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On the geometry of quadrics and their degenerations

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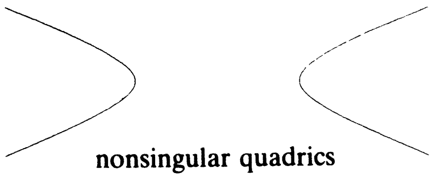
Chapter 1. Introduction

The spaces X^n of complete quadrics is a compactification of the space of nonsingular quadric hypersurfaces in complex projective $n-1$ space. It was introduced by Chasles for $n = 3$ in 1864 [C] and by Schubert for general n in 1879 [Sch]. In this paper, we give a conceptual formula for the rational cohomology groups of X^n, including its ring structure.

1.1. In our view, the variety X^n of complete quadrics ranks with Grassmannians and flag manifolds as one of the most important special varieties. Since they are less well known, we summarize some background about them in the introduction.

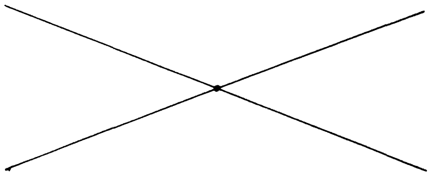
First we identify X^n as a set. The points of the variety X_0^n of nondegenerate quadrics represent nonsingular quadrics (degree two hypersurfaces) in complex projective $n-1$ space $\mathbb{P}^{n-1}$. The idea behind the variety of complete quadrics X^n is to add to X_0^n points representing certain geometric objects in $\mathbb{P}^{n-1}$ called *degenerate* quadrics. A degenerate quadric is a partial flag $\phi = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_k = \mathbb{P}^{n-1}$ of linear subspaces of $\mathbb{P}^{n-1}$ together with, for each $i > 0$, a nonsingular quadric in the projective space of planes of dimension $(\dim F_{i-1} + 1)$ which contain F_{i-1} and are contained in F_i. (We take the natural conventions that $\dim F_0 = -1$, that $\mathbb{P}^0$ consists of a single point and contains a unique quadric, and that a nondegenerate quadric in a one dimensional projective space is a pair of distinct points). A complete quadric is by definition either a nonsingular quadric or a degenerate quadric. As a set, X^n is just the set of complete quadrics. Complete quadrics are classified into strata according to the type of flag. For $n = 3$, there are four strata:



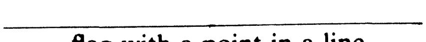
nonsingular quadrics

trivial flag with no point or line

flag with one line

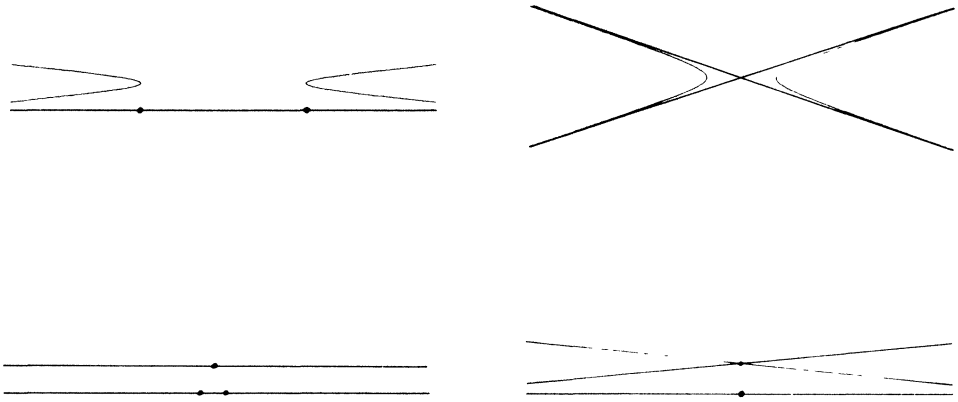


flag with one point



flag with a point in a line

Now we need to put a topology on the set of complete quadrics. The topology within each stratum is clear. The intuitive idea behind convergence of sequences in one stratum to a point in another is illustrated by the following four pictures of pairs of complete quadrics which represent points close together in X^3.

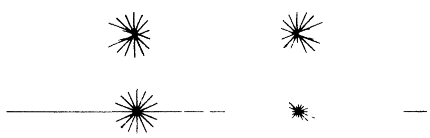


Perhaps the simplest precise definition of X^n as a topological space is the following one in the spirit of [FKM]: Let $\mathbf{F}$ be the flag variety of points contained in hyperplanes in $\mathbb{P}^{n-1}$. Map the variety X_0^n of quadrics into the space Sub($\mathbf{F}$) of closed subsets of $\mathbf{F}$ by associating to a quadric the set of flags consisting of a point and the tangent plane at that point. Then X^n is the closure of X_0^n in Sub($\mathbf{F}$) endowed with the Hausdorff topology.

1.2. Complete quadrics have arisen in both algebraic geometry and algebraic group theory, and the space X^n has modern constructions as an algebraic variety from both points of view. We sketch these in this section, but we have minimized our reliance on either of them in this paper since most readers are familiar with at most one construction.

Constructions of X^n in Algebraic Geometry

The original use of the variety of complete quadrics was in enumerative geometry, where they are used to count quadrics with certain sets of tangency conditions. See [K1], [K2] for a historical account. The importance of it there stems from the following characterization of X^n: For each integer j, we can associate to any nonsingular quadric the variety of its i-dimensional tangent planes. This turns out to be a nonsingular degree two subvariety of G_i, the Grassmannian of all i-planes in $\mathbb{P}^{n-1}$ (with respect to the Plucker coordinates on the Grassmannian). So for each i, we have a map φ_i from X_0^n to $\bar{Q}_i$, the variety of all degree two subvarieties of G_i (singular or not). Then X^n is the minimal compactification of X_0^n over which all of the maps φ_i extend. This means that a degenerate complete quadric has ideal tangent planes which are limits of tangent planes of nearby nonsingular quadrics. For example, if Q is a quadric in the stratum of X^3 consisting of a line with a nondegenerate quadric (two points) in it, the ideal tangent lines to Q are all lines through those two points.



The algebraically simplest construction of X_n is a realization of this universal property: A nondegenerate quadric is most conveniently represented by symmetric $n \times n$ complex matrix M with nonzero determinant, with two being considered equivalent if one is a scalar multiple of the other. This gives an embedding of X_0^n in the projectivization of the space of all $n \times n$ matrices. The closure X_{naive}^n of X_0^n in this space is called the naive compactification. If we consider not only M but also of all its exterior powers (or adjugates, in the language of [T]), we similarly get an embedding in a product of projective spaces, one for each exterior power, and X^n is the closure of X_0^n in this product of projective spaces ([T], [V]).

The variety X^n also has a construction by blowups from the naive compactification X_{naive}^n which is stratified by the rank of the matrix M. To obtain X^n, first blow X_{naive}^n up along the rank 0 stratum, then blow the result up along the proper transform of the rank 1 stratum, and so on [V].

Constructions of X^n in Algebraic Group Theory

The variety X_0^n of nonsingular quadrics is an example of a symmetric variety. A symmetric variety is a particular type of homogeneous space that is the analogue in algebraic geometry of a Riemannian symmetric space. Any symmetric variety X has a natural compactification $C(X)$ which may be characterized as the minimal one that is *wonderful* [DP1]. Here wonderful means that $C(X)$ is equivariant and $C(X) - X$ is a union of nonsingular divisors which intersect transversally and whose natural stratification as a divisor with normal crossings coincides with its stratification by orbits. The compactification $C(X)$ has a construction as the closure of X_0^n in the space L of subalgebras of the Lie algebra of the automorphism group, where X_0^n is embedded in L by sending each point to the Lie algebra of its stabilizer group [DP1], [DP2]. It also has a construction as the closure of an orbit homeomorphic to X_0^n in the projectivization of an irreducible representation of the automorphism group which is general (in the sense that the highest weight is not in a wall of the Weyl chamber [DP1], [DP2]. (We note that the compactification $C(X)$ is the analogue for symmetric varieties of the largest Satake compactification of a symmetric space of negative curvature [Sa].)

In our case, $X^n = C(X_0^n)$. We admit our bias in favor of the algebraic group theoretic approach because of its generality and the beauty of its conceptual framework. However, this paper does not rely on it. Only section 5.3 makes explicit reference to any preexisting construction of X^n.

1.3. The cohomology $H^*(X^n)$ of the space X^n of complete quadrics has been the subject of many studies. The manipulations of Chasles and Schubert, done many years before homology was even defined, may be interpreted as calculations inside the ring $H^*(X^n)$, as was first pointed out by van der Waerden [VW]. In fact, Schubert's formulas represent an extensive understanding of the subring R of the cohomology ring generated by classes in degree two. A number of people have worked on making these formulas rigorous in the language of cohomology (see [K1], [K2]) and in [DP1] there is a complete calculation of R.

There are two modern approaches to the complete calculation of $H^*(X^n)$. One is to realize X^n as a projective space with a sequence of varieties blown up as described above, and to iteratively use the formula for the cohomology of a blow up (Vainsencher [V]). This gives the ring structure in principle, but following the formula through the interations leads to combinatorial difficulties ([V], p. 201). The other is to find a paving of X^n by affine spaces (Strickland [Str]). See also [Dr1] and [Dr2].

Our object here is to give a formula which is conceptual and non-iterative in the sense that Borel's calculation of the cohomology of the flag variety ([B01], [B02]) or Danilov's calculation of the cohomology of a toric variety [D] are. We have succeeded only at the expense of sacrificing the integers: our formula holds only for rational cohomology $H^*(X^n; \mathbb{Q})$. This is, of course, sufficient for the development of Schubert's calculus.

1.4. We now give the statement of our first main result, which is a formula for the cohomology of X^n as a rational vector space. It involves a flag variety $\mathscr{F}_s$ and a toric variety $\bar{T}_s$.

Let $\mathscr{F}_s$ be the space of all direct sum decompositions $\mathbb{C}^n = P_1 \oplus P_2 \oplus \cdots \oplus P_s \oplus L_{2s+1} \oplus L_{2s+2} \oplus \cdots \oplus L_n$ where the P_i are two dimensional subspaces, the L_k are one dimensional subspaces, and all of the subspaces are orthogonal with respect to the standard Hermitian inner product on $\mathbb{C}^n$. The product of symmetric groups $\Sigma_s \times \Sigma_{n-2s}$ acts on $\mathscr{F}_s$ by permuting the labelling of the planes P_i and the labelling of the lines L_i. The space $\mathscr{F}_s$ is homeomorphic to the space of $(2, 4, \ldots, 2s, 2s+1, 2s+2, \ldots, n-1)$-flags in $\mathbb{C}^n$. Hence its cohomology together with the action of $\Sigma_s \times \Sigma_{n-2s}$ on it is conceptually computable using [Bo1] or [Bo2]. (See [Ste] and [BM] for explicit calculations of the action.)

In order to specify a Toric variety, we need the following data: a real vector space V, an integral lattice L in V, and a rational polyhedral cone decomposition Y of V. For any integer m, let V_m be the hyperplane in $\mathbb{R}^m$ with coordinates $a_1, a_2, \ldots, a_m$ defined by the equation $\Sigma_i a_i = 0$. Let L_m be the lattice of points on which each a_i takes on integral values, and let Y_m be the cone decomposition generated by the hyperplanes in V which are defined by $a_i = a_j$ for all pairs $i \neq j$. (This is a description of the decomposition into Weyl chambers for the root system A_{m-1}.) Let $\bar{T}_s$ be the toric variety corresponding to the data V_{n-s}, L_{n-s}, and Y_{n-s}. An action of the permutation group $\Sigma_s \times \Sigma_{n-2s}$ on $\bar{T}_s$ is induced from the following action on V_{n-s}: Σ_s permutes the s coordinates $a_1, a_2, \ldots, a_s$ and Σ_{n-2s} permutes the $n-2s$ coordinates $a_{s+1}, \ldots, a_{n-2s}$. The cohomology of $\bar{T}_s$ together with the action of $\Sigma_s \times \Sigma_{n-2s}$ on it is conceptually computable by [D] §10. Also, there is a more explicit calculation of it in [P].

THEOREM. *There is an isomorphism of groups*

$$H^k(X^n; \mathbb{Q}) \cong \bigoplus_{s=1}^{[n/2]} \left\{ \bigoplus_{a+b=k-4s} [H^a(\mathscr{F}_s; \mathbb{Q}) \otimes H^b(\bar{T}_s; \mathbb{Q})]^{\Sigma_s \times \Sigma_{n-2s}} \right\}$$

In this formula, the superscript $\Sigma_s \times \Sigma_{n-2s}$ means to take the invariants. Note that this could not be a ring isomorphism because the dimension shift of $4s$ would contradict the grading.

