH^k(V_2) = H^k(V_2).$$

Then in any C^o-neighborhood of f there is a map $g: V_1 \to V_2$ which is homotopic to f and for which $g^{-1}(W)$ is a geometric cocycle in V_1 representing the cohomology class

$$\{g^{-1}(W)\} = g^*\{W\} = f^*\{W\} \in GH^k(V_1) = H^k(V_1).$$

Proof: According to Appendix 1 there is an embedding $f: V_1 \to \mathbb{R}^n$ so that $f(V_1)$ is a Whitney object, with one stratum $f(X)$ for each stratum X of V_1, and there is a system of control data $\{T_{f(X)}, \pi_{f(X)}, \rho_{f(X)}\}$ on $f(V_1)$ so that for each stratum X of V_1 and each $p \in T_X$,

$$\pi_{f(X)}(f(p)) = f(\pi_X(p)).$$

For the remainder of this proof we identify V_1 with its embedded image $f(V_1)$ and similarly assume V_2 is a Whitney object in some Euclidean space.

By multiplying each distance function ρ_X by a smooth scale factor $f: X \to \mathbb{R}$, we may choose $\epsilon_X > 0$ so that W satisfies the π-fibre condition in all of $T_X(\epsilon_X)$ whenever X is a stratum of V_2 (rather than in some possibly smaller neighborhood of each stratum). Let

$$U_2 \equiv \cup\{T_X(\epsilon_X) \mid X \text{ is a stratum of } V_2\}.$$

U_2 is a regular neighborhood of V_2. Let U_1 be a regular neighborhood of V_1 and let $H_2: U_2 \to V_2$ be the canonical retraction (see 0.5).

Then $H_2^{-1}(W)$ is an extrinsic π-fibre object with $H_2^{-1}(W) \cap V = W$ and $H_2^{-1}(W)$ represents $\{V\} \in H^k(V_2)$ as described in section 4.4.

Choose a continuous extension of f to a mapping $U_1 \to V_2$ and let $h: U_1 \to U_2$ be a C^0-close smoothing of this map which takes V_1 transversally to $H_2^{-1}(W)$ (using the transversality theorem). Then $h^{-1}(H_2^{-1}(W))$ is transverse to V_1 and represents the cohomology class $f^*\{W\}$ as outlined in the preceeding Proposition 5.1.1.

Furthermore, as in Lemma 4.3, there is a smooth map $\psi: U_1 \to U_1$ transverse to $h^{-1}H_2^{-1}(W)$ for which $\psi^{-1}h^{-1}H_2^{-1}(W)$ is a π-fibre stratified object which is (Whitney)-cobordant to $h^{-1}H_2^{-1}(W)$.

Therefore, as in 4.4, $V_1 \cap \psi^{-1}h^{-1}H_2^{-1}(W)$ is a π-fibre geometric cocycle in V_1 representing $f^*\{W\}$ and in summary, then, we need only define $g: V_1 \to V_2$ to be the composition

$$V_1 \hookrightarrow U_1 \xrightarrow{\psi} U_1 \xrightarrow{h} U_2 \xrightarrow{H} V_2$$

in order to prove the proposition.

The maps ψ and H can be chosen C^0 close to the identity if U_i are chosen small, and h can be chosen C^0 close to f.

Note that any C^1-small perturbation of h will give a composition $g': V_1 \to V_2$ which also satisfies the conclusion of the theorem and in this sense the map g is "generic among π-fibre maps". Q.E.D.

5.3. <u>The Homotopy Axiom</u>

There is a nice proof of the homotopy axiom in the context of either Whitney or Thom-Mather stratied objects. It is proven here, for example, in the following form:

5.3.1. <u>Proposition</u>. Suppose V_1 and V_2 are Whitney objects in manifolds M_1 and M_2 respectively. Choose systems of control data for V_1 and V_2 and let U_i be regular neighborhoods of V_i in M_i. Suppose ${}^kW \subset U_2$ is a (representative of an) extrinsic π-fibre geometric cocycle representing the cohomology class $\{W\} \in H^k(V_2)$. Suppose f and g are smooth maps from U_1 to U_2 each of which takes V_1 transversally to W. Suppose f and g are homotopic. Then $f^{-1}(W)$ is cobordant to $g^{-1}(W)$ by a cobordism transverse to $V_1 \times (0,1)$ in $U_1 \times (0,1)$.

<u>Proof</u>: Choose a homotopy $H: U_1 \times [0,1] \to U_2$ between f and g. We may assume

$$H(p,t) = H(p,0) = f(p) \quad \text{if} \quad t \le \epsilon$$
$$H(p,t) = H(p,1) = g(p) \quad \text{if} \quad t \ge 1 - \epsilon$$

for some $\epsilon > 0$. There is a smoothing H' of H which agrees with H on $U_1 \times ([0,\epsilon/2] \cup [1-\epsilon/2,1])$, and which takes V_1 transversally to W on this set. Therefore, there is a smooth map $G: U_1 \times [0,1] \to U_2$ which coincides with H' on $U_1 \times ([0,\epsilon/2] \cup [1-\epsilon/2,1])$ and which takes $V_1 \times (0,1)$ transversally to W, (by a direct application of the transversality) G extends to a map

$$G: \; U_1 \times \mathbb{R} \to U_2 \;\text{ by}$$

$$G(x,t) = \begin{cases} G(x,0) & \text{if} \quad t < 0 \\ G(x,1) & \text{if} \quad t > 1 \end{cases}$$

then $G^{-1}(W) \subset U_1 \times \mathbb{R}$ is the desired cobordism between $f^{-1}(W)$

and $g^{-1}(W)$ since $\delta(G^{-1}(W)) = G^{-1}(\delta W) = 0$.

5.3.2. Remark. A very similar proof shows the cohomology pull-back is well defined: a cobordism between two geometric cocycles in the target pulls back to a cobordism between their pre-images in the domain.

CHAPTER 6

PRODUCTS

In this chapter the homology and cohomology cross products are interpreted as Carterian products of geometric cycles and cocycles. This makes it possible to show that transverse intersection of two geometric cocycles in a stratified object defines the cohomology cup product. The cap product is similarly interpreted.

6.1. Homology Cross Product

6.1.1. Definition. Suppose W_1 and W_2 are Thom-Mather stratified objects, then the product Thom-Mather structure on $W_1 \times W_2$ is defined as follows:

A stratum of $W_1 \times W_2$ is a product $A_1 \times A_2$ of a stratum A_1 of W_1 with a stratum A_2 of W_2.

Define

$$\pi_{A_1 \times A_2}(p,q) \equiv (\pi_{A_1}(p), \pi_{A_2}(q))$$

$$\rho_{A_1 \times A_2}(p,q) \equiv \rho_{A_1}(p) + \rho_{A_2}(q)$$

whenever $p \in T_{A_1}$ and $q \in T_{A_2}$. Recall that $\rho_A(p)$ is the "square of the distance from p to $\pi_A(p)$ ".

6.1.2. Definition. If W_1 and W_2 are Thom-Mather objects with families of lines then the product family of lines on $W_1 \times W_2$ is defined as follows:

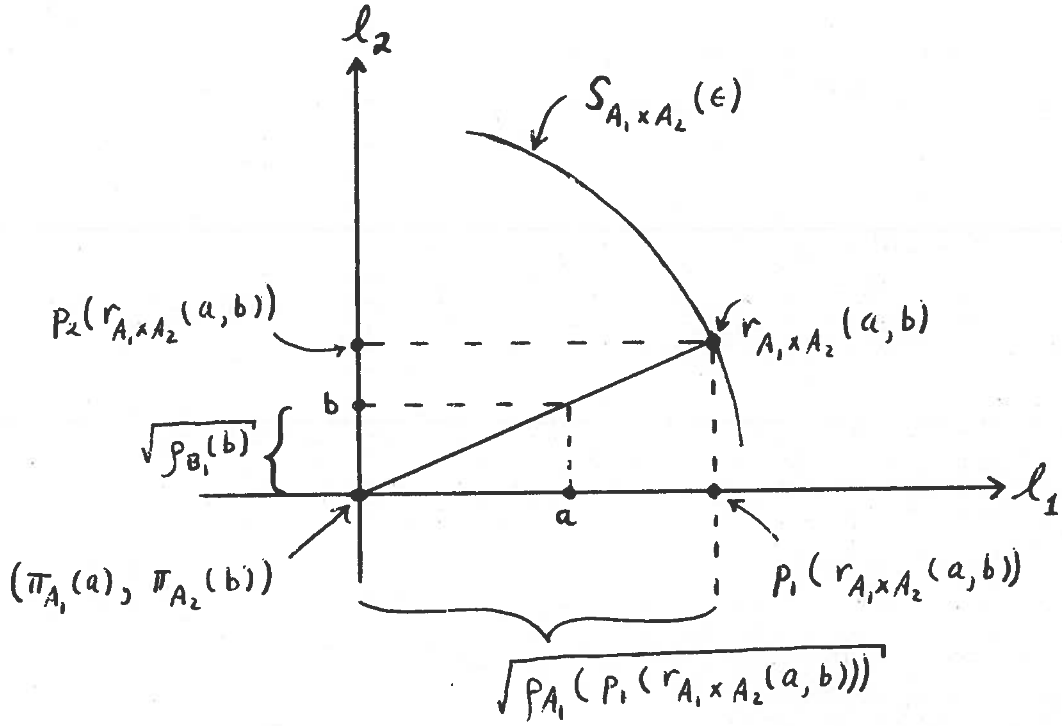
There is an obvious "family of planes" on $W_1 \times W_2$ and whenever A_i is a stratum of W_i we require the $A_1 \times A_2$-lines to be <u>subordinate</u> to this family of planes. The radial retractions $r_{A_1 \times A_2}$ is then uniquely determined by requiring that the two ratios

$$\rho_{A_1}(p_1(r_{A_1 \times A_2}(a,b))) : \rho_{A_2}(p_2(r_{A_1 \times A_2}(a,b)))$$

and

$$\rho_{A_1}(a) : \rho_{A_2}(b)$$

be equal whenever $a \in T_{A_1} \subset V_1$ and $b \in T_{A_2} \subset V_2$. Here, $p_i\colon V_1 \times V_2 \to V_i$ denotes the projection to the first or second factor.



A single plane in $W_1 \times W_2$ is a product of the two lines ℓ_1 in W_1 with ℓ_2 in W_2.

6.1.3. Definition. If W_1 and W_2 are oriented Thom-Mather stratified objects then the product orientation on $W_1 \times W_2$ is defined as follows:

If X_1^j is a top stratum of W_1 and X_2^k is a top stratum of W_2 and if $p \in X_1$ and $q \in X_2$ then the orientation of T_pX_1 followed by the orientation of T_qX_2 defines an orientation of $T_{(p,q)}(X_1 \times X_2)$. Let the multiplicity of $X_1 \times X_2$ be the product of the multiplicity of X_1 with the multiplicity of X_2.

6.1.4. Proposition. Suppose V_1 and V_2 are Thom-Mather stratified objects with a family of lines and let $W_1^j \subset V_1$ and $W_2^k \subset V_2$ be geometric cycles. Then with the product orientation, $W_1 \times W_2$ is a geometric cycle in $V_1 \times V_2$ (where $V_1 \times V_2$ is given the product Thom-Mather structure with the product family of lines) and

$$[W_1 \times W_2] = [W_1] \times [W_2] \in H_{i+j}(V_1 \times V_2)$$

where X denotes the homology cross product.

Proof: $W_1 \times W_2$ clearly follows the family of lines on $V_1 \times V_2$ (in fact, $W_1 \times W_2$ follows the product family of planes). The conditions that $W_1 \times W_2$ be a geometric chain are easily verified.

If $Y_1^{j-1} < X_1^j$ are strata of W_1 and $Y_2^{k-1} < X_2^k$ are strata of W_2 then the induced orientation (and multiplicities) cancel on Y_1 and therefore, the induced orientation (and multiplicities) cancel on $Y_1 \times X_2$. (Similarly for $X_1 \times Y_2$). Therefore $W_1 \times W_2$ is a cocycle.

If W_i is cobordant to W_i' then $W_1 \times W_2$ is cobordant to $W_1 \times W_2'$ which is cobordant to $W_1' \times W_2'$. Thus, the Cartesian product gives a well-defined map

$$GH_j(V_1) \times GH_k(V_2) \to GH_{j+k}(V_1 \times V_2) \to H_{j+k}(V_1 \times V_2).$$

It is, therefore, only necessary to verify the proposition on two particular geometric cycles representing given subcomplexes of a d-cellulation of V_1 and V_2. It is well-known that in this case the homology cross product coincides with the Cartesian product of (cellular) cycles. Q.E.D.

Remark. The above proposition holds if V_1 and V_2 are replaced with manifolds U_1 and U_2 with boundaries ∂U_1 and ∂U_2 and if relative geometric cycles on $(U_i, \partial U_i)$ are used.

6.2. Cohomology Cross Product

6.2.1. Definition. Let V_1 and V_2 be Thom-Mather stratified objects and suppose $W_i \subset V_i$ $(i = 1,2)$ are intrinsic geometric cocycles. The product coorientation on $W_1 \times W_2$ is defined as follows:

If X_i are top costrata of W_i with normal bundles E_i $(i = 1,2)$ then the normal bundle of $X_1 \times X_2$ in $V_1 \times V_2$ is $E_1 \times E_2$ and, therefore, the orientation of E_1 followed by the orientation of E_2 defines an orientation on $E_1 \times E_2$. Let the multiplicity of $X_1 \times X_2$ be the product of the multiplicity of X_1 with the multiplicity of X_2.

6.2.2. Proposition. Let V_1 and V_2 be Thom-Mather stratified objects and suppose $W_i \subset V_i$ $(i = 1,2)$ are intrinsic geometric cocycles with codimension $(W_1) = j$ and codimension $(W_2) = k$. Then $W_1 \times W_2$ is a geometric cocycle in $V_1 \times V_2$ (under the product concentration) and

$$\{W_1 \times W_2\} = \{W_1\} \times \{W_2\} \in H^{j+k}(V_1 \times V_2)$$

where $\times$ denotes the cohomology cross product.