(The constructions above have the following interpretation in terms of algebraic groups. Let G be PGL(n), the automorphism group of the projective plane $\mathbb{P}^{n-1}$. Then the space $\mathscr{F}_s$ is G/P_s for an appropriate parabolic subgroup P_s. The vector space V_{n-s} is the subspace of the Cartan subalgebra of G given by zeros of s orthogonal roots and Y is the restriction to V of the decomposition into Weyl chambers. This formula is the extension to the case of complete quadrics of a formula of deConcini and Procesi for the cohomology of the completion of an adjoint group viewed as a symmetric variety [DP3]. In the case of the adjoint group, the first direct sum was unnecessary since there was only one term.)

1.5. Our second main result is a determination of the ring structure on the rational cohomology of the variety of complete quadrics.

Fix an integer s in $\{1, 2, \ldots, [n/2]\}$. We will define an algebra R_s over the rationals (see also §1.6). Consider the one dimensional cones $\sigma_1, \sigma_2, \ldots, \sigma_r$ in the cone decomposition Y_n of V_n (defined above) which lie in the closed quadrant defined by $a_1 \le a_2$, $a_3 \le a_4, \ldots, a_{2s-1} \le a_{2s}$. In each σ_i, let $\vec{D}_i$ be the vector to the first lattice point in L_i.

Recall that the cohomology ring $H^*(\mathcal{F}_s, \mathbb{Q})$ of the flag variety $\mathcal{F}_s$ is generated by the cohomology classes $c^1(P_j)$, $c^2(P_j)$, and $c^1(L_k)$ where P_j, (for $j = 1, 2, \ldots, s$) and L_k, (for $k = s+1, \ldots, n$) are the tautological plane bundles and line bundles over $\mathcal{F}_s$ and where c^i denotes the Chern classes.

DEFINITION. The ring R_s is the polynomial ring

$$H^*(\mathcal{F}_s; \mathbb{Q})[D_1, D_2, \ldots, D_r]$$

(in commuting variables $D_1, \ldots, D_r$ of degree 2), divided by the following four relations:

1. The monomial $D_{i_1} D_{i_2} \cdots D_{i_k} = 0$ whenever $\sigma_{i_1}, \sigma_{i_2}, \ldots,$ and σ_{i_k} do not all lie in some cone of Y_n.

2. For each D in $\{D_1, D_2, \ldots, D_r\}$ and each integer j in $\{1, 2, \ldots, s\}$ such that $a_{2j-1}(\vec{D}) < a_{2j}(\vec{D})$, let $\tilde{D}$ denote the sum of all the D_i such that $a_{2j-1}(\vec{D}_i) < a_{2j}(\vec{D}_i)$. Then

$$(\tilde{D}^2 - 4c^1(P_j)^2 + 16c^2(P_j))D = 0.$$

3. For each j in $\{1, 2, \ldots, s\}$.

$$\sum_{i=1}^{r} D_i(a_{2j}(\vec{D}_i) + a_{2j-1}(\vec{D}_i)) = c^1(P_j)$$

4. For each k in $\{2s+1, 2s+2, \ldots, n\}$,

$$\sum_{i=1}^{r} D_i a_k(\vec{D}_i) = c^1(L_k)$$

The permutation group $\Sigma_s \times \Sigma_{n-2s}$ acts on R_s. It acts on $H^*(\mathcal{F}_s)$ as described earlier, and it permutes the generators D_i by acting on V_n as follows: Σ_s permutes the pairs of coordinates $(a_1, a_2), (a_3, a_4), \ldots, (a_{2s-1}, a_{2s})$ and Σ_{n-2s} permutes the coordinates $a_{2s+1}, \ldots, a_n$.

Next, we define a sequence of ring homomorphisms

$$R_{[n/2]} \to R_{[n/2]-1} \to \cdots \to R_1 \to R_0$$

Since $H^*(\mathscr{F}_s) \subset H^*(\mathscr{F}_{s-1})$, to define the map $R_s \to R_{s-1}$ we only need to describe where the generators D_i of R_s go. If $a_{s-1}(\vec{D}_i) = a_s(\vec{D}_i)$ then D_i goes to itself. If $a_{s-1}(\vec{D}_i) < a_s(\vec{D}_i)$, then D_i goes to $D_i + D_j$ where $\vec{D}_j$ is the reflection of $\vec{D}_i$ in the hyperplane $a_{s-1} = a_s$.

THEOREM. *The ring $H^*(X_n; \mathbb{Q})$ is isomorphic to the subring of $R_{[n/2]}$* consisting of elements r whose image in the ring R_s is invariant under $\Sigma_s \times \Sigma_{n-2s}$ for each s in $\{1, 2, \ldots, [n/2]\}$.

1.6. The main technical tool of this paper is the idea of a diagonal quadric with respect to a direct sum decomposition of the space $\mathbb{C}^n$. This notion has a natural extension to complete quadrics (see §2.2). The toric variety $\bar{T}_n$ is isomorphic to the variety of all complete quadrics which are diagonal with respect to a fixed decomposition of $\mathbb{C}^n$ into lines. The ring R_s is the cohomology ring of the variety M_s which is the fiber bundle over the flag manifold $\mathscr{F}_s$ whose fiber over the point $P_1 \oplus \cdots \oplus P_s \oplus L_{2s+1} \oplus \cdots \oplus L_n$ is the space of complete quadrics which are diagonal with respect to this decomposition of $\mathbb{C}^n$.

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Chapter 2. Definitions

Throughout this paper, we fix an integer $n > 0$. We will make use of the standard basis and Hermitian product on $\mathbb{C}^n$. Cohomology groups will always be taken with coefficients in the rational numbers $\mathbb{Q}$.

§2.1. Complete quadrics. A *nondegenerate quadric cone Q* in a vectorspace $V \cong \mathbb{C}^N$ is a cone whose equations are given by the (multiples of) a complex valued nondegenerate quadratic form on V. (i.e., with respect to the standard basis, it is given by (all multiples of) a nonsingular symmetric matrix of complex numbers). The nondegenerate quadric cones in $\mathbb{C}^n$ are in one to one correspondence with nonsingular subvarieties of degree 2 in $\mathbb{P}^{n-1} = \mathbb{P}(C^n)$, which are called *nonsingular quadrics*.

Remark. By convention, a one dimensional vectorspace V contains a unique quadric cone. A double hyperplane (i.e. a hyperplane with multiplicity 2) is considered to be a singular quadric (even though its underlying space is nonsingular).

Let X^0 denote the variety of nonsingular quadric hypersurfaces of $\mathbb{P}^{n-1}$. It has a smooth, projective compactification X, which is called the variety of complete quadrics in $\mathbb{P}^{n-1}$ ([DP1], [DP2], [Se], [T], [V]) which we now describe.

Let $I = \{1_i, i_2, \ldots, i_{k-1}\}$ be any subset (possibly empty) of the numbers $\{1, 2, 3, \ldots, n-1\}$. A *partial flag F of type I* is a sequence of subspaces,

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{k-1} \subset F_k = \mathbb{C}^n$$

such that for each p (with $1 \le p \le k-1$), $\dim(F_p) = i_p$.

DEFINITION. The variety X_I of *complete quadrics of type I* is the variety of all pairs (F, Q), where $F = \{F_p\}$ is a partial flag of type I, and Q is a collection which consists of a nondegenerate quadric cone Q_p in each F_p/F_{p-1} (where $1 \le p \le k$). (We make the convention that $X_\Phi = X^0$.)

PROPOSITION 1.1. *The variety X of complete quadrics (as defined in* [DP2], [Se], [Sch], [T], *or* [V]) *is the union*

$$X = \bigcup_I X_I$$

Proof. The proof depends on which rigorous definition of X is used (see the introduction to this paper for a list of such definitions). For a proof starting from the definitions in algebraic geometry, see [V]. For a proof starting from the group theoretic construction of [DP1], see §5.6.

Remark. In fact the space X is a "wonderful" compactification ([DP1]) of X^0, i.e. it is nonsingular, $X - X^0$ is a union of nonsingular divisors $D_1, D_2, \ldots, D_n$ which meet transversally. (This was proven by Severi [Sv] for the case of conics, by Semple [Se] for the case of quadrics, in space, by Alguneid [A] for quadrics in $\mathbb{P}^4$, by Tyrrell [T] in the general case of quadrics, and by DeConcini and Procesi [DP1] for general symmetric varieties). If $I = \{i_1, i_2, \ldots, i_{k-1}\}$ is any subset (possibly empty) of the numbers $\{1, 2, 3, \ldots, n-1\}$, then (for an appropriate renumbering of the divisors D_i) the closure $\bar{X}_I$ of the set of complete quadrics of type I is precisely the intersection $\bar{X}_I = D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k}$ (See §5.6, or [V] theorem 6.3).

EXAMPLE. The variety X of complete quadrics in $\mathbb{P}^1$ consists of all unordered pairs of (not necessarily distinct) points in $\mathbb{P}^1$. It is isomorphic to $\mathbb{P}^2$. A single point $q \in \mathbb{P}^1$ with multiplicity 2 is called a degenerate quadric: such points form a subvariety *which we will denote by* $\partial\mathbb{P}^2$. It is isomorphic to $\mathbb{P}^1$ but is itself embedded as a quadric hypersurface in X. (See §5.8 for more details).

§2.2. Diagonal Quadrics. Suppose we are given a direct sum decomposition,

$$\mathbb{C}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

of $\mathbb{C}^n$ into a sum of complex vectorspaces. A nondegenerate quadric cone in $\mathbb{C}^n$ is *diagonal* with respect to this decomposition if (with respect to a basis adapted to this decomposition) the symmetric matrix corresponding to the quadric has no nonzero off diagonal blocks.

We now define the notion of a diagonal complete quadric (which will turn out to be the limit in X of a sequence of diagonal nondegenerate quadrics). Fix $I = \{i_1, i_2, \ldots, i_{k-1}\} \subset \{1, 2, \ldots, n-1\}$ as above. An *I-filtration* of this direct sum decomposition of $\mathbb{C}^n$ is a k-step filtration of each V_m,

$$0 = V_m^0 \subset V_m^1 \subset V_m^2 \subset \cdots \subset V_m^{k-1} \subset V_m^k = V_m$$

such that for each p (where $1 \le p \le k-1$) we have,

$$\sum_{m=1}^{r} \dim (V_m^p / V_m^{p-1}) = i_p$$

Each I-filtration of this direct sum decomposition gives rise to a flag F of type I by setting

$$F_p = \bigoplus_{m=1}^{r} V_m^p$$

(but some flags of type I are not obtained in this way).

DEFINITION. A complete quadric (F, Q) (of type I) is diagonal with respect to the direct sum decomposition

$$\mathbb{C}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

if there exists an I-filtration $\{V_m^p\}$ of this direct sum decomposition such that

(a) The flag F arises from this I-filtration as described above.

(b) For each p (where $1 \le p \le k$), the nondegenerate quadric cone Q_p on F_p/F_{p-1} is diagonal with respect to the induced decomposition

$$F_p/F_{p-1} \cong \bigoplus_{m=1}^{r} (V_m^p/V_m^{p-1})$$

PROPOSITION 2.2. *The set of complete quadrics which are diagonal with respect to the decomposition* $\mathbb{C}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ *is precisely the closure* (*in* X) *of the set of nondegenerate quadrics which are diagonal with respect to this direct sum decomposition. It is an irreducible algebraic subvariety of* X.

Proof. The proof will appear in §5.6.

§2.3. The spaces Z_s and ∂Z_s. The following construction will be made with respect to the standard basis of $\mathbb{C}^n$.

DEFINITION. The variety Z_s is the set of all complete quadrics which are diagonal with respect to the decomposition of $\mathbb{C}^n$ into the first s coordinate planes and the last $n-2s$ coordinate lines,

$$\mathbb{C}^n = P_1 \oplus P_2 \oplus \cdots \oplus P_s \oplus L_{2s+1} \oplus L_{2s+2} \oplus \cdots \oplus L_n$$

There is an action of $\Gamma_s = \Sigma_s \times \Sigma_{n-2s}$ on Z_s: Fix $\sigma \in \Sigma_s$ and $\tau \in \Sigma_{n-2s}$ (which we will think of as the permutation group on the numbers $\{2s+1, 2s+2, \ldots, n\}$. Since we have chosen a basis of $\mathbb{C}^n$ compatible with the direct sum decomposition, we obtain isomorphisms

$$P_j \cong P_{\sigma(j)}$$

$$L_i \cong L_{\tau(i)}$$

which induces an isomorphism

$$\mathbb{C}^n = P_1 \oplus \cdots \oplus L_n \cong P_{\sigma(1)} \oplus \cdots \oplus L_{\tau(n)} = \mathbb{C}^n$$

This isomorphism takes a complete quadric Q to some complete quadric Q', so it induces an action on Z_s. (A different choice of basis of $\mathbb{C}^n$ and of a direct sum decomposition of $\mathbb{C}^n$ which was compatible with that basis would induce a homotopic action of Γ_s).

We obtain a canonical map

$$\mu : Z_s \to \prod_{i=1}^{s} \mathbb{P}^2$$

by associating to each complete quadric (F, Q) (which is diagonal with respect to the above decomposition), its intersections

$$((F \cap P_1, Q \cap P_1), (F \cap P_2, Q \cap P_2), \ldots, (F \cap P_s, Q \cap P_s))$$

(See the example of §2.1.) In other words, if (F, Q) is a complete quadric of type $I = \{i_1, i_2, \ldots, i_r\}$ which is diagonal with respect to this decomposition of $\mathbb{C}^n$, then each partial flag

$$0 \subset F_1 \cap P_j \subset F_2 \cap P_j \subset \cdots \subset F_r \cap P_j$$

reduces to either a two-step flag

$$0 \subset F_k \cap P_j \subset P_j$$

(which gives an element of $\partial\mathscr{P}^2 \subset \mathbb{P}^2$, and in which we say the intersection $(F \cap P_j, Q \cap P_j)$ is *degenerate*) or else to a one step flag,

$$0 = F_k \cap P_j \subset F_l \cap P_j = P_j$$

and in this case, $Q_l \cap P_j$ is a *nondegenerate* quadric in P_j, so it also gives an element of $\mathbb{P}^2$.

PROPOSITION 2.3. *This map is well defined, continuous, and algebraic.*

Proof. The proof will appear in §5.9.1 (and its corollary).

DEFINITION. We define the divisors

$$\Delta_i = \mathbb{P}^2 \times \mathbb{P}^2 \times \cdots \times \partial\mathbb{P}^2 \times \cdots \mathbb{P}^2$$

(where $\partial\mathbb{P}^2$ appears in the ith position, and denotes the set of degenerate quadrics in $\mathbb{P}^1$), their union

$$\Delta = \bigcup_{i=1}^{s} \Delta_i$$

and the pre-images,

$$\partial_i Z_s = \mu^{-1}(\Delta_i)$$

$$\partial Z_s = \mu^{-1}(\Delta)$$

It is easy to see that Σ_s permutes the boundary divisors $\partial_i Z_s$.

§2.4. The spaces M_s. We will use the standard Hermitian metric on $\mathbb{C}^n$. For each integer s, where $0 \le s \le [n/2]$ we define a space M_s of pairs ((P, L), (F, Q)) where (P, L) is an *orthogonal* direct sum decomposition,

$$\mathbb{C}^n = P_1 \oplus P_2 \oplus \cdots \oplus P_s \oplus L_{2s+1} \oplus L_{2s+2} \cdots \oplus L_n$$

into two dimensional subspaces $P_1, P_2, \ldots, P_n$ and one dimensional subspaces $L_{2s+1}, L_{2s+2}, \ldots, L_n$, and where (F, Q) is a complete quadric which is diagonal with respect to this decomposition of $\mathbb{C}^n$. The space M_s is an algebraic variety, but in an unnatural way, and the canonical map

$$\Phi_s : M_s \to X$$

(which is given by $\Phi_s((P, L), (F, Q)) = (F, Q)$) is not an algebraic map. There is a canonical (real analytic but not algebraic) map $\beta : M_{s-1} \to M_s$ which is obtained by setting

$$\beta((P, L), (F, Q)) = ((P', L'), (F, Q))$$

where $P'_i = P_i$ for $1 \le i \le s-1$, and $L'_i = L_i$ for $2s+1 \le i \le n$ and

$$P'_s = L_{2s-1} \oplus L_{2s}$$

For each i, (where $1 \le i \le s$) we define the space

$$\partial_i M_s = \{((P, L), (F, Q)) \in M_s \mid (F \cap P_i, Q \cap P_i) \text{ is degenerate}\}$$

(see §2.3). The symmetric group Σ_s on s letters acts on M_s by permuting the (labelling of the) P's, and the symmetric group Σ_{n-2s} acts on M_s by permuting the L's. These actions do not affect the quadric Q, so the fibres of the map Φ_s are invariant under the group $\Gamma_s = \Sigma_s \times \Sigma_{n-2s}$. It is easy to see that Σ_s permutes the components $\partial_i M_s$, so it leaves invariant the subvariety

$$\partial M_s = \bigcup_{i=1}^{s} \partial_i M_s$$

There is a canonical $\Gamma_s = \Sigma_s \times \Sigma_{n-2s}$—equivariant fibre bundle map

$$\pi_1 : M_s \to \mathscr{F}_s$$

to the manifold $\mathscr{F}_s$ of partial flags of type $I_s = \{2, 4, \ldots, 2s, 2s+1, 2s+2, \ldots, n-1\}$ which associates to each pair $((P, L), (F, Q))$ the flag of partial sums of the P's and L's. (Here $\Gamma_s = \Sigma_s \times \Sigma_{n-2s}$ acts on $\mathscr{F}_s$ by permuting the P's and L's, which can be recovered from the partial flag by using the Hermitian metric.) It follows that Γ_s acts on the cohomology sheaves $R^i(\Phi_s)_*(\mathbb{Q})$ of the fibre $Z_s = \pi_1^{-1}(p)$. Since $\mathscr{F}_s$ is simply connected, these sheaves are constant, and we obtain an action of Γ_s on the cohomology $H^*(Z_s)$ of the fibre.

PROPOSITION 2.4. *This action of $\Gamma_s = \Sigma_s \times \Sigma_{n-2s}$ on $H^*(Z_s)$ coincides with the action defined in* §2.3.

Proof. The proof will appear in §5.7.4.

§2.5. The toric varieties $\bar{T}$ and $\bar{T}_s$. Consider the torus $(\mathbb{C}^*)^s \times (\mathbb{C}^*)^{n-2s}$ which acts on $\mathbb{C}^n$ by scalar multiplication on each of the factors in the above direct sum decomposition. Each $x \in (\mathbb{C}^*)^s \times (\mathbb{C}^*)^{n-2s}$ thus corresponds to an $n \times n$ matrix. This torus acts by projective transformations on $\mathbb{P}^{n-1}$ and it transforms the set Z_s into itself under the induced action on quadrics. However the subtorus

$$D = \{x \in (\mathbb{C}^*)^s \times (\mathbb{C}^*)^{n-2s} \mid xx^T = x^2 = \lambda I \text{ for some } \lambda\}$$

acts on the symmetric matrices which represent nondegenerate quadrics in Z_s by homotheties, so it induces a trivial action on Z_s. Therefore the action factors through an action of the quotient

$$T_s = ((\mathbb{C}^*)^s \times (\mathbb{C}^*)^{n-2s}))/D$$

It is easy to see that the torus T_s preserves the fibres of the map μ and that T_s acts with an open dense orbit on any fibre of μ. We fix the "basepoint" $p = \mu(Q_0)$, where Q_0 denotes the homogeneous quadric cone in $\mathbb{C}^n$ which corresponds to the identity matrix.

DEFINITION. We define the toric variety $\bar{T}_s$ to be the fibre $\mu^{-1}(p)$, together with its action of T_s. We denote by $\bar{T}$ the maximal torus, $\bar{T} = \bar{T}_0 = Z_0$.

The action of $\Gamma_s = \Sigma_s \times \Sigma_{n-2s}$ on Z_s which was described in §2.4 restricts to an action on $\bar{T}_s = \mu^{-1}\mu(Q_0)$ (because each of the isomorphisms $P_j \cong P_{\sigma(j)}$ take $Q_0 \cap P_j$ to $Q_0 \cap P_{\sigma(j)}$).

These spaces and maps can be arranged in the following diagram:

$$\begin{tikzcd}
 & \bar{T}_s \subset \bar{T} \subset Z_s \arrow[r, "\mu"] \arrow[d, hook] & \prod_{i=1}^{s} \mathbb{P}^2 \supset \Delta_i \\
X & M_s \arrow[l, "\Phi_s"'] \arrow[d, "\pi_1"'] & \\
 & \mathcal{F}_s &
\end{tikzcd}$$

Chapter 3. Statement of results

§3.1. Statement of the main theorems. We use the notation of chapter 2. All cohomology groups will be taken with rational coefficients. Let $m = [n/2]$. The maps $\Phi_s : M_s \to X$ fit together in a tower of spaces,

$$X \xleftarrow{\theta_m} M_m \xleftarrow{\theta_{m-1}} M_{m-1} \xleftarrow{\theta_{m-2}} \cdots \xleftarrow{\theta_1} M_1 \xleftarrow{\theta_0} M_0$$

(The group actions Γ_s are not compatible with these maps. They are K-equivariant maps, but are not G-invariant, where $G = \mathrm{PGL}_n(\mathbb{C})$ and K is the maximal compact subgroup of G, i.e. the special unitary group).

Let $\Phi_s^* : H^*(X) \to H^*(M_s)$ and $\Psi_s^* : H^*(M_m) \to H^*(M_s)$ denote the induced homomorphisms on cohomology (where $0 \le s \le m$).

THEOREM 1. *The homomorphism $\Phi_m : H^*(X) \to H^*(M_m)$ is injective. Furthermore the image is precisely those cohomology classes which, for all s, pull back* *to Γ_s-invariant classes, i.e.*

$$H^*(X) \cong \{\xi \in H^*(M_m) \mid \Psi_s^*(\xi) \in H^*(M_s)^{\Gamma_s} \text{ for each } s,\ 0 \le s \le m\}$$

The cohomology ring $H^*(M_s)$ is completely described in theorem 6.1.2.

Now define the ideals, $I_s = \ker \Phi_s^* \subset H^*(X)$. These filter $H^*(X)$,

$$0 = I_m \subset I_{m-1} \subset \cdots \subset I_1 \subset I_0 \subset I_{-1} = H^*(X)$$

For each s (where $0 \le s \le m$) the subquotient $I_{s-1}/I_s \subset H^*(M_s)$ lies in the Σ_s-invariants.

THEOREM 2. *The homomorphism $H^*(M_s, \partial M_s) \to H^*(M_s)$ is injective and*

$$I_{s-1}/I_s = (H^*(M_s, \partial M_s))^{\Gamma_s}$$

THEOREM 3. *There is a canonical isomorphism,*

$$H^i(M_s, \partial M_s) \cong \bigoplus_{a+b=i} H^a(\mathcal{F}_s) \otimes H^{b-4s}(\bar{T}_s)$$

which is an isomorphism of representations of $\Gamma_s = \Sigma_s \times \Sigma_{n-2s}$.

Our proof of these results depends on the following two results (§3.2 and §3.3).

§3.2. The Main Lemma. *The long exact cohomology sequence for the pair $(M_s, \partial M_s)$ breaks into a series of $\Sigma_s \times \Sigma_{n-2s}$—equivariant short exact sequences,*

$$0 \to H^i(M_s, \partial M_s) \to H^i(M_s) \xrightarrow[\alpha]{} H^i(\partial M_s) \to 0$$

Furthermore, if we restrict to invariant cohomology, then we obtain a diagram

$$\begin{tikzcd} 0 \arrow[r] & H^i(M_s, \partial M_s)^{\Gamma_s} \arrow[r] & H^i(M_s)^{\Gamma_s} \arrow[r, "\alpha"'] \arrow[d, "\beta"'] & H^i(\partial M_s)^{\Gamma_s} \arrow[r] & 0 \\ & & H^i(M_{s-1}) & & \end{tikzcd}$$

and $\ker(\alpha) = \ker(\beta)$.

§3.3. Rational Cell Decomposition of M_s and Z_s.

DEFINITION. An algebraic rational cell C in an algebraic variety M is a locally closed algebraic subset such that $H_c^i(C; \mathbb{Q}) = 0$ for all but one value of i, which is called the dimension of the cell. (It is twice the dimension of C as an algebraic variety).