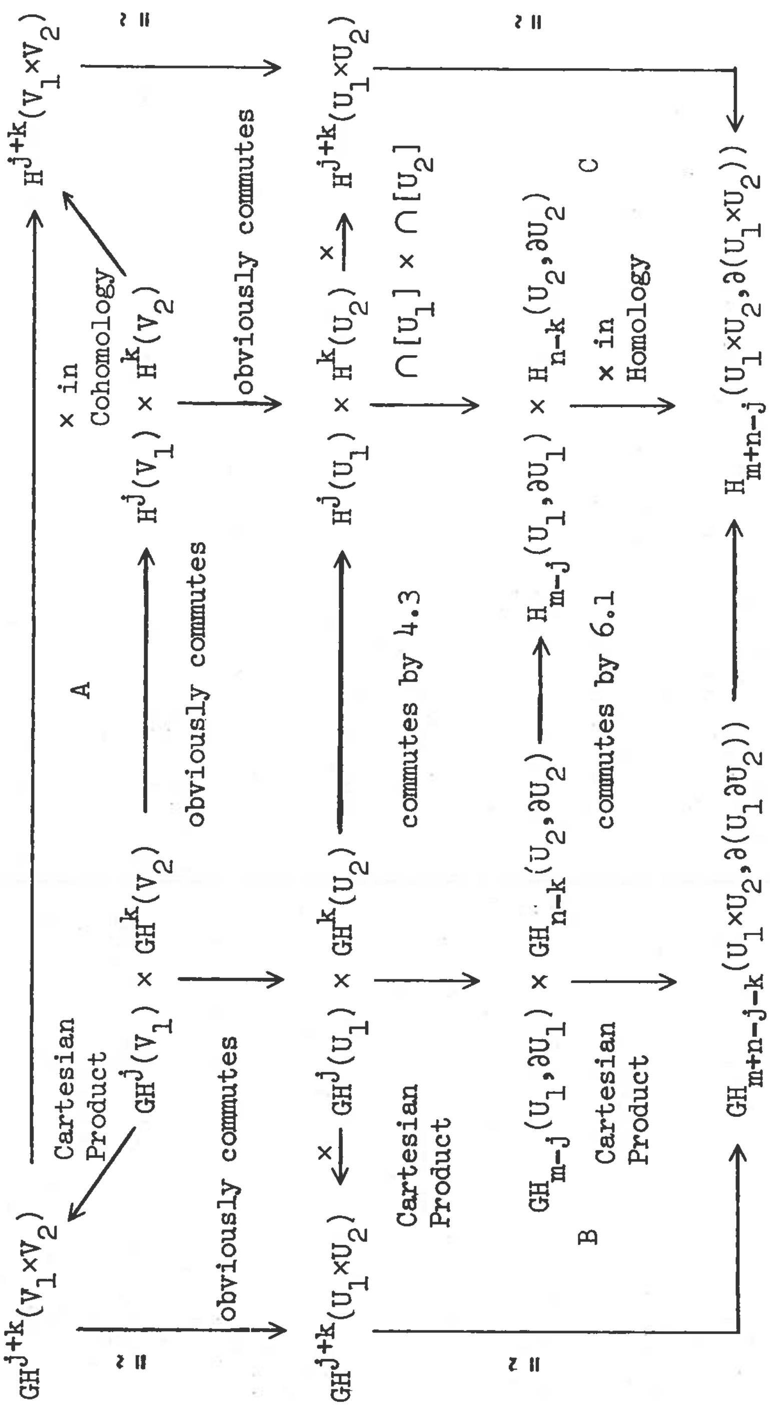
<u>Proof</u>: The cocycle and π-fibre conditions are easily verified, so $W_1 \times W_2$ is a geometric cocycle.

Choose Whitney embeddings of V_1 and V_2 in Euclidean space (see Appendix) and let U_i be a regular neighborhood of V_i. In 4.4 the canonical identification between intrinsic cochains on V_i and extrinsic cochains on the embedded image of V_i is described, and we may therefore assume W_1 and W_2 are extrinsic Whitney cochains on $V_1 \subset U_1^m$ and $V_2 \subset U_2^n$ respectively.

The proposition will be proven if we can show the square labelled "A" in the following diagram commutes. (The middle six groups are the product of two copies of the diagram in section 4.5.)

The horizontal arrow on the top row commutes with the composition around the edge of the diagram by section 4.5. Thus, it is only necessary to show the rectangles labelled "B" and "C" commute. Actually, according to Spanier [13], (p. 255, section 6.21), the rectangle "C" commutes except for a sign change of $(-1)^{n(n-k)}$. We shall show the rectangle "B" involves the same sign change.

Let ${}^jW_1$ and ${}^kW_2$ be geometric cochains in U_1^m and


$GH^{j+k}(V_1 \times V_2)$
$H^{j+k}(V_1 \times V_2)$
$GH^j(V_1) \times GH^k(V_2)$
$H^j(V_1) \times H^k(V_2)$
Cartesian Product
× in Cohomology
A
obviously commutes
$GH^{j+k}(U_1 \times U_2)$
$GH^j(U_1) \times GH^k(U_2)$
$H^j(U_1) \times H^k(U_2)$
$H^{j+k}(U_1 \times U_2)$
commutes by 4.3
$\cap[U_1] \times \cap[U_2]$
$GH_{m-j}(U_1,\partial U_1) \times GH_{n-k}(U_2,\partial U_2)$
$H_{m-j}(U_1,\partial U_1) \times H_{n-k}(U_2,\partial U_2)$
commutes by 6.1
B
C
× in Homology
$GH_{m+n-j-k}(U_1 \times U_2, \partial(U_1 \partial U_2))$
$H_{m+n-j}(U_1 \times U_2, \partial(U_1 \times U_2))$
≅

U_2^n respectively, and suppose X_1 and X_2 are top costrata of W_1 and W_2, with normal bundles E_1 and E_2. Let $p \in X_1$ and $q \in X_2$. Suppose $(e_1,\ldots,e_j)$ is an ordered basis for the fibre of E_1 over $p \in X_1$, defining the given orientation of E_1, and suppose $(v_1,\ldots,v_{m-j})$ is an ordered basis of T_pX_1, chosen so that $(v_1,\ldots,v_{m-j}, e_1,\ldots,e_j)$ is the orientation of T_pU_1. Choose $(v_1',\ldots,v_{n-k}')$ and $(e_1',\ldots,e_k')$ in an analogous manner for $q \in X_2$. Then the orientation $U_1 \times U_2$ is given by the ordered basis

$$(v_1,\ldots,v_{n-j},e_1,\ldots,e_j,v_1',\ldots,v_{n-k}',e_1',\ldots,e_k')$$

whereas the orientation of $X_1 \times X_2$ followed by the orientation of $E_1 \times E_2$ is

$$(v_1,\ldots,v_{n-j},v_1',\ldots,v_{n-k}',e_1,\ldots,e_j,e_1',\ldots,e_k').$$

these two ordered bases are related by a permutation of parity $(-1)^{j(n-k)}$. Q.E.D.

6.3.1. <u>Definition</u>. Suppose W_1 and W_2 are cooriented Whitney objects in some manifold M and suppose W_1 is transverse to W_2. Then the <u>product coorientation</u> on $W_1 \cap W_2$ is defined as follows:

If X_i are top costrata of W_i with normal bundles E_i in M $(i = 1,2)$, then the differential of the diagonal map defines an isomorphism between the normal bundle of $X_1 \times X_2$ and

$$(E_1|\ X_1 \cap X_2) \oplus (E_2|\ X_1 \cap X_2).$$

The desired orientation is then given by the orientation of E_1 followed by the orientation of E_2. The desired multiplicity is the product of the two multiplicities.

6.3.2. Proposition. Suppose V is a Whitney stratified object with a system of control data, in some manifold M. Let W_1 and W_2 be (representatives of) extrinsic geometric co-cycles on V. Suppose W_1 is transverse to W_2 and $W_1 \cap W_2$ is transverse to V. Let $\Delta: M \to M \times M$ be the diagonal map. Then

(1) Δ takes V transversally to $W_1 \times W_2$ in $M \times M$

(2) $\Delta^{-1}(W_1 \times W_2) = W_1 \cap W_2$ is a (π-fibre) extrinsic geometric cocycle on V, under the product co-orientation.

(3) $\{W_1\} \cup \{W_2\} = \Delta^*\{W_1 \times W_2\} = \{\Delta^{-1}(W_1 \times W_2)\} = \{W_1 \cap W_2\}$ where "$\cup$" represents the cohomology cup product.