A *paving* of M by (algebraic) rational cells is a decomposition

$$M = \coprod_{\alpha \in J} C_\alpha$$

into finitely many rational cells C_α, together with a total ordering of the index set J such that for each $\beta \in J$, the set

$$M_\beta = \bigcup_{\alpha \leq \beta} C_\alpha$$

is closed in M.

If M has a paving by rational algebraic cells, then these cells are even dimensional and form a basis for the rational cohomology of M.

For each subset $I \subset \{1, 2, \ldots, s\}$ we define subsets

$$\overline{\partial_I M_S} = \bigcap_{i \in I} \partial_i M_s$$

$$\partial_I M_s = \overline{\partial_I M_S} - \bigcup_{J \supsetneq I} \overline{\partial_J M_s}$$

THEOREM 4. *There exists a paving of M_s into algebraic rational cells,*

$$M_s = \dot{\bigcup_{i=1}^{m}} C_i$$

such that

(a) *each $\partial_I M_S$ is a union of cells*

(b) *The subvariety $Z_s \subset M_s$ is a union of rational cells, each $\partial_i Z_s$ is a union of rational cells, and the map $\mu : Z_s \to \prod_{i=1}^{s} \mathbb{P}^2$ takes cells to cells.*

Proof. The proof will appear in §5.11.

§3.4. Numerology. We will often use the following simple facts about representations of a finite group G:

PROPOSITION 3.4. (1) *If V and W are two representations of G, and if V is isomorphic to a sum of copies of the regular representation of G, then we have*

$$\dim (V \otimes W)^G = \frac{\dim (V) \dim (W)}{|G|}$$

(2) *If V is the regular representation of G, and $H \subset G$ is a subgroup with normalizer N_H in G, then the space V^H (as a representation of N_H/H) is isomorphic to $(G:N_H)$ copies of the regular representation.*

We will apply these facts to the cohomology $V = H^*(\mathscr{F}; \mathbb{Q})$ of the flag variety in $\mathbb{P}^{n-1}$, which is the regular representation of Σ_n. The cohomology of a partial flag variety is of the form V^H, for a suitable subgroup H. In the case of the variety $\mathscr{F}_s$, we have

$$H = (\mathbb{Z}/(2))^s \quad \text{and} \quad N_H/H = \Sigma_s \times \Sigma_{n-2s}$$

and

$$H^*(\mathscr{F}_s; \mathbb{Q}) = V^H.$$

DEFINITION. We define

$$\mathscr{X}_s = \dim (H^*(\mathscr{F}_s) \otimes H^*(\bar{T}_s))^{\Sigma_s \times \Sigma_{n-2s}}$$

Remark. The variety $\bar{T}_s$ is nonsingular and admits an action of the algebraic torus T_s, with $(n-s)!$ fixed points, so by [BB1], $\dim H^*(\bar{T}_s) = (n-s)!$. Since $H^*(\mathscr{F}_s)$ is a sum of regular representations of $\Gamma_s = \Sigma_s \times \Sigma_{n-s}$ we have

$$\mathscr{X}_s = \frac{\dim H^*(\mathscr{F}_s) \dim H^*(\bar{T}_s)}{|\Sigma_s \times \Sigma_{n-2s}|} = \binom{n-s}{s} \frac{n!}{2^s}$$

PROPOSITION 3.5. *The Euler characteristic, $\mathscr{X}(X)$ satisfies*

$$\mathscr{X}(X) = \sum_{s=0}^{m} \mathscr{X}_s$$

Proof. See [Str]

Chapter 4. Proof of theorems 1, 2, and 3

§4.1. Φ_m is injective. In fact we will show that the map $M_m \to X$ is a map of finite and positive degree between nonsingular spaces of the same dimension (so by Poincaré duality the induced homomorphism on cohomology is injective). It is easy to see that the map $\Phi_m : M_m \to X$ is generically $\tau = n!/(2^m)$ to 1: Let Q be a generic diagonal quadric. Then a decomposition of $\mathbb{C}^n$ into m planes (and a line, if n is odd) which diagonalizes Q is just a decomposition of $\mathbb{C}^n$ into m coordinate planes (and possibly a line). There are τ such decompositions.

The group $\Sigma_m = \Gamma_m$ acts freely on the generic fibre $\Phi_m^{-1}(Q)$ so we get a partition of this fibre into $\tau/(m!)$ orbits of Σ_m. Since $\tau/(m!)$ is an odd integer, the map Φ_m will have nonzero degree provided Σ_m acts in an orientation preserving manner. This is a consequence of the following facts: (a) $\pi_m : M_m \to \mathscr{F}_m$ is a Σ_m—equivariant fibration whose fibres are algebraic varieties which are permuted by Σ_m in an orientation preserving way; and (b) Σ_m acts on $\mathscr{F}_m$ in an orientation preserving way. (This argument even shows that the degree of Φ_m is exactly $m!$).

We will prove theorems 1, 2, and 3 in reverse order.

§4.2. Proof of theorem 3. The fibre bundle $\pi_1 : M_s \to \mathscr{F}_s$ (with fibre Z_s) restricts to a bundle

$$\bar{\pi}_1 : M_s - \partial M_s \to \mathscr{F}_s$$

with fibre $Z - \partial Z_s$. Since each of these spaces has a "rational cell decomposition" (§3.3) with even dimensional cells, the cohomology spectral sequence for the map $\bar{\pi}_i$ degenerates at E^2. Furthermore, $\mathscr{F}_s$ is simply connected. Thus,

$$H^i(M_s, \partial M_s) = H^i_c(M_s - \partial M_s) \cong \bigoplus_{a+b=i} H^a_c(Z_s - \partial Z_s) \otimes H^b(\mathscr{F}_s)$$

and (by proposition 2.4) this is an isomorphism of representations of $\Sigma_s \times \Sigma_{n-2s}$. On the other hand, the map $\mu : Z_s \to (\mathbb{P}^2)^s$ restricts to a fibre bundle,

$$\bar{\mu} : Z_s - \partial Z_s \to (\mathbb{P}^2)^s - \Delta$$

with fibre $\bar{T}_s$ (where Δ was defined in §2.3). The fundamental group $\pi_1((\mathbb{P}^2)^s - \Delta)$ is a product $(\mathbb{Z}/(2))^s$ (see §5.8: $\mathbb{P}^2 - \partial\mathbb{P}^2$ is homotopy equivalent to $\mathbb{P}^2(\mathbb{R})$), but we will show (§5.9.7) that it acts trivially on $H^i(\bar{T}_s)$, from which we conclude that

$$H^a_c(Z_s - \partial Z_s) = H^{a-4s}(\bar{T}_s).$$

§4.3. ***Proof of theorems* 1 *and* 2.** *Recall that* $\Phi_s^*: H^*(X) \to H^*(M_s)$ denotes the induced homomorphism on cohomology. We now define the following groups:

DEFINITION.

$J_s = \text{Image}\,(\Phi_s^*) \subset H^*(M_s)$

$I_s = \text{Kernel}\,(\Phi_s^*) \subset H^*(X)$

$I_{-1} = H^*(X)$

$A_s = \{\xi \in H^*(M_s) \mid \text{for each } r \geq s,\ \xi \text{ pulls back to a } \Gamma_r \text{ invariant class in } H^*(M_r\}$

Since each J_s is contained in the invariant cohomology (see §1.1) we have a diagram

$$\begin{tikzcd}
 & H^*(M_m) \arrow[r, "\theta^*_{m-1}"'] & H^*(M_{m-1}) \arrow[r, "\theta^*_{m-2}"'] & \cdots \arrow[r, "\theta^*_0"'] & H^*(M_0) & \\
 & \cup & \cup & & \cup & \\
 & H^*(M_m)^{\Gamma_m} & H^*(M_{m-1})^{\Gamma_{m-1}} & & H^*(M_0)^{\Gamma_0} & \\
 & \cup & \cup & & \cup & \\
 & A_m \arrow[r] & A_{m-1} \arrow[r] & \cdots \arrow[r] & A_0 \arrow[r] & A_{-1} = 0 \\
 & \cup & \cup & & \cup & \cup \\
H^*(X) \arrow[r, two heads, "\cong"] & J_m \arrow[r, two heads] & J_{m-1} \arrow[r, two heads] & \cdots \arrow[r, two heads] & J_0 \arrow[r, two heads] & J_{-1} = 0
\end{tikzcd}$$

Where $\theta_s^*: H^*(M_{s+1}) \to H^*(M_s)$ is the induced homomorphism on cohomology. We also have the following inclusions,

$$I_{s-1}/I_s = (J_s \cap \ker(\theta^*_{s-1})) \subset (A_s \cap \ker(\theta^*_{s-1})) \subset (H^*(M_s)^{\Gamma_s} \cap \ker(\theta^*_{s-1}))$$
$$= H^*(M_s, \partial M_s)^{\Gamma_s}$$

by the main lemma (§3.2). Since $I_m = 0$ (§4.1), we have

$$\mathscr{X}(X) = \dim(J_m) = \sum_{s=0}^{m} \dim(J_s \cap \ker(\theta^*_{s-1}))$$
$$= \sum_{s=0}^{m} \dim(A_s \cap \ker(\theta^*_{s-1})$$
$$\leq \sum_{s=0}^{m} \dim H^*(M_s, \partial M_s)^{\Gamma_s}$$

$$= \sum_{s=0}^{m} \dim ((H^*(\mathcal{F}_s) \otimes H^*(\bar{T}_s))^{\Gamma_s}$$

$$= \sum_{s=0}^{m} \mathcal{X}_s = \mathcal{X}(X)$$

(By theorem 3). Consequently the above inclusions are all equalities, so

$$I_{s-1}/I_s \cong (H^*(\mathcal{F}_s) \otimes H^*(\bar{T}_s))^{\Gamma_s}$$

and the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A_s \cap \ker(\theta^*_{s-1}) \arrow[r] & A_s \arrow[r] & A_{s-1} & \\
0 \arrow[r] & J_s \cap \ker(\theta^*_{s-1}) \arrow[r] \arrow[u, equal] & J_s \arrow[r] \arrow[u] & J_{s-1} \arrow[r] \arrow[u] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

shows (by induction) that $A_s = J_s$ and, in particular,

$$H^*(X) \cong J_m \cong A_m$$

which proves theorems 1 and 2.

COROLLARY. *The homomorphism $H^*(X) \to A_0 = H^*(M_0)^{\Gamma_0}$ is surjective.*

§4.4. Proof of main lemma. We have seen (§3.3) that both M_s and ∂M_s have pavings by even dimensional "rational cells". It follows that the odd rational cohomology of each of these spaces vanishes. Therefore the long exact cohomology sequence for the pair $(M_s, \partial M_s)$ splits into short exact sequences.

We now prove that $\ker(\alpha) = \ker(\beta)$ by analyzing the map $\beta: M_{s-1} \to M_s$ of §2.4, which is obtained by setting $\beta((P, L), (F, Q)) = ((P', L'), (F, Q))$, where $P'_i = P_i$ for $1 \le i \le s-1$, and $L'_i = L_i$ for $2s+1 \le i \le n$, and

$$P'_s = L_{2s-1} \oplus L_{2s}$$

The group $\mathbb{Z}/(2)$ acts freely on M_{s-1} by switching the (labelling of) the subspaces L_{2s-1} and L_{2s}, and the map β is equivariant with respect to this action. It is easy to see that $\beta^{-1}(\partial_s M_s)$ consists of two disjoint copies of $\partial_s M_s$ which are switched under the $\mathbb{Z}/(2)$ action. Thus we have a commutative diagram, where β' is an isomorphism:

$$\begin{tikzcd}
H^*(M_s) \arrow[r, "\tilde{\alpha}"] \arrow[d, "\tilde{\beta}"'] & H^*(\partial_s M_s) \arrow[d, "\beta'"] \\
H^*(M_{s-1})^{\mathbb{Z}/(2)} \arrow[r, "\tilde{\gamma}"'] \arrow[d, phantom, "\cap"] & H^*(\beta^{-1}(\partial_s M_s))^{\mathbb{Z}/(2)} \arrow[d, phantom, "\cap"] \\
H^*(M_{s-1}) \arrow[r] & H^*(\beta^{-1}(\partial_s M_s))
\end{tikzcd}$$

LEMMA. $\ker(\tilde{\alpha}) = \ker(\tilde{\beta})$.

Proof of lemma. The homomorphism $\tilde{\alpha}$ is surjective since (§5.11) M_s has a cell decomposition with even dimensional cells such that $\partial_s M_s$ is a union of cells. Since β' is an isomorphism, it follows that $\tilde{\gamma}$ is surjective. But this means that $\tilde{\tau}$ is an isomorphism since

$$\sum_i \dim H^i(M_{s-1})^{\mathbb{Z}/(2)} = \sum_i \dim H^i(\partial_s M_s)$$

(by corollary 5.9.4). This proves the lemma.