Proof: To prove (1) we may assume W_1, W_2 and V are manifolds. Let $p \in W_1 \cap W_2 \cap V$ and let $(\xi, \eta) \in T_{(p,p)}(M \times M)$. We must find $\alpha_i \in T_pW_i$ and $v \in T_pV$ so that $d\Delta(p)(v) + (\alpha_1, \alpha_2) = (\xi, \eta)$. First,

choose $v_1 \in T_pV$ and $\alpha_1' \in T_pW_1$ so that

$\xi = v_1 + \alpha_1'$ by transversality

choose $v_2 \in T_pV$ and $\alpha_2' \in T_pW_2$ so that

$\eta = v_2 + \alpha_2'$ by transversality

choose $\alpha_1'' \in T_pV \cap T_pW_1$ and $\alpha_2'' \in T_pV \cap T_pW_2$ so that

$$v_1 - v_2 = \alpha_1'' - \alpha_2''.$$

(This is possible since $(T_pW_1 \cap T_pV) + (T_pW_2 \cap T_pV) \subseteq (T_pW_1 + T_pW_2) \cap T_pV = T_pV$ and $T_pW_1 \cap T_pW_2$ is transverse to T_pV so $(T_pW_1 \cap T_pV) + (T_pW_2 \cap T_pV) = T_pV$). Let $v = v_1 - \alpha_1'' = v_2 - \alpha_2''$. Then

$$(\xi,\eta) = (v_1+\alpha_1', v_2+\alpha_2') = (v+\alpha_1'+\alpha_1'', v+\alpha_2'+\alpha_2'')$$

$$= d\Delta(p)(v) + (\alpha_1'+\alpha_1'', \alpha_2'+\alpha_2'') \quad \text{as desired.}$$

The proof of (2) is obvious since an intersection of π-fibre objects is clearly π-fibre. The final statement follows from the fact that Δ takes V transversally to $W_1 \times W_2$, Theorem 5.1 on the cohomology pullback, Proposition 6.2 on the cohomology cross product, and the formula

$$\Delta^*(\{W_1\} \times \{W_2\}) = \{W_1\} \cup \{W_2\}$$

as in Spanier (p. 251), i.e.

$$\{W_1 \cap W_2\} = \{\Delta^{-1}(W_1 \times W_2)\} = \Delta^*(\{W_1 \times W_2\})$$
$$\Delta^*(\{W_1\} \times \{W_2\}) = \{W_1\} \cup \{W_2\}.$$

The next two technical lemmas are used in Theorem 6.4 which states that cup and cap products in a Thom-Mather stratified object can be interpreted as transverse intersections of (intrinsic) geometric cocycles and cycles.

6.3.3. <u>Lemma</u>. Suppose V is a Thom-Mather stratified object with a family of lines. Let W_1 be an (intrinsic π-fibre) geometric cochain in V, and let W_2 be a geometric chain in V which, therefore, follows the lines of V.

Suppose X is a stratum of V and $W_1 \cap X$ is transverse to $W_2 \cap X$ in X. Let $Y > X$ be another stratum of V and let $\epsilon > 0$ be sufficiently small that W_1 is π-fibre in $T_X(\epsilon)$ and W_2 follows X-lines in $T_X(\epsilon)$. Then $W_1 \cap Y \cap T_X(\epsilon)$ is transverse to $W_2 \cap Y \cap T_X(\epsilon)$ in Y.

Proof: If $p \in W_1$ then the tangent space at p to the stratum of W_1 containing p will be denoted T_pW_1 for simplicity.

Let $p \in W_1 \cap W_2 \cap Y \cap T_X(\epsilon)$ and let $y \in T_pY$. Then $d\pi_X(p)(y) \in T_{\pi_X(p)}(W_1 \cap W_2 \cap X)$ so by transversality, there are vectors $w_1 \in T_{\pi_X(p)}(W_1 \cap X)$ and $w_2 \in T_{\pi_X(p)}(W_2 \cap X)$ so that

$$d\pi_X(p)(y) = w_1 + w_2.$$

Choose $w_i' \in T_p(W_i \cap Y)$ so that

$$d\pi_X(p)(w_i') = w_i \qquad (\text{for } i = 1,2).$$

(Recall that $\pi_X| W_1 \cap Y: W_1 \cap Y \to W_1 \cap X$ is a submersion on each stratum.)

Then define $w_3 \equiv y - w_1' - w_2' \in \ker d\pi_X(p) \subset T_pW_1$ since W_1 is a π-fibre object. Thus

$$y = (w_1'+w_3) + w_2' \in T_p(W_1 \cap Y) + T_p(W_2 \cap Y). \quad \text{Q.E.D.}$$

6.3.4. Lemma. Suppose W_1 and W_2 are Whitney objects in a manifold M and suppose U is an open subset of M for which $W_1 \cap U$ is transverse to $W_2 \cap U$. Let K be a closed subset of U. Then there is a smooth vectorfield η on M so that $\eta(x) = 0$ if $x \in K$ and so that $\phi_1(W_1)$ is transverse to W_2, where $\phi_1 \colon M \to M$ is the diffeomorphism generated by the flow of η after unit time. (Note, however, that if η does not have compact support then the one-parameter group of diffeomorphisms $\phi_t \colon M \to M$ do <u>not</u> describe a continuous path in Diff(M) under the Whitney C^1-topology. This lemma is, therefore not a suitable substitute for the transversality theorem.)

<u>Proof</u>: Choose finitely many vectorfields $X_1,\ldots,X_r$ on M which span the tangent space to M at every point. Let $g_i \colon M \to \mathbb{R}$ be smooth functions vanishing on K and <u>not</u> vanishing outside U for which

$$Y_i(p) \equiv g_i(p)X_i(p) \qquad (1 \leq i \leq r)$$

has a smooth flow defined at least for unit time. Let V be the free vectorspace on $Y_1,\ldots,Y_r$ (i.e., the set of formal linear combinations of the Y_i). There is a neighborhood V' of 0 in V for which the map

$$F \colon V' \times M \to M$$
$$F(Y,p) = \text{the image of } p \text{ under the flow of } Y \text{ after time } 1$$

is defined. For each $p \notin U$ the partial map $F_p \colon V' \to M$ has surjective differential at $0 \in V'$, and thus in a neighborhood of 0 as well. Thus there is a neighborhood $Q \subset V' \times M$ of

$\{0\} \times M$ for which

$$F|Q \text{ takes } Q \cap (V\times W_1) \text{ transversally to } W_2.$$

(If $p \in U$ then

$$dF(0,p)(\{0\} \times T_pW_1) + T_{F(0,p)}W_2 = T_pW_1 + T_pW_2 = T_pM.$$

If $p \notin U$ then

$$dF(0,p)(T_0V \times \{p\}) + T_{F(0,p)}W_2 = \text{span}\{Y_i(p)\} = T_pM,$$

where T_pW_i denotes the tangent space at p to the stratum of W_i containing p.)

Then $F^{-1}(W_2) \cap Q \cap (V\times W_1)$ is a Whitney object in $V \times M$. Let $\pi\colon Q \to V'$ be the projection, and let A_1 be a stratum of W_1 and A_2 a stratum of W_2.

If $\dim A_1 < \text{cod } A_2$ then

$$\dim(F^{-1}(A_2) \cap (V\times A_1) \cap Q) < \dim V = r$$

and $\pi(F^{-1}(A_2) \cap (V\times A_1) \cap Q)$ has measure 0. (Note that $a \notin \pi(F^{-1}(A_2) \cap (V\times A_1))$ if and only if $F_a(A_1) \cap A_2 = \phi$ where $F_a\colon M \to M$ is the partial map defined by $a \in V$). Otherwise, a point $a \in V'$ is a critical value of

$$\pi|(F^{-1}(A_2) \cap (V\times A_1) \cap Q)$$

if and only if the partial map $F_a\colon M \to M$ does not take A_1 transversally to A_2. The set of such critical values also has measure 0, by Sard's theorem. We conclude that in any neighborhood of $0 \in V'$ there is a vectorfield η which is not in either of these sets of measure 0 for <u>any</u> such pair of strata A_1 and A_2. This vectorfield η therefore has a time 1 flow $\phi_1 = F_\eta: M \to M$ which takes each stratum of W_1 transversally to each stratum of W_2. Q.E.D.

(This is the typical transversality technique of Thom and Mather - see, for example, Golubitsky and Guillemin [2]).