Now consider the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & H^*(M_s, \partial M_s)^{\Gamma_s} \arrow[r] & H^*(M_s)^{\Gamma_s} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & H^*(\partial_s M_s)^{\Gamma_s} \arrow[r] \arrow[d, "\gamma"] & 0 \\
& & H^*(M_{s-1}) & \bigoplus_{i=1}^{s} H^*(\partial_i M_s) &
\end{tikzcd}$$

The homomorphism γ is injective since ∂M has a paving by rational cells such that each intersection $\partial_i M_s \cap \partial_j M_s$ is a union of cells (and so that the same is true for all multiple intersections). Thus, for any $\xi \in H^*(M_s)^{\Gamma_s}$ we have,

$$\begin{aligned}
&\alpha(\xi) = 0 \\
&\Leftrightarrow (\gamma \circ \alpha)(\xi) = 0 \\
&\Leftrightarrow \tilde{\alpha}(\xi) = 0 \\
&\Leftrightarrow \beta(\xi) = 0
\end{aligned}$$

(since ξ is Σ_s-invariant). This completes the proof of the main lemma.

Chapter 5. Complete quadrics and algebraic groups

§5.1. PGL_n acts on X

Recall (§2.1) that X is a union of varieties X_I, consisting of pairs (F, Q), where $F = \{F_p\}$ is a partial flag of type I, and Q is a collection of nonsingular quadric hypersurfaces Q_p in each $\mathbb{P}(F_p/F_{p-1})$. There is an obvious action of the group $\mathrm{PGL}_n(\mathbb{C})$ of projective transformations on the variety X_I: a transformation $A : \mathbb{C}^n \to \mathbb{C}^n$ takes each subspace F_p into some other subspace $A(F_p)$ and induces a projective transformation $\tilde{A}_p : \mathbb{P}(F_p/F_{p-1}) \to \mathbb{P}(A(F_p)/A(F_{p-1}))$ which takes the quadric hypersurface Q_p into some other quadric hypersurface $\tilde{A}(Q_p)$. Since all nondegenerate quadric hypersurfaces in projective space are projectively equivalent, we see that $G = \mathrm{PGL}_n(\mathbb{C})$ acts transitively on X_I.

PROPOSITION. *This action of G on X is algebraic, and the orbits of G are precisely the varieties X_I.*

Proof. This is precisely theorem 6.3 of [V] (p. 214); see also [Dr1], [Dr2].

Remark. In this section we give an explicit construction of the pair (F, Q) which corresponds to a complete quadric, as defined in §1.2 ("Constructions of X in algebraic geometry"). We refer to §5.3, or [T] and [V] for proofs. Suppose (as in §1.2) that $C = (C_0, C_1, C_2, \ldots, C_{n-2}) \in X$, where each C_i is a point in the variety $\tilde{Q}_i$. (Here, $\tilde{Q}_i$ is the set of all degree 2 subvarieties of the Grassmannian G_i of i-planes in $\mathbb{P}^{n-1}$.) Now consider those C_i which are totally degenerate, i.e., which consist of the "double hyperplane sections" of i-dimensional subspaces which meet a fixed $n-i-2$ dimensional subspace F_i. Let $j_1 \le j_2 \le \cdots \le j_r$ be the indices for which this occurs. Then $F_{j_1} \supset F_{j_2} \supset \cdots \supset F_{j_r}$ is the required flag F.

Now let k be any other index such that $j_r < k < j_{r+1}$ (where $-1 = j_0$ and $n-1 = j_n$). Then C_k is the closure of its open part,

$$C_k^0 = \{\pi \in C_k \mid \pi \cap F_{j_r} \text{ is proper}\}$$

This open part has an alternate description

$$C_k^0 = \{\pi \mid \pi \cap F_{j_r} \text{ is proper and is a tangent subspace to } Q_r \text{ in } F_{j_r} \text{ with kernel } F_{j_{r+1}}\}$$

for some fixed quadric Q_r. This sequence of quadrics gives the element Q.

§5.2. Review of toric varieties. To fix notation, we review some basic facts about torus embeddings ([D], [Ash], [K]).

5.2.1. Cone Decompositions. Let $S = (\mathbb{C}^*)^r$ be an r-dimensional torus. We denote the group of one parameter subgroups of S by $X^*(S)$, and the dual group of characters of S by $X_*(S)$. An *S-embedding* is a complete normal algebraic variety $\bar{S}$ which contains S as a dense open subset, and which is an S-equivariant compactification of S.

A rational polyhedral cone decomposition (R.P.D.) Σ of the vectorspace $X^*(S) \otimes \mathbb{R}$ is a decomposition,

$$X^*(S) \otimes \mathbb{R} = \bigcup_{\alpha} c_\alpha$$

into finitely many closed rational polyhedral convex cones, c_α which are centered at the origin, such that

(1) no c_α contains a line (i.e. a 1 dimensional linear subspace)
(2) for each α, every face of c_α is a cone $c_\beta \in \Sigma$
(3) for any α, β, the intersection $c_\alpha \cap c_\beta$ is a face of both c_α and c_β.

PROPOSITION. *The possible S-embeddings are in one to one correspondence with the possible R.P.D. of the vectorspace $X^*(S) \otimes \mathbb{R}$. If $\bar{S}$ denotes such an S-embedding with associated R.P.D. Σ, then each closed cone $c_\alpha \in \Sigma$ corresponds to a unique S-orbit, $\langle c_\alpha \rangle$ in $\bar{S}$, and*

$$c_\alpha \subset c_\beta \Leftrightarrow \overline{\langle c_\alpha \rangle} \supset \langle c_\beta \rangle$$

In fact, the points in the orbit $\langle c_\alpha \rangle$ can be identified as follows; there is a canonical isomorphism

$$S/R \to \langle c_\alpha \rangle$$

where R is the subgroup of S which is generated by all the one-parameter subgroups in the closed cone c_α. With this identification, convergence from the largest orbit $S = \langle 0 \rangle$ to a smaller orbit $\langle c_\alpha \rangle$ is given as follows: If $A \in c_\alpha$ is a one parameter subgroup, and if $b \in S$, then

$$\lim_{\substack{a \to \infty \\ a \in A}} ba = bR \in S/R$$

5.2.2. Maps between toric varieties. Suppose $\{\mu_1, \mu_2, \ldots, \mu_r\}$ is a basis for $X_*(S)$, and let Γ be a subset of these basis elements. Define

$$K = \{s \in S \mid s^\mu = 1, \text{ for all } \mu \in \Gamma\}$$

to be the associated subtorus, and let

$$\rho : S \to T = S/K$$

be the quotient mapping. We obtain an induced homomorphism of vectorspaces,

$$\rho_{\#} : X^*(S) \otimes \mathbb{R} \to X^*(T) \otimes \mathbb{R}$$

Now suppose we are given two toric varieties,

(a) $\bar{S}$ compactifying S, with associated R.P.D. $\Sigma \subset X^*(S) \otimes \mathbb{R}$
(b) $\bar{T}$ compactifying T, with associated R.P.D. $\Omega \subset X^*(T) \otimes \mathbb{R}$

PROPOSITION. *The homomorphism $\rho : S \to T$ extends to an S-equivariant homomorphism $\bar{\rho} : \bar{S} \to \bar{T}$ iff for every cone $\omega \in \Omega$, the preimage $\rho_{\#}^{-1}(\omega)$ is a union of cones in Σ.*

In this case, the preimage of any orbit in $\bar{T}$ is a union of orbits in $\bar{S}$, i.e.

$$(\bar{\rho})^{-1}(\langle \omega \rangle) = \bigcup \{\langle \sigma \rangle \mid \sigma \in \Sigma \text{ and } \rho_{\#}(\sigma) \subset \omega\}$$

In fact one can say more: if $\sigma \in \Sigma$ and if $\rho_{\#}(\sigma) \subset \omega$, and if a point $p \in \langle \sigma \rangle$ is represented by a coset aR (where R is the subtorus of S which is generated by the one parameter subgroups in σ), then $\bar{\rho}(p)$ is represented by the coset $\rho(a)R'$, where R' is the subtorus of T which is generated by the one parameter subgroups in ω. If $T' = \ker(\rho)$ is connected, then the fibre $\bar{\rho}^{-1}(1)$ is a T' torus embedding with associated R.P.D. equal to $\Sigma \cap \ker(\rho_{\#})$.

§5.3. The closure of the diagonal matrices. Define the hyperplane

$$V = \left\{ a \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} a_i = 0 \right\}$$

together with the linear functionals

$$\alpha_i(a) = a_{i+1} - a_i$$

Define the integral points in V to be the points a such that each $\alpha_i(a) \in \mathbb{Z}$. With respect to this integral structure on V, we define the following decomposition Σ of V into rational polyhedral cones: Σ is generated by the hyperplanes $a_i = a_j$ for

$i \neq j$. In other words, for each ordered partition $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_r\}$ of the numbers $\{1, 2, \ldots, n\}$, there is a closed cone $\sigma \subset V$, which is given by

$$\sigma = \left\{ a \in V \;\middle|\; \begin{array}{l} \text{if } i, j \in \Gamma_s \text{ then } a_i = a_j \\ \text{if } i \in \Gamma_{s_1} \text{ and } j \in \Gamma_{s_2} \text{ and } s_1 < s_2 \text{ then } a_i < a_j \end{array} \right\}$$

(An ordered partition is an ordered collection of disjoint, nonempty subsets whose union is the whole set $\{1, 2, \ldots, n\}$. There is no relation between the ordering of the Γ's and the ordering of the numbers between 1 and n.)

Let $T \subset X$ denote the diagonal nondegenerate quadrics (i.e. the nonsingular diagonal matrices, modulo scalar multiples). This is a torus T (under the operation of matrix multiplication), and a basis for the character group $X_*(T)$ is given by $\{\exp(\alpha_1), \exp(\alpha_2), \ldots, (\alpha_{n-1})\}$ where $\exp(\alpha_i)(t) = t_{i+1}/t_i$. (Here $t_1, t_2, \ldots, t_n$ denote the diagonal entries in the matrix t.) Thus the vectorspace $X^*(T) \otimes \mathbb{R}$ has been identified with the hyperplane V above.

PROPOSITION 5.3.1. *This identification extends to an identification of the closure* $\bar{T}$ (in X) *of the diagonal matrices with the toric variety associated to the cone decomposition* Σ

Proof. We recall the proof of [DP2] theorem 5.3.) In [DP1] there is constructed a basic T-stable affine open set A of $\bar{T}$, whose associated polyhedral cone is the fundamental Weyl chamber. The action of Σ_n stabilizes $\bar{T}$ and hence also $\bigcup_{w \in \Sigma_n} wA$. The open sets wA correspond to distinct Weyl chambers and since the chambers decompose V, we see that $\bigcup_{w \in \Sigma_n} wA$ is complete and hence coincides with $\bar{T}$.

Recall that one of the properties of A is that each orbit of the action of the projective group on the space of complete quadrics intersects A in a T orbit.

PROPOSITION 5.3.2. *If* $\sigma \in \Sigma$ *is a cone corresponding to a partition* $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_r\}$ *of the set* $\{1, 2, \ldots, n\}$ *then this identification takes the* T*-orbit* $\langle\sigma\rangle$ *into the stratum* (*or* G*-orbit*) *of* X *corresponding to the subset* $I = \{i_1, i_2, \ldots, i_{r-1}\} \subset \{1, 2, \ldots, n-1\}$ *where*

$$i_k = \sum_{j=1}^{k} |\Gamma_j|$$

Remark. As an immediate corollary of 5.3.2, we see that two T-orbits in $\bar{T}$ are in the same G-orbit of X if and only if they are in the same orbit under the symmetric group.

The points in the toric variety $\bar{T}$ are identified with points in X in an explicit way: A point $p \in \langle \sigma \rangle$ is given by a coset tR (where $t = (t_1, t_2, \ldots, t_n)$ and where R is the subgroup of T generated by the one parameter subgroups which are in σ).

PROPOSITION 5.3.3. *The point tR is identified with the following diagonal complete quadric (F, Q): F is the partial flag which is given by the direct sum decomposition*

$$\mathbb{C}^n = \langle e_{\Gamma_1} \rangle \oplus \langle e_{\Gamma_2} \rangle \oplus \cdots \oplus \langle e_{\Gamma_r} \rangle$$

where $\langle e_{\Gamma_i} \rangle$ denotes the span of the basis vectors $\{e_\lambda \mid \lambda \in \Gamma_i\}$. Within each $\langle e_{\Gamma_i} \rangle$ the nondegenerate quadric Q_i is given by the diagonal symmetric matrix (t_{Γ_i}) whose diagonal entries are the numbers $\{t_\lambda \mid \lambda \in \Gamma_i\}$.

Remark. Observe that this identification depends only on the coset of $t \,(\operatorname{mod} R)$ because R is the subgroup of diagonal matrices (t) such that each (t_{Γ_i}) is some multiple of the identity matrix.

Remark. Since the boundary divisors (i.e. the codimension 1 orbits) in $\bar{T}$ are given by two step flags $0 \subset F \subset \mathbb{C}^n$, these correspond to ordered partitions with two elements,

$$\Gamma_1 \cup \Gamma_2 = \{1, 2, \ldots, n\}$$

Proof of §5.3.2 *and* §5.3.3. We will use the construction of X which was described in §1.2 and was explained in §5.1. The map $\Phi_i : X_0^n \to \bar{Q}_i$ (from the space of nondegenerate quadrics to the space of quadric hypersurfaces of the Grassmannian G_i) is given in projective coordinates by the formula

$$M \to \Lambda^i M$$

where M is a symmetric n by n matrix and $\Lambda^i M$ is the matrix of determinants of i by i minors. This may be thought of as the matrix of a quadratic form on $\Lambda^i \mathbb{C}^n$ (i.e. a quadric in Plucker coordinates). The open set $A \subset \bar{T}$ (see the proof of §5.1) is then given as follows:

Let $\mathbb{A}^{n-1}$ be the affine space with coordinates $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$. We map $\mathbb{A}^{n-1}$ to the diagonal n by n matrices by

$$(\lambda_1, \ldots, \lambda_{n-1}) \to \operatorname{diag}(\lambda_1 \lambda_2 \cdots \lambda_{n-1}, \ldots, \lambda_1 \lambda_2, \lambda_1, 1) = M$$

Each entry of $\Lambda^i M$ is divisible by $\lambda_1^{i-1}\lambda_2^{i-2}\cdots\lambda_{i-1}$. Since we are using projective coordinates, we may divide by this and obtain a well defined map $\tilde{\phi}_i:\mathbb{A}^{n-1}\to\tilde{Q}_i$. Then the map $\tilde{\phi}=(\tilde{\phi}_1,\ldots,\tilde{\phi}_{n-1})$ maps $\mathbb{A}^{n-1}$ isomorphically to its image, which we call $\mathbb{A}$. (cf. [DP1], [Se], [T]). It is now easy to check that if $p\in\mathbb{A}^{n-1}$ and if the ith coordinate of p is 0, then $\tilde{\phi}_i(p)$ is the set of $i-1$ dimensional subspaces in $\mathbb{P}^{n-1}$ which meet the subspace defined by the vanishing of the last $n-i$ coordinates. Following the explanation given in §5.1, one can verify that the flag obtained is a part of the standard flag, and that the quadrics Q_i are diagonal. As remarked in §5.3.1, the orbit of $\mathbb{A}$ under the symmetric group is all of $\bar{T}$. On the other hand, this action of the symmetric group amounts to modifying the flag by permuting the subspaces which are given by the coordinate axes. This shows that $\bar{T}$ describes all diagonal complete quadrics. The rest of §5.3.2 and §5.3.3 can be verified by inspection.

§5.4. Partially diagonal quadrics. For any subset $I=\{i_1,i_2,\ldots,i_r\}\subset\{1,2,\ldots,n\}$ there is a canonical decomposition of $\mathbb{C}^n$ into coordinate planes,

$$\mathbb{C}^n=V_1\oplus V_2\oplus\cdots\oplus V_r$$

with $\dim(V_j)=v_j=i_j-i_{j-1}$. We define

$$G_I\cong\mathbb{P}(\mathrm{Gl}_{v_1}(\mathbb{C})\times\cdots\times\mathrm{GL}_{v_r}(\mathbb{C}))$$

to be the group of projective transformations preserving each of the subspaces in this decomposition. We denote by G the projective linear group $\mathrm{PGL}_n(\mathbb{C})$ and by $K\subset G$ the projective unitary group and $K_I=K\cap G_I$.

LEMMA 5.4.1. *The subset $Z_I\subset X$ (of complete quadrics which are diagonal with respect to the above decomposition of $\mathbb{C}^n$) is stable under the action of G_I and of K_I. Furthermore, two points (F,Q) and (F',Q') of Z_I lie in the same G_I orbit if and only if for each i,*

$$\dim(F_i)=\dim(F_i')$$

(i.e., the flags F and F' are of the same type) and for all i and j,

$$\dim(F_i\cap V_j)=\dim(F_i'\cap V_j).$$

Proof. These conditions are clearly necessary since G_I leaves the spaces V_j stable. To see that they are sufficient, suppose we are given two such points

(F, Q) and (F', Q'). Choose $g_j \in \mathrm{GL}(V_j)$ so that for each i,

$$g_j(F_i \cap V_j) = F_i' \cap V_j$$

The element $g = (g_1, g_2, \ldots, g_r) \in G_I$ takes (F, Q) to (F', Q''). But this point is G_I-conjugate to (F', Q') because the stabilizer of F acts on each F_j/F_{j-1} as the full linear group $\mathrm{GL}(F_j/F_{j-1})$.

In the next two sections we will prove proposition 2.2.

§5.5. Diagonalization by K. Let Z_I^0 denote the collection of *nondegenerate* quadrics which are diagonal with respect to the above decomposition.

PROPOSITION 5.5. *For any nondegenerate quadric* $Q \in Z_I^0$ *there is an element* $k \in K_I$ *such that* $k.Q$ *is* (completely) *diagonal, i.e.* $k.Q. \in Z_0^0 = T$.

Proof of proposition. It suffices to show that each of the intersections $Q \cap V_i$ can be diagonalized separately, and since the argument is the same for each V_i we will replace it with $\mathbb{C}^n$. Thus, it suffices to show that for any nondegenerate quadric Q in $\mathbb{C}^n$, there exists an orthonormal basis of $\mathbb{C}^n$ with respect to which the matrix representing Q is a diagonal matrix. (Here, orthonormal refers to the standard Hermitian metric on $\mathbb{C}^n$ which is given by

$$H(x, y) = \sum_{i=1}^{n} \bar{x}_i y_i$$

We will also use the standard identification $\mathbb{R}^{2n} \cong \mathbb{C}^n$ as $\mathbb{R}$-vectorspaces.)

This is a consequence of the generalized Cartan decomposition

$$G = KA^+H$$

of [Fl] (theorem 4.1, page 118) and [R] (theorem 10, page 169). Take $G = \mathrm{Gl}_n(\mathbb{C})$ (viewed as a real Lie group), $H = \mathrm{O}(n, \mathbb{C})$, $K = U(n)$, and $A^+ \subset A =$ the group of real diagonal matrices with positive entries. The unitary matrix $u \in U(n)$ takes the standard basis to the desired orthonormal basis.

Remark. There is also an ordering (which is possibly degenerate) induced on this basis of $\mathbb{C}^n$ which is given by $v_j \leq v_k$ iff $|\lambda_j| \leq |\lambda_k|$.

CORROLLARY. *For any G-equivariant compactification Y of the space X^0 of nondegenerate quadrics, the closure $\bar{T}$ of the diagonal quadrics meets every G orbit.*

Proof of Corollary. By the diagonalization lemma, the image of $K \times \bar{T}$ contains X^0. But X^0 is dense in Y, and $K \times \bar{T}$ is compact. Therefore $K \times \bar{T} \to Y$ is surjective. So $\bar{T}$ meets every K orbit and hence it meets every G orbit.

Remark. A similar corollary holds for any G_I equivariant compactification of Z_I^0.

§5.6. ***Proof of proposition* 2.1 *and* 2.2.** Proposition 2.1 now follows directly: Each X_I is a G-homogeneous space, and every G orbit on X intersects $\bar{T}$ nontrivially (§5.5). But the degenerate quadrics in $\bar{T}$ are described explicitly in §5.3.

We now give the proof of proposition 2.2. The orbit $G_I \cdot Q_0$ of G_I through the identity matrix is the set of all nondegenerate quadrics whose corresponding symmetric matrices are in diagonal block form with blocks of size $v_1 \times v_1$, $v_2 \times v_2, \ldots,$ and $v_r \times v_r$.

PROPOSITION 2.2. *The closure* (in X) *of this orbit* $G_I \cdot Q_0$ *is precisely the variety* Z_I.

Proof. Lemma (5.4) shows that $G_I \times \bar{T} \to Z_I$ is surjective. Thus Z_I is a constructible set in an *irreducible* algebraic subvariety of X. Furthermore the diagonalization lemma shows that the image of the map $K_I\bar{T} \to Z_I$ contains Z_I^0 which is an open subset of Z_I. Since the image of this map is compact, it must equal Z_I (which is compact and hence closed), so it contains the closure of $G_I \cdot Q_0$, which is also irreducible. Therefore Z_I coincides with the closure of $G_I \cdot Q_0$.

THEOREM 5.6. ([Ab]) *Z_I is a "wonderful" G_I-equivariant compactification of Z_I^0, i.e. each orbit closure in Z_I is smooth* (*in particular, Z_I is smooth*), *and $Z_I - Z_I^0$ is a union of divisors S_j, each of which is an orbit closure, and which all meet transversally.*

Proof. The proof follows the corresponding statement for X, which is proved in [DP1]. Recall ([DP2]) that $\bar{T}$ intersects the closed orbit G/B of X in the $n!$ points fixed by T. Let $p \in G/B$ be one of these points and let U be the unipotent radical of the Borel subgroup of G whose opposite Borel subgroup fixes p.

In [DP1] it was shown that there is an affine U stable open cell V_p centered at p in X with the following properties:

(a) there is an isomorphism, $V_p \cap \bar{T} \to \mathbb{C}^{n-1}$ which takes the T orbit closures of $V_p \cap \bar{T}$ to the coordinate subspaces of $\mathbb{C}^{n-1}$

(b) the map $\psi: U \times \mathbb{C}^{n-1} \to V_p$ which is defined by

$$\varphi(u, Q) = u \cdot Q$$

is a U-equivariant isomorphism.

(c) The union $\bigcup_p (V_p \cap \bar{T}) = \bar{T}$ where p varies over the T-fixed points in G/B.

Since each G_l orbit meets $\bar{T}$, it suffices to study Z_l and its orbits in each of the open sets V_p. For such a fixed point p, let U be the corresponding unipotent subgroup and let U_l be the intersection $U_l = U \cap G_l$. Since the map φ is U-equivariant and $\bar{T} \subset Z_l$ we have

$$\varphi(U_l \times \mathbb{C}^{n-1}) \subset Z_l \cap V_p$$

But U_l is a maximal unipotent subgroup in G_l so

$$\dim(\varphi(U_l \times \mathbb{C}^{n-1})) = \dim(Z_l) = \tfrac{1}{2} \sum_{i=1}^{r} v_i(v_i + 1) - 1$$

But these are two irreducible closed subsets of the same dimension in V_p, so they coincide, i.e.

$$\varphi(U_l \times \mathbb{C}^{n-1}) = Z_l \cap V_p$$

It follows that Z_l is smooth (since $\varphi(U_l \times \mathbb{C}^{n-1})$ is an affine space).

We verify the other properties of this compactification locally. We shall show that given $p \in G/B$ as above and given an orbit $\mathcal{O} \subset Z_l$ such that $V_p \cap \mathcal{O} \neq \varnothing$, there exists a unique T orbit S in $V_p \cap \bar{T}$ such that

$$V_p \cap \mathcal{O} = \varphi(U_l \times S)$$

Since the map φ is U_l-equivariant, and $V \cap \mathcal{O}$ is stable under U_l we have,

$$V_p \cap \mathcal{O} = \varphi(U_l \times S)$$

where S is a T stable subset of $\mathbb{C}^{n-1}$. But two elements of the form $\varphi(u_1, Q_1)$, $\varphi(u_2, Q_2)$ are G-conjugate if and only if Q_1 and Q_2 are T conjugate. (cf. [DP1] prop. 2.8) It follows that S is a unique T orbit.