6.4. <u>Cup and Cap Product</u>

6.4.1. <u>Definition</u>. Suppose V is a Thom-Mather stratified object with a family of lines and $W_1 \subset V$ is an intrinsic geometric cocycle in V. Suppose $W_2 \cap V$ is a geometric cycle in V and suppose that, for each stratum A of V, $W_1 \cap A$ is tranverse to $W_2 \cap A$. Then we say W_1 is transverse to W_2 in V and the product orientation on $W_1 \cap W_2$ is defined as follows:

Let X_1 be a top costratum of W_1 and suppose E_1 is the normal bundle to X_1. Let X_2 be a top stratum of W_2. Then $X_1 \cap X_2$ is oriented compatible with the exact sequence of bundles over $X_1 \cap X_2$:

$$0 \to T(X_1 \cap X_2) \to TX_2|(X_1 \cap X_2) \to E_1|(X_1 \cap X_2) \to 0.$$

The multiplicity of $X_1 \cap X_2$ is defined as the product of the multiplicities of X_1 and of X_2.

6.4.2. <u>Definition</u>: Let V be a Thom-Mather stratified object and suppose W_1 and W_2 are geometric cocycles in V and W_1 is transverse to W_2. Then the product coorientation of $W_1 \cap W_2$ is defined by analogy with definition 6.3.1:

If X_i are top costrata of W_i having normal bundles E_i $(i = 1,2)$ then the normal bundle of $X_1 \cap X_2$ identifies naturally with $E_1|(X_1 \cap X_2) \oplus E_2|(X_1 \cap X_2)$ and its orientation is defined to be the orientation of E_1 followed by the orientation of E_2. The multiplicity of $X_1 \cap X_2$ is the product of the two multiplicities.

6.4.3. Proposition. Suppose V is a Thom-Mather stratified object and $W_1 \subset V$ is an intrinsic geometric cocycle in V. Suppose $W_2 \subset V$ is a substratified object in V which follows a given family of lines on V. Then there is a (π-fibre) cobordism between W_1 and a geometric cocycle W_1' so that:

(1) W_1' is transverse to W_2 (meaning that each stratum of W_1' is transverse to each stratum of W_2 within the appropriate stratum of V).

(2) $W_1' \cap W_2$ is a substratified object which follows the family of lines in V.

(3) If W_2 is a π-fibre geometric cocycle then so is $W_1' \cap W_2$ and $\{W_1' \cap W_2\} = \{W_1'\} \cup \{W_2\}$ in the cohomology of V, if $W_1' \cap W_2$ is assigned the product coorientation.

(4) If W_2 is a geometric cycle then so is $W_1' \cap W_2$ and $[W_1' \cap W_2] = \{W_1'\} \cap [W_2]$ in the homology of V, where $W_1' \cap W_2$ is assigned the product orientation.

6.4.4. Remark. When geometric cycles were defined in section 2.4 a strong technical condition was placed on the strata of the cycle. This condition ensures that if W_1 is a π-fibre object and W_2 is a cycle then $W_1 \cap W_2$ will carry the structure of a Thom-Mather object. However, there are several possible conditions on W_2 which are preserved under intersecting with a π-fibre object W_2 and therefore the technical restriction could be weakened to any such condition provided it also implies that a geometric cycle satisfying the condition admits a Thom-Mather structure.

Proof of 6.4.2. Let X_i be the (union of) the strata of V with dimension i. Then $W_1 \cap X_0$ is transverse to $W_2 \cap X_0$ in X_0. Assume by induction that W_1 is (π-fibre) cobordant to a π-fibre stratified object W_1' for which

$$W_1' \cap X_j \text{ is transverse to } W_2 \cap X_j \text{ in } X_j \text{ whenever } j \leq i.$$

According to Lemma 6.3.3 there is a neighborhood $T_i(\epsilon)$ of $X_1 \cup X_2 \cup \ldots \cup X_i$ so that whenever Y is any stratum of V,

$$W_1' \cap Y \cap T_i(\epsilon) \text{ is transverse to } W_2 \cap Y \cap T_i(\epsilon) \text{ in } Y.$$

Lemma 6.3.4 then provides a vectorfield η on X_{i+1} which vanishes on $X_{i+1} \cap T_i(\epsilon/2)$ such that the unit time flow of η takes $W_1' \cap X_{i+1}$ transversally to $W_2 \cap X_{i+1}$ in X_{i+1}. Then by Mather (9), η can be lifted to a controlled vectorfield η' on V, i.e., a family of vectorfields tangent to each stratum of V for which

(a) $d\rho_Y(p)(\eta'(p)) = 0$

(b) $d\pi_Y(p)(\eta'(p)) = \eta'(\pi_Y(p))$

(c) $\eta' | X_{i+1} = \eta$

whenever Y is a stratum of V and $p \in Y$. (In particular, $\eta'(p) = 0$ if $p \in T_i(\epsilon/2)$.) Mather proves (in (9)) that such a controlled vectorfield generates a one-paramter group $\{\phi_t\}$ of homeomorphisms of V with

$$\phi_t | Y: Y \to Y$$

a diffeomorphism on each stratum Y of V.

Thus, $\phi_t(W')$ is a π-fibre stratified object (recall that $\pi_Y \circ \phi_t = \phi_t \circ \pi_Y$ for any stratum Y of V) and $\phi_1(W')$ is transverse to V_2 on $X_1 \cup X_2 \cup \ldots \cup X_i \cup X_{i+1}$. The π-fibre cobordism between $W' = \phi_0(W')$ and $\phi_1(W')$ is simply

$$\{(\phi_t(p),t) \in V \times [0,1] | \; p \in W', \; t \in [0,1]\}$$

which completes the inductive step, proving part (1).

(Actually, the above cobordism must be made constant near the ends of the interval $[0,1]$ but this is a trivial matter.)

Part (2) of the theorem involves showing two properties of $W_1' \cap W_2$: it must admit the structure of a Thom-Mather stratified object and it must satisfy property (D) of section 1.4.

Suppose $X < Y$ are strata of V and let $A < B \subset Y$ be strata of $W_2 \cap Y$ and $C < D \subset X$ be strata of $W_2 \cap X$. Suppose $\pi_X(A \cap T_X(\epsilon)) = C$ and $\pi_X(B \cap T_X(\epsilon)) = D$ for some

$$\pi_X(A \cap T_X(\epsilon)) = C \quad \text{and} \quad \pi_X(B \cap T_X(\epsilon)) = D \quad \text{for some} \quad \epsilon > 0.$$

Similarly, suppose $A' < B'$ are strata of $W_1' \cap Y$ and $C' < D'$

$C' < D$ are strata of $W_1' \cap X$ with $\pi_X(A' \cap T_X(\epsilon)) = C'$ and $\pi_X(B' \cap T_X(\epsilon)) = D'$.

Then $A \cap A' < B \cap B'$ are strata of $W_1' \cap W_2 \cap Y$ and $C \cap C' < D \cap D'$ are strata of $W_1' \cap W_2 \cap X$ and condition D must be verified for the mapping

$$\pi_X\colon S_X(\epsilon) \cap (A \cap A' \cup B \cap B') \to (C \cap C') \cup (D \cap D').$$

Choose a sequence of points $p_i \in S_X(\epsilon) \cap B \cap B'$ converging to a point $p \in S_X(\epsilon) \cap A \cap A'$ and suppose all tangent planes converge. Then

$$d\pi_X(p)(\lim T_{p_i}(B \cap B')) = d\pi_X(p)(\lim T_{p_i} B \cap \lim T_{p_i} B')$$

$$= d\pi_X(p)(\lim T_{p_i} B \cap d\pi_X(\pi_X(p))^{-1}(\lim T_{\pi_X(p_i)} D'))$$

(since W_1' is π-fibre)

$$= d\pi_X(p)(\lim T_{p_i} B) \cap \lim T_{\pi_X(p_i)} D'$$

$$= \lim T_{\pi_X(p_i)} D \cap \lim T_{\pi_X(p_i)} D' = \lim T_{\pi_X(p_i)} (D \cap D')$$

(since W_2 satisfies condition D).