§5.7. The spaces M_s. From now on we restrict to a particular decomposition of

$\mathbb{C}^n$ into s coordinate planes and $n-2s$ coordinate lines,

$$\mathbb{C}^n = P_1 \oplus \cdots \oplus P_s \oplus L_{2s+1} \oplus \cdots \oplus L_n$$

and we denote the corresponding space Z_I by Z_s and the corresponding groups G_I by G_S, etc.

Remark. The space M_s is canonically identified with the space $K \times_{K_s} Z_s$.

We obtain a commutative diagram where the horizontal maps are natural identifications and the vertical maps are fibrations:

$$\begin{tikzcd} G \times_{P_s} Z_s \arrow[r] \arrow[d] & K \times_{K_s} Z_s \arrow[r, "\theta"] \arrow[d] & M_s \arrow[d] \\ G/P_s \arrow[r] & K/K_s \arrow[r] & \mathscr{F}_s \end{tikzcd}$$

The map θ identifies $K \times_{K_s} Z_s$ with all pairs $(f, (F, Q))$ (where f is a partial flag of type $(2, 2, 2, \ldots, 2, 1, 1, \ldots, 1)$ and (F, Q) is a complete quadric which is diagonal with respect to this flag) by assigning

$$\theta(k, z) = (\bar{k}, kz)$$

Here, $\bar{k} \in K/K_s = \mathscr{F}_s$ denotes the corresponding partial flag. Since G, P_s, and Z_s are algebraic, we see

PROPOSITION 5.7.1. *The space M_s is homeomorphic to an algebraic variety.*

PROPOSITION 5.7.2. *Each of the maps $\Phi_s : K \times_{K_s} Z_s = M_s \to X$ is surjective.*

Proof of proposition 5.7.2. Since $Z_0 \subset Z_s$ for each s, it suffices to show that $K \times_{K_0} Z_0 \to X$ is surjective. But applying proposition 5.5 (with the trivial decomposition $\mathbb{C}^n = \mathbb{C}^n$) we find that for any quadric $Q \in X^0$, there exists $k \in K$ so that the quadric $Q' = k \cdot Q$ is an element of Z_0, i.e. $k^{-1} \cdot Q' = Q$. Thus Φ_0 is surjective to the nondegenerate quadrics, which form a dense open subset of X. But $W_0 = K \times_{K_0} Z_0$ is compact, so Φ_0 is surjective.

6.7.3. *Remark.* The quotient X/K is homeomorphic to the m-cube. The quotient map takes points of a given orbit type to a fixed face: Let $\bar{K}_0$ denote the normalizer of K_0 in K, i.e. the extension of K_0 which is given by allowing permutations of the lines L_i. From the surjective map $K \times_{\bar{K}_0} Z_0 \to X$ we obtain a homeomorphism

$$Z_0/\tilde{K}_0 \to X/K$$

by dividing by K. But $Z_0/\tilde{K}_0 = (Z_0/K_0)/(\tilde{K}_0/K_0)$ where $P = Z_0/K_0$ is the polyhedron which is the image of Z_0 under the Atiyah moment map (which is induced from the T action) ([At]). In other words, $Z_0 = \bar{T}$ is a toric variety and P is the associated convex polyhedron.) But this is a cube. In fact this map can be seen directly, orbit by orbit as follows: the image X_I/K is a union of faces of the cube, with each face corresponding to certain coincidences of the eigenvalues $|\lambda_i|$ of §5.4. For example, X^0/K contains all the faces whose closure contains the origin (which is a vertex of the cube); the interior corresponds to all eigenvalues different, the codimension 1 faces correspond to single coincidences of eigenvalues, etc.

5.7.4. *Proof of proposition* 2.4. Let $\tilde{G}_s = N(G_s)$ be the normalizer of G_s in G. This is the subgroup of G which preserves the union of the subspaces in the decomposition of $\mathbb{C}^n$, i.e. it includes permutations of the P's and permutations of the L's. Then $\tilde{G}_s/G_s = \Gamma_s = \Sigma_s \times \Sigma_{n-2s}$. Similarly the group $\tilde{K}_s = N(K_s) = K \cap \tilde{G}_s$ acts on Z_s and induces the action of $\Gamma_s = \tilde{K}_s/K_s$ on Z_s which was described in §2.3. On the other hand, $\tilde{K}_s$ acts on M_s by

$$(k, (h, z)) \to (hk^{-1}, k \cdot z)$$

For each $g \in G$ define $i_g : Z_s \to M_s$ to be the inclusion of the fibre,

$$i_g(z) = (g, z)$$

(where we identify $M_s = G \times_{P_s} Z_s$). Since G is connected, these maps are all homotopic, i.e. they all induce the same homomorphism on cohomology. Therefore, for any $k \in \tilde{K}_s$ we have the following commutative diagram:

$$\begin{array}{ccc} Z_s & \xrightarrow{i_I} & M_s \\ {\scriptstyle \cdot k}\downarrow & & \downarrow{\scriptstyle \cdot k} \\ Z_s & \xrightarrow[i_{k^{-1}}]{} & M_s \end{array}$$

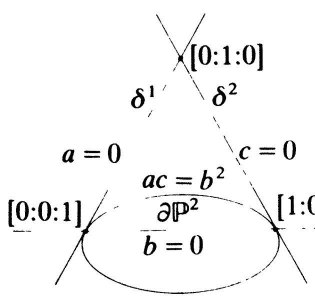
Thus, $i^* : H^*(M_s) \to H^*(Z_s)$ is equivariant with respect to the action of Γ_s. However this homomorphism is also surjective because the spectral sequence for the fibration $M_s \to \mathscr{F}_s$ collapses and Z_s is the fibre. (It can also be seen to be surjective because M_s has a decomposition into rational cells such that Z_s is a union of cells. See §6.) It follows that the two actions of Γ_s which were defined on the cohomology of Z_s coincide.

§5.8. Complete quadrics in $\mathbb{P}^1$. The following remarks about the variety $\mathbb{P}^2$ of quadrics in $\mathbb{P}^1$ will be used in the analysis of the map $\mu: Z_S \to (\mathbb{P}^2)^s$. Using the standard basis of $\mathbb{C}^2$, a complete quadric in $\mathbb{P}^1$ (i.e. a quadric cone in $\mathbb{C}^2$) corresponds to a symmetric matrix,

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

The solutions to the equation $xAx' = 0$ consist of two lines, so a complete quadric in $\mathbb{P}^1$ is given by two (not necessarily distinct) points in $\mathbb{P}^1$. This variety of complete quadrics is the symmetric product of $\mathbb{P}^1$ and $\mathbb{P}^1$, and is naturally isomorphic to $\mathbb{P}^2$, with homogeneous coordinates $[a:b:c]$. A quadric is degenerate if $b^2 = ac$. (In this case the two points in the quadric coincide.) The degenerate quadrics form a subvariety which we have been denoting by $\partial\mathbb{P}^2$. The diagonal complete quadrics (i.e., $b = 0$) form a (flat) hyperplane which we will denote by $\mathbb{P}^1 \subset \mathbb{P}^2$. The action of the group $G = \mathrm{PGL}_2(\mathbb{C})$ on the variety of complete quadrics has two orbits: $\partial\mathbb{P}^2$ and $\mathbb{P}^2 - \partial\mathbb{P}^2$. This second orbit is a rational cell: it deformation retracts to $\mathbb{R}\mathbb{P}^2 \subset \mathbb{C}\mathbb{P}^2$ because a quadric in $\mathbb{P}^2 - \partial\mathbb{P}^2$ consists of a pair of distinct points in $\mathbb{P}^1$, and these may be moved apart (along the unique geodesic which joins them) until they are antipodal. This $\mathbb{R}\mathbb{P}^2$ is a minimal K-orbit.

The action of the torus $\mathbb{C}^* \subset G$ stabilizes $\mathbb{P}^1$ and also $\partial\mathbb{P}^2$. It has three fixed points: $[1:0:0]$, $[0:0:1]$, and $[0:1:0]$. The first two of these points constitute the intersection $\partial\mathbb{P}^2 \cap \mathbb{P}^1$ and the third point lies in the $\mathbb{R}\mathbb{P}^2$. These three points are joined by two other flat hyperplanes, δ^1 (the set where $a = 0$) and δ^2 (the set where $c = 0$) which are tangent to $\partial\mathbb{P}^2$. This geometry may be summarized in the following picture:



Furthermore, let

$$U = \left\{ \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \middle| \, u \in \mathbb{C} \right\}$$

Then U is the unipotent radical of a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C})$. The line δ^1 is left fixed under U and so is the complement of δ^1 in $\mathbb{P}^2$ which we will denote by $\mathbb{A}^2$ and we will set $\mathbb{A}^1 = \mathbb{P}^1 \cap \mathbb{A}^2$. It is easy to verify that

PROPOSITION. *The map* $j: U \times \mathbb{A}^1 \to \mathbb{A}^2$ *which is given by* $j((u, x)) = u \cdot x$ *is a* U*-equivariant isomorphism.*

Note also that $\mathbb{P}^1 \cap \mathbb{A}^2$ *is identified with the standard* $\mathbb{A}^1$ *by the coordinate* c/a.

§5.9. The map $\mu : Z_s \to (\mathbb{P}^2)^s$. In this section we will show that the map μ defined in §2.3 is well defined, continuous, and algebraic. As in §5.3, we denote by T the (completely) diagonal quadrics, and we let $Q_0 \in X$ denote the quadric corresponding to the identity matrix. Let $\exp(\alpha_i)$ be the character $\exp(\alpha_i)(t) = t_{i+1}/t_i$ for any $t \in T$.

PROPOSITION. (a) *The map* $\exp(\alpha_i): T \to \mathbb{C}^*$ *has a unique extension,*

$$\overline{\exp(\alpha_i)}: \bar{T} \to \mathbb{P}^1$$

(b) *The product mapping*

$$\bar{\pi}_s = (\overline{\exp(\alpha_1)}, \overline{exp(\alpha_3)}, \ldots, \overline{\exp(\alpha_{2s-1})}: \bar{T} \to \prod_{j=1}^{s} \mathbb{P}^1$$

is a morphism of toric varieties and the pre-image,

$$\bar{T}_s = \bar{\pi}_s^{-1}(\bar{\pi}_s(Q_0))$$

is an $R = \bigcap_{j=1}^{k} \ker(\exp(\alpha_{2j-1}))$ *embedding, which is isomorphic to the torus embedding of* (completely) *diagonal quadrics in* $\mathbb{P}^{n-s-1}$.

(c) *The composition*

$$\bar{T} \to \prod_{j=1}^{s} \mathbb{P}^1 \to \prod_{j=1}^{s} \mathbb{P}^2$$

coincides with the restriction of the map μ (*of* §2.3) *to the toric variety* $\bar{T}$.

Proof. (a) Since the R.P.D. corresponding to a product of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is simply the decomposition of $\mathbb{R}^s$ into quadrants (generated by the hyperplanes $\alpha_{2i-1} = 0$, for $1 \le i \le s$), it suffices to show that either α_{2i-1} has constant sign, or else is identically zero on each cone $\sigma \in \Sigma$ of the R.P.D. corresponding to $\bar{T}$. But this is clear from the explicit description (§5.3) of the cones in Σ.

(b) The torus embedding associated to the Weyl chambers of the root system A_{n-1} can be combinatorially described as follows: Let v be a real vectorspace of dimension $n-1$ and let $\Phi \subset V^*$ be a set of vectors (the positive roots) indexed by pairs (i, j) of numbers such that $1 \le i < j \le n$. We denote such a vector by $\varepsilon_{ij} \in \Phi$. Assume that

(1) If $i < j < k$ then $\varepsilon_{ik} = \varepsilon_{ij} + \varepsilon_{jk}$

(2) The set $\{\varepsilon_{12}, \varepsilon_{23}, \ldots, \varepsilon_{n-1,n}\}$ forms a basis of V^*.

We define an integral structure on V by setting

$$\Lambda = \{v \in V \mid \varepsilon_{ij}(v) \in \mathbb{Z} \text{ (for all } (i, j))\}$$

and a cone decomposition which is given by the walls

$$H_{ij} = \{v \in V \mid \varepsilon_{ij}(v) = 0\}$$

i.e., we consider as open cones of the R.P.D., the connected components of any intersection of a set of H_{ij} minus the intersections of this set with any other H_{mn}. For each integer k with $1 \le 2k \le n$, define

$$V^k = \{v \in V \mid \varepsilon_{12}(v) = \varepsilon_{34}(v) = \cdots = \varepsilon_{2k-1,2k}(v) = 0\}$$

i.e.,

$$V^k = H_{12} \cap H_{34} \cap \cdots \cap H_{2k-1,2k}$$

Given any $\varepsilon_{ij} \in \Phi$, denote its restriction to V^k by $\bar{\varepsilon}_{ij}$. For each $j \le k$ we have

(1) $\bar{\varepsilon}_{2j-1,2j} = 0$

(2) $\bar{\varepsilon}_{2j-1,h} = \bar{\varepsilon}_{2j,h}$ (if $h > 2k$)

(3) $\bar{\varepsilon}_{2j-1,2h-1} = \bar{\varepsilon}_{2j,2h-1} = \bar{\varepsilon}_{2j-1,2h} = \bar{\varepsilon}_{2j,2h}$ if $j < h \le k$.