To show $W_1' \cap W_2$ is a stratified object, it is necessary to use Appendix 1 in which a specific embedding

$$f\colon V \to \mathbb{R}^m$$

with $f(v)$ a Whitney stratified object, is constructed. Then $W_1' \cap W_2$ follows the lines of V and satisfies condition (D), so by Corollary A2 of Appendix 1, $f(W_1' \cap W_2)$ is a Whitney stratified object (and, therefore $W_1' \cap W_2$ admits the structure of a Thom-Mather stratified object).

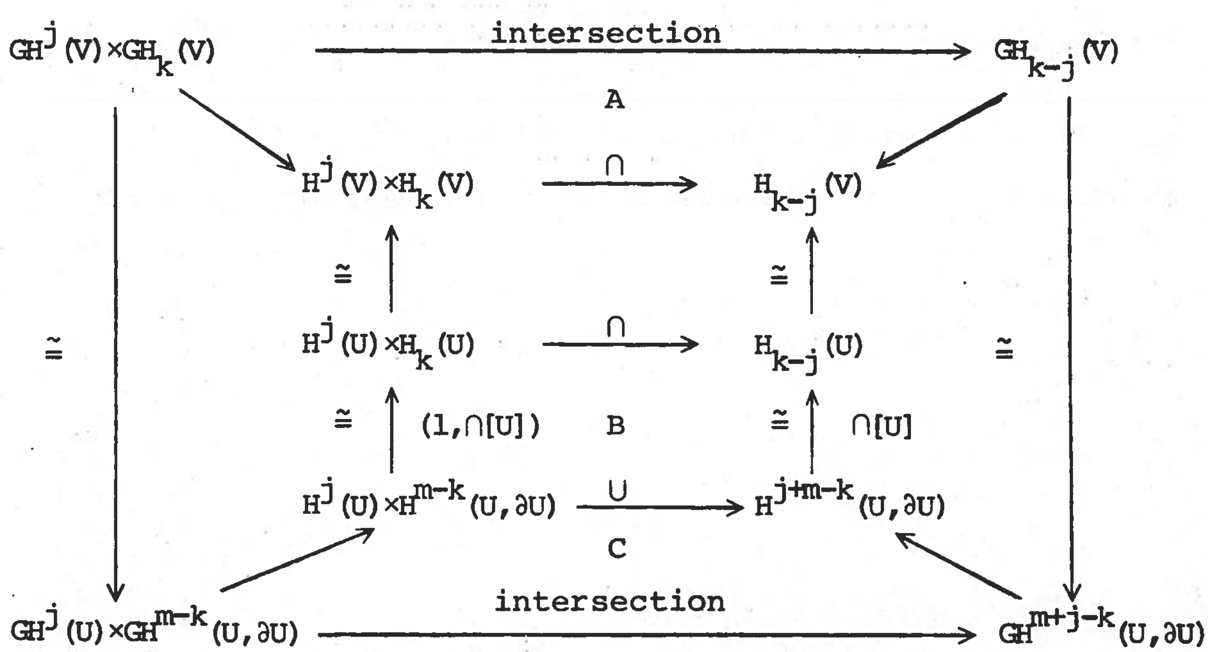
Part (3) of the theorem is proven by embedding V into

$\mathbb{R}^m$ as above, choosing extrinsic π-fibre geometric cocycles Z_1 and Z_2 for which $Z_1 \cap V = W_1'$ and $Z_2 \cap V = W_2$, and then applying Proposition 6.3 to Z_1 and Z_2 noting that $Z_1 \cap Z_2$ is transverse to V_1 from which we conclude

$$\{W_1' \cap W_2\} = \{Z_1 \cap Z_2\} = \{Z_1\} \cup \{Z_2\} = \{W_1'\} \cup \{W_2\}$$

in the cohomology of V.

Part (4) of the theorem follows from a procedure very similar to that of section 6.2. Choose a Whitney embedding of V into Euclidean space and let U^m be a regular neighborhood of the image of V. Replace the intrinsic geometric cocycle ${}^jW_1'$ by an extrinsic geometric cocycle and note that the geometric cycle $W_2^k \subset V$ defines a relative geometric cocycle in $GH^{m-k}(U, \partial U)$. It is necessary to show that the square labelled "A" in the following diagram commutes:



The square labelled "B" commutes by Spanier, p. 254.

The square labelled "C" commutes by a relative version of Proposition 6.3.

The large square around the edge of the diagram commutes provided $W_1^! \cap W_2$ is oriented in the following manner:

Let X_1 be a top costratum of $W_1^!$ and suppose E_1 is the normal bundle of X_1. Let X_2 be a top stratum of W_2. Then $X_1 \cap X_2$ must be oriented compatible with the exact sequence of bundles over $X_1 \cap X_2$:

$$0 \to T(X_1 \cap X_2) \to TX_2 | (X_1 \cap X_2) \to E_1 | (X_1 \cap X_2) \to 0$$

as defined in section 6.4.1. Q.E.D.

APPENDIX 1

WHITNEY EMBEDDINGS OF THOM-MATHER OBJECTS

A.1.1. <u>Definition</u>. Suppose W is a Thom-Mather stratified object and $f: W \to \mathbb{R}^m$ is an embedding which is a smooth embedding of each stratum. Then f is said to embed W with straight lines, provided for each stratum X of W,

(1) there is an $\in > 0$ so that whenever $p \in S_X(\in)$, the map $g_p: [0,\in) \to \mathbb{R}^m$

$$g_p(t) = f(h_X^{-1}(p,t))$$

is an affine map, where $h_X: T_X - X \to S_X(\in) \times (0,\infty)$ is the homeomorphism defined by the family of lines, as in section 0.4.

(2) $\frac{d}{dt} g_p(0)$ is not tangent to $f(X)$.

(3) There is an open neighborhood $T_{f(X)}$ of $f(X)$ in R^m and a smooth projection.

$$\pi_{f(X)}: T_{f(X)} \to f(X)$$

so that

$$\pi_{f(X)}(f(p)) = f(\pi_X(p)) \quad \text{for all} \quad p \in T_X \subset W$$

and

$\pi_{f(X)}^{-1}(f(p))$ is an open subset of an affine Euclidean plane in R^m for all $p \in X$.

(4) If $X < Y$ then $\pi_{f(X)}\pi_{f(Y)} = \pi_{f(X)}$

(5) If $\epsilon > 0$ is sufficiently small then

$$f|\,\overline{S_X(\epsilon)}: \overline{S_X(\epsilon)} \to \mathbb{R}^m$$

is an embedding with straight lines (defined inductively).

It follows from section 3.7 that in this case, $f(W)$ is a Whitney stratified object in $\mathbb{R}^m$.

<u>Proposition $P_{k,i}$</u>: Suppose W^k is a Thom-Mather stratified object with a family of lines. Let

$$T_j(\epsilon) = \cup\{T_X(\epsilon)\,|\; X \text{ is a stratum of } W \text{ and } \dim(X) \le j\}.$$

Suppose $f: \overline{T_i(\epsilon)} \to \mathbb{R}^m$ is an embedding with straight lines which extends to an embedding

$$f: \overline{T_i(\epsilon)} \cup X \to \mathbb{R}^m$$

where X is the stratum of dimension $i+1$. Suppose a tubular neighborhood $\pi_{f(X)}: T_{f(X)} \to f(X)$ is given, for which

(a) $\pi^{-1}_{f(X)}(x)$ is an open subset of a Euclidean affine plane in $\mathbb{R}^m$ (for each $x \in X$).

(b) $\pi_{f(X)}(f(p)) = f(\pi_X(p))$ if $p \in \overline{T_i(\epsilon)} \cap T_X$.

Then for any $\epsilon' < \epsilon$ there is an embedding with straight lines

$$g: S_X(\epsilon) \to \mathbb{R}^m \times \mathbb{R}^n \qquad \text{(for some } n)$$

$$g(p) = (g_1(p), g_2(p))$$

such that

(a) for all $p \in S_X(\epsilon)$, $g_1(p) \in T_{f(X)}$ and

$$f(\pi_X(p)) = \pi_{f(X)} g_1(p)$$

(b) If $p \in S_X(\epsilon) \cap T_i(\epsilon')$ then $g(p) = (f(p),0)$.

<u>Proof</u>. The propositions are proven in the following order:

$$\begin{array}{llll} P_{0,-1} & & & \\ P_{1,-1} & P_{1,0} & & \\ \vdots & & & \\ P_{k,-1} & P_{k,0} & \cdots & P_{k,k-1} \\ P_{k+1,-1} & P_{k+1,0} & \cdots & P_{k+1,k} \\ \vdots & & & \end{array}$$

To prove Proposition $P_{k,i}$ we shall assume all previous propositions are proven. (Note that $P_{0,-1}$ is trivial.)

Now suppose $\dim(W) = k$ and W satisfies the hypotheses of the proposition. Therefore X is a stratum of W, $\dim(X) = i + 1$, and we wish to embed $S_X(\epsilon)$.

Let

$$U_n(\delta) \equiv \cup\{\overline{T_Y(\delta)} \cap S_X(\epsilon) \mid \dim(Y) \leq n + 1 \text{ and } Y \neq X \text{ is a stratum of } W\}.$$

To embed $S_X(\epsilon)$ it is necessary to embed $U_n(\delta)$ for $n = 1,2,\ldots,k-1$. This is accomplished by increasing induction on n. Let

$$\epsilon' < \delta_k < \delta_{k-1} < \ldots < \delta_1 < \epsilon'' < \epsilon.$$

Claim: There is an integer a^* for which an embedding $h: U_n(\delta_n) \to \mathbb{R}^m \times \mathbb{R}^{a^*}$ can be found, where

$$h(p) = (h_1(p), h_2(p))$$

satisfies

(1) $h_1(p) \in T_{f(X)}$ for all $p \in U_n(\delta_n)$.

(2) h embeds $U_n(\delta_n)$ with straight lines (and in particular, for each stratum Q of $h(U_n(\delta_n))$ there is a tubular neighborhood $\pi_{h(Q)}: T_{h(Q)} \to h(Q)$ so that $\pi^{-1}_{h(Q)}(q)$ is a Euclidean plane contained in $\pi^{-1}_{f(X)}(\pi_X(q))$.

(3) $\pi_{f(X)} h_1(p) = f(\pi_X(p))$ if $p \in U_n(\delta_n)$.

(4) $\pi_{f(X)}\pi_{h(Q)}(p) = \pi_{f(X)}(p_1)$ if $p = (p_1, p_2) \in R^m \times R^{m'}$ and Q is a stratum of $h(U_n(\delta_n))$.

(5) $h_1(p) = f(p)$ and $h_2(p) = 0$ if $p \in U_n(\delta_n) \cap T_i(\epsilon')$.

Completing this claim for $n = k - 1$ will embed $S_X(\epsilon)$ with the desired properties. Assume the claim is true with $n - 1$ in place of n and suppose Y is a stratum of W with $\dim(Y) = n + 1$. To prove the claim we must extend the domain of h to include $\overline{T_Y(\delta_n)} \cap S_X(\epsilon)$. This will be accomplished in the following manner: first $Y \cap S_X(\epsilon)$ will be embedded, then a tubular neighborhood of $Y \cap S_X(\epsilon)$ in Euclidean space will be constructed. This is sufficient data to apply the proposition (inductive hypothesis) to obtain an embedding of $S_Y(\delta_n) \cap S_X(\epsilon)$ compatible with the chosen tubular neighborhood.

Finally, $T_Y(\delta_n) \cap S_X(\epsilon)$ is embedded as the (Euclidean) cone over $S_Y(\delta_n) \cap S_X(\epsilon)$. This will be the desired embedding of $U_n(\delta_n)$.

Convention: As the construction proceeds by adding Euclidean space factors to the space in which we are embedding, at each stage we will assume all previous work is automatically embedded in the larger Euclidean space by setting new co-ordinates equal to 0, and all previous tubular neighborhoods are automatically extended to tubular neighborhoods in the larger Euclidean space in the obvious way.

The Embedding of $Y \cap S_X(\epsilon)$:

Let

$$\phi: Y \cap S_X(\epsilon) \to [0,1]$$

be a smooth function for which

$$\phi(p) = 0 \quad \text{if} \quad p \in Y \cap S_X(\epsilon) \cap [U_{n-1}(\delta_n) \cup T_i(\delta_n)]$$

$$\phi(p) = 1 \quad \text{if} \quad p \notin Y \cap S_X(\epsilon) \cap [U_{n-1}(\delta_{n-1}) \cup T_i(\delta_{n-1})].$$

Choose any smooth embedding

$$g_1: Y \cap S_X(\epsilon) \to \mathbb{R}^{S-1} \times \{1\} \subset \mathbb{R}^S.$$

Since we assume by induction the existence of

$$h\colon U_{n-1}(\delta_{n-1}) \to \mathbb{R}^m \times \mathbb{R}^a = \mathbb{R}^{m+a}$$

we may define

$$g_2\colon Y \cap S_X(\epsilon) \to \mathbb{R}^{m+a} \times \mathbb{R}^S$$

by

$$g_2(p) = ((1-\phi(p))h(p) + \phi(p)f(\pi_X(p)),\ \phi(p)g_1(p)).$$

Then g_2 is an embedding and

$$g_2(p) = (h(p),0) \quad \text{if} \quad p \in Y \cap S_X(\epsilon) \cap [U_{n-1}(\delta_n) \cup T_i(\delta_n)]$$

$$g_2(p) = (f\pi_X(p), g_1(p)) \quad \text{if} \quad p \notin Y \cap S_X(\epsilon) \cap [U_{n-1}(\delta_{n-1}) \cup T_i(\delta_{n-1})]$$

and $\pi_{f(X)}g_2(p) = f\pi_X(p)$. (This is obvious if p is in either of the two subsets of $Y \cap S_X(\epsilon)$ described above, and follows for general p from the fact that $\pi^{-1}_{f(X)}(x)$ is a Euclidean plane whenever $x \in f(X)$).

<u>Construction of $T_{g_2(Y \cap S_X(\epsilon))}$:</u> A tubular neighborhood of $h(Y \cap S_X(\epsilon) \cap U_{n-1}(\delta_{n-1}))$ has been constructed whose fibres are open subsets of Euclidean planes contained in the fibres of $\pi_{f(X)}$. This tubular neighborhood defines a smooth section of the Grassmann bundle over $Y \cap S_X(\epsilon) \cap U_{n-1}(\delta_{n-1})$ whose fibre over a point $y \in Y \cap S_X(\epsilon) \cap U_{n-1}(\delta_{n-1})$ is the Grassmannian of α-planes in the plane determined by $\pi^{-1}_{f(X)}(\pi_{f(X)}(y))$ where

$\alpha = m + a + S - \dim(Y)$.

This section lies in the open set of planes transverse to T_yY and so after first restricting to $Y \cap S_X(\epsilon) \cap U_{n-1}(\delta_n)$, it can be extended smoothly across the rest of $Y \cap S_X(\epsilon)$, (the section always determining a plane transverse to T_yY). In an open neighborhood of $g_2(Y)$ containing no focal points of $g_2(Y)$ this section determines a tubular neighborhood $\pi_{g_2(Y)}: T_{g_2(Y)} \to g_2(Y)$. By shrinking $T_{g_2(Y)}$ we may assume $T_{g_2(Y)} \subset T_{f(X)}$.

Embedding $S_Y(\delta_n) \cap S_X(\epsilon)$: The Proposition $P_{k-1,n-1}$ can now be applied to the stratified object $\overline{S_X(\epsilon)}$ (which has dimension 1 less than that of W) by induction. We have an embedding of the union $U_{n-1}(\delta_{n-1})$ of the tubular neighborhoods of strata of dimension $\leq n - 1$ and an extension of the domain to an embedding of $Y \cap S_X(\epsilon)$ with a tubular neighborhood of the image. Therefore, there is an embedding with straight lines

$$g_3: S_Y(\delta_n) \cap S_X(\epsilon) \to \mathbb{R}^{m+a} \times \mathbb{R}^s \times \mathbb{R}^t$$

for which

(a) $g_3(p) = (h(p),0,0)$ if $p \in U_{n-1}(\delta_n)$

(b) $\pi_{g_2(Y)} g_3(p) = g_2(\pi_Y(p))$ if $p \in S_Y(\delta_n) \cap S_X(\epsilon)$

and the commutation relations amongst the various tubular neighborhoods are valid.

Embedding $T_Y(\delta_n) \cap S_X(\epsilon)$: Using the same function

$\phi: Y \cap S_X(\epsilon) \to \mathbb{R}$ as above, define

$$g_4: T_Y(\delta_n) \to \mathbb{R}^{m+a+s+t} \times \mathbb{R}$$

by

$$g_4(p) = (g_3(\pi_Y(p)), \phi(\pi_Y(p))\rho_Y(p))$$

where $r_Y: T_Y - Y \to S_Y(\delta_n)$ is the retraction given by the family of Y-lines

If $p \in U_n(\delta_n)$ then

$$g_4(p) = (g_3(p), 0) = (h(p), 0, 0)$$

and

$$\pi_{h(Y)} g_4(p) = \pi_{h(Y)} g_3 r_Y(p) = h\pi_Y(p).$$

Every Y-line is straight and, in fact,

$$g_4(h_Y^{-1}(p,t)) = (\text{const.}, \text{const. } \rho_Y h_Y^{-1}(p,t))$$
$$= (\text{const.}, (\text{const.})\cdot t) \quad \text{if} \quad p \in S_Y(\delta_n)$$

which is an affine function of t.

Each Z-line for $Z > Y$ remains straight since such a line lies in a single (π_Y, ρ_Y) fibre. Furthermore, the condition that $\frac{d}{dt} g_4 h_Y^{-1}(p,t)$ is not tangent to $g_4(Y)$ holds since $g_4(h_Y^{-1}(p,t))$ has a nonzero last coordinate (unless

$p \in U_n(\delta_{n-1})$ where the tangent condition was assumed to hold by induction).

This completes the construction of $h: U_n(\delta_n) \to \mathbb{R}^{m+a+s+t+1}$ and thus the inductive step of the proof.

A.1.2. Corollary. Any stratified object of dimension k-1 with a family of lines can be embedded in Euclidean space as a Whitney object with straight lines.

Proof: Given the Thom-Mather object W^{k-1}, the cone over W is a Thom-Mather object of dimension k. Embed the cone point X into $\mathbb{R}^1$ and apply Proposition $P_{k,-1}$ to find an embedding of $S_X = W$. Q.E.D.

Conjecture: If W^{k-1} is embedded in some $\mathbb{R}^N$ as described above then the generic projection of the embedded image to $\mathbb{R}^{2k-1}$ will be an embedding whose image is again a Whitney stratified object with straight lines. If this is true, then every Thom-Mather object of dimension n can be embedded as a Whitney object in $\mathbb{R}^{2n+1}$.

A.1.3. Corollary. Suppose V is a Thom-Mather stratified object and $W \subset V$ is a closed subset with the following properties:

(a) W has a decomposition into a locally finite family of smooth submanifolds of the strata of V.

(b) For each stratum X of V, $W \cap X$ satisfies the Whitney conditions and for all $\epsilon > 0$ sufficiently small,

$W \cap T_X(\epsilon)$ follows the X-lines, i.e. $W \cap (T_X(\epsilon) - X) = C^o_X(W \cap S_X(\epsilon))$.

(c) If $X < Y$ are strata of V then for all $\epsilon > 0$ sufficiently small,

$$\pi_X: W \cap Y \cap S_X(\epsilon) \to W \cap X$$

is a stratified map satisfying condition (D) of section 2.3.1.

Then W admits the structure of a Thom-Mather stratified object.

Proof: Choose an embedding with straight lines $f: V \to \mathbb{R}^m$. We claim $f(W)$ is a Whitney object in $\mathbb{R}^m$. It is only necessary to verify the Whitney conditions between strata of $f(W \cap X)$ and strata of $f(W \cap Y)$ whenever $X < Y$ are strata of V. However, Corollary 3.7 applies to the Whitney objects $f(W \cap Y \cap S_X(\epsilon)) \subset f(Y \cap S_X(\epsilon))$ and $f(W \cap X) \subset f(X)$ and the map $\pi_{f(X)}: f(Y \cap S_X(\epsilon)) \to f(X)$. We conclude $f(W \cap T_X(\epsilon))$ is a Whitney object in $\mathbb{R}^n$.

Since $f(W)$ is Whitney, it admits a Thom-Mather structure which is therefore a Thom-Mather structure on W as well. Q.E.D.

BIBLIOGRAPHY

1. BAAS, N., On Bordism Theory of Manifolds with Singularities. Math. Scand. 33(1973), 279-302.

2. GOLUBITSKY, M. and GUILLEMIN, V., Stable Mappings and Their Singularities. Graduate Texts in Mathematics, Springer-Verlag (1973).

3. HENDRICKS, E., Triangulation of Stratified Sets. Ph.D. Thesis, Massachusetts Institute of Technology (1973).

4. HUDSON, J., Piecewise Linear Topology. W.A. Benjamin, Inc. (1969).

5. MC CRORY, Clint, Poincaré Duality in Spaces with Singularities. Thesis, Brandeis University (1972).

6. MATHER, John, Structural Stability of Mappings. Mimeographed Notes, Princeton University (1966).

7. MATHER, John, Stability of C^∞ Mappings II: Infinitesimal Stability Implies Stability. Ann. of Math. 89(1969).

8. MATHER, John, Stratifications and Mappings in Proceedings of Symposium on Dynamical Systems at Salvador, Brazil (1971). Ed. M.M. Peixoto, Academic Press (1973).

9. MATHER, John, Notes on Topological Stability. Mimeographed Notes, Harvard (1970).

10. MILNOR, John and STASHEFF, James, Characteristic Classes, Princeton University Press (1974).

11. PUTZ, ., Triangulation of Fibre Bundles. Canadian Journal of Math., 19(1967), 499-513.

12. ROURKE, C., SANDERSON, B. and BUONCRISTIANO, S., A Geometric Approach to Homology Theory. Mimeographed Notes, University of Warwick.

13. SPANIER, E., Algebraic Topology, McGraw-Hill (1966).

14. SULLIVAN, D., Geometric Periodicity and the Invariants of Manifolds also A General Problem in Manifolds. Amsterdam 1970, Lecture Notes in Mathematics #197, Springer-Verlag (1971).

15. SULLIVAN, D., Singularities in Spaces - Proceedings of Liverpool Singularities Symposium II. Lecture Notes in Mathematics #209, Srpinger-Verlag (1971).

16. THOM, René, Quelques Proprietes Globales Des Varietes Differentiables. Commentari Math. Helvetici, 28(1954), 17-86.

17. THOM, René, Ensembles et Morphismes Stratifies. Bull. Amer. Math. Soc. 75(1969), 240-284.

18. THOM, René, Stratified Sets and Morphisms: Local Models, in Proceedings of Liverpool Singularities, vol. I, Lecture Notes in Mathematics #192, Springer-Verlag (1971).

19. VERONA, Andrei, Integration on Whitney Prestratifications. Rev. Roum. Math. Pures et Appl. 17(1972), 1473-1480.

20. WHITNEY, H., Geometric Methods in Cohomology. Proc. Nat. Acad. Sci., U.S.A., 33(1947), 7-9.