Let us rename the nonzero forms $\bar{\varepsilon}_{ij}$ as follows: For $1 \le i < j \le n-k$ define η_{ij} by

$$\eta_{ij} = \begin{cases} \bar{\varepsilon}_{2i,2j} & \text{if } i < j \le k \\ \bar{\varepsilon}_{2i,j+k} & \text{if } i \le k < j \\ \bar{\varepsilon}_{i+k,j+k} & \text{if } k < i < j \end{cases}$$

It is easily seen that

(i) All the nonzero $\bar{\varepsilon}_{ij}$ appear in this list

(ii) the vectors η_{ij} satisfy the axioms for the positive roots of type A_{n-1-k}

(iii) The integral lattice $\Lambda^k \subset V^k$ defined by the η_{ij} is $\Lambda \cap V^k$.

These facts are exactly those required to verify that the R.P.D. on V intersects $V^k \subset V$ in an R.P.D. of type A_{n-1-k}.

(c) First we shall show that the projection to the jth factor agrees with the composition of μ with the projection to the jth factor. Let $H = \text{span}\,(e_{2j-1}, e_{2j})$. Suppose $Q \in T$ is a *nondegenerate* quadric, i.e. Q is given by

$$\Sigma t_i x_i^2 = 0$$

where $t_i \neq 0$ for each i. Then $Q \cap \mathbb{P}(H)$ consists of two distinct points which are defined by the equation

$$t_{2j-1}x_{2j-1}^2 + t_{2j}x_{2j}^2 = 0$$

or

$$x_{2j-1}^2 + \frac{t_{2j}}{t_{2j-1}} x_{2j}^2 = 0$$

In other words, $\exp(\alpha_{2j-1})(t) = t_{2j}/t_{2j-1}$ is the map which associates to each nondegenerate diagonal quadric its intersection with $\mathbb{P}(H)$. This is exactly the map μ.

Using the explicit description (§5.2) of convergence in the toric variety $\bar{T}$, it is easy to see that the restriction of $\overline{\exp(\alpha_{2j-1})}$ to the toric variety $\bar{T}$ coincides with the map μ also.

PROPOSITION 5.9.1. *There is a unique G_s-equivariant map $\mu: Z_s \to \prod_{i=1}^s \mathbb{P}^2$ such that the following diagram commutes*:

$$\begin{array}{ccc} \bar{T} & \longrightarrow & Z_s \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mu} \\ \prod_{i=1}^{s} \mathbb{P}^1 & \longrightarrow & \prod_{i=1}^{s} \mathbb{P}^2 \end{array}$$

COROLLARY. *The above map μ coincides with the map μ of* §2.3. *In* *particular, the map μ is well defined, continuous, and algebraic; and the above map $\bar{\pi}$ has the geometric description which was given in* §2.3.

Proof of corollary. The map μ of §2.3 is G_s-equivariant and its restriction to $\bar{T}$ coincides with the map $\bar{\pi}$.

Proof of proposition 5.9.1. Uniqueness follows from lemma 5.4. 1 (every G_s orbit in Z_s meets $\bar{T}$ in a nonempty set). Existence is by construction: $\overline{\exp(\alpha_{2i-1})}$ extends equivariantly to Z_s^0 (the set of nondegenerate quadrics) because for any $Q \in Z_s^0$, the quadric $Q \cap P_i$ is nondegenerate, i.e. it consists of two lines, giving a point in $\mathbb{P}^2$. Let A be the set of all points $p \in Z_s$ such that $\overline{\exp(\alpha_{2i-1})}$ is defined as a morphism in a neighborhood of p. Then $A \supset Z_0$ and it is G_s stable. Thus, in order to show that $A = Z_s$, it suffices to show that A meets each G_s orbit. In §5.6 we found an open set $V_p \cap Z_s$, where p denotes a T-fixed point in the closed orbit in X. Within this open set, each element can be written uniquely as uQ, where $q \in V_p \cap \bar{T}$ and with $u \in U_s$. (Here, U_s is a suitably chosen maximal unipotent subgroup of G_s.) We have seen that each G_s orbit in Z_s meets at least one of these open sets in a nonempty subset. Thus, it suffices to extend μ to $V_p \cap Z_s$ for each such p. But this can be done by the formula

$$\mu(u \cdot Q) = u \cdot \overline{\exp(\alpha_{2i-1})}\,(Q)$$

The rest of this section contains technical results needed in the proof of the main lemma.

Let $q : \prod_{i=1}^{s} \mathbb{P}^2 \to \mathbb{P}^2$ denote the projection to the last factor, and let $\mathbb{P}^1 \subset \mathbb{P}^2$ denote the hyperplane of §5.6 above.

PROPOSITION 5.9.2. $(q\mu)^{-1}(\mathbb{P}^1) = Z_{s-1}$

Proof. This is immediate from the description (§1.3) of the map μ, because

$$(q\mu)^{-1}(\mathbb{P}^1) = \mu^{-1}\left(\prod_{i=1}^{s-1} \mathbb{P}^2 \times \mathbb{P}^1\right)$$

which is the set of complete quadrics (F, Q) which are diagonal with respect to the decomposition

$$\mathbb{C}^n = P_1 \oplus \cdots \oplus P_s \oplus L_{2s+1} \oplus \cdots \oplus L_n$$

and such that the intersection $(F, Q) \cap P_s$ is diagonal with respect to the decomposition

$$P_s = L_{2s-1} \oplus L_{2s}$$

But this is precisely Z_{s-1}.

Remark. We have the following diagram, where the top row is the $(q\mu)$-preimage of the bottom row:

$$\begin{array}{ccccc} Z_{s-1} & \subset & Z_s & \supset & \partial_s Z_s \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^1 & \subset & \mathbb{P}^2 & \supset & \partial\mathbb{P}^2 \end{array}$$

PROPOSITION 5.9.3. $\mathscr{X}(Z_{s-1}) = \mathscr{X}(\partial_s Z_s)$.

Proof. Both Z_{s-1} and $\partial_s Z_s$ are stable under the action of T on Z_s, and this action has finitely many fixed points. Furthermore, Z_{s-1} is smooth and $\partial_s Z_s$ is a union of nonsingular divisors with normal crossings (§5.6 and §5.9.7), and each has cohomology in even dimensions only. Therefore, by the Bialynicki–Birula decomposition ([BB1], [BB2]) and Mayer Vietoris, it suffices to show that both Z_{s-1} and $\partial_s Z_s$ have the same number of T-fixed points. In fact they have the *same* fixed points: Define

$$J = Z_{s-1} \cap \partial_s Z_s = (q\mu)^{-1}(\mathbb{P}^1 \cap \partial\mathbb{P}^2)$$

This is a T-stable set, since $\mathbb{P}^1 \cap \partial\mathbb{P}^2$ is fixed under T. It follows that

$$J^T = (Z_{s-1})^T = (\partial_s Z_s)^T$$

because every T-fixed point in Z_s must lie over a T-fixed point in $\mathbb{P}^2$, and these are just the points $[0:1:0]$ and $\mathbb{P}^1 \cap \partial\mathbb{P}^2$ (see §5.8).

COROLLARY 5.9.4. $\Sigma \operatorname{rank} H^i(M_{s-1})^{\mathbb{Z}/(2)} = \Sigma \operatorname{rank} H^i(\partial_s M_s)$.

Proof. The space M_{s-1} is a fibre bundle over $\mathscr{F}_{s-1}$ with fibre Z_{s-1}, while the space $\partial_s M_s$ is a fibre bundle over $\mathscr{F}_s$, with fibre $\partial_s Z_s$. The spectral sequence for these fibrations collapsed long ago, and $\mathscr{F}_s$ is simply connected. Therefore

$$\mathscr{X}(\partial_s M_s) = \mathscr{X}(\partial_s Z_s)\mathscr{X}(\mathscr{F}_s) = \mathscr{X}(Z_{s-1})\mathscr{X}(\mathscr{F}_s) = \mathscr{X}(Z_{s-1})\mathscr{X}(\mathscr{F}_{s-1})/2 = \mathscr{X}(M_{s-1})/2$$

since (by §3.4), $\mathbb{Z}/(2)$ acts on $H^*(F_{s-1})$ by a multiple of the regular representation.

COROLLARY 5.9.5. $\mu^{-1}(\mu(\bar{T})) = \mu^{-1}(\prod_{i=1}^{s} \mathbb{P}^1) = \bar{T}$

Proof. This follows by induction (the case $s = 0$ is trivial since $Z_0 = \bar{T}$ and μ is the constant map), and proposition 5.9.2.

COROLLARY 5.9.6. $\mathscr{X}(Z_s) = \sum_{h=0}^{s} \binom{s}{n}(n-h)!$

Proof. Divide the variety Z_s into T stable subvarieties, $Z^\gamma = \mu^{-1}(N^\gamma)$ where $\gamma \subset \{1, 2, \ldots, s\}$ and

$$N^\gamma = \{(p_1, p_2, \ldots, p_s) \in (\mathbb{P}^2)^s \mid p_i \in \mathbb{P}^1 \text{ if } i \notin \gamma \text{ and } p_i = [0:1:0] \text{ if } i \in \gamma\}$$

Each T-fixed point of Z_s is contained in some Z^γ, and these Z^γ are disjoint. Thus,

$$\mathscr{X}(Z_s) = \sum_\gamma \mathscr{X}(Z^\gamma) = \sum_\gamma (n - |\gamma|)!$$

But for each $h = |\eta|$, there are $\binom{s}{h}$ possible choices for γ.

PROPOSITION 5.9.7. *The action of $\pi_1((\mathbb{P}^2)^s - \Delta)$ on the cohomology of the fibre of μ is trivial.*

Proof. The group G_s acts on Z_s and on $(\mathbb{P}^2)^s$ and the map μ is equivariant with respect to this action. Therefore, over the large open orbit $U \subset (\mathbb{P}^2)^s$ we have

$$\mu^{-1}(U) \cong G_s \times_H F$$

where F is the fibre $\mu^{-1}(p)$ of a generic point p, and H is the stabilizer of F in G_s. We must show that H acts trivially on $H^*(F)$. Since $H \subset G_s$ (which is connected), it follows that H acts trivially on $H^*(Z_s)$. So it suffices to show that the map $H^*(Z_s) \to H^*(F)$ is surjective. This follows from the analysis in the next chapter where it will be clear that the cohomology of F is generated by the classes which are dual to the boundary divisors of F, and these are transversal intersections with boundary divisors of Z_s.

§5.10. Counting the G_s orbits in Z_s. In this section we will give a one to one correspondence between the codimension one G_s orbits $\{D_1, D_2, \ldots, D_r\}$ in Z_s

and the one dimensional cones λ in the cone decomposition Σ of V (corresponding to the torus $\bar{T}$ §5.2) such that $\alpha_{2j-1}(\lambda) \geq 0$ for each j $(1 \leq j \leq s)$.

We have seen in §5.8 how the standard $\mathbb{P}^2$ contains a standard $\mathbb{A}^2$ which is U-stable and isomorphic to $U \times \mathbb{A}^1$. Taking products we have

$$(\mathbb{P}^2)^s \supset (\mathbb{A}^2)^s \cong (U \times \mathbb{A}^1)^s \cong (U \times U \times \cdots \times U) \times (\mathbb{A}^1)^s$$

where the group U^s is a unipotent radical of G_s. Let us now consider the open set

$$A_s = \mu^{-1}((\mathbb{A}^2)^s)$$

in Z_s and $C_s = A_s \cap \bar{T} = (\mu \mid \bar{T})^{-1}((\mathbb{A}^1)^s)$. We have seen in §5.9 that C_s is the torus embedding corresponding to the R.P.D. of the quadrant $Q^s \subset V$, which is defined by

$$\alpha_{2i-1} \geq 0 \quad \text{for} \quad 1 \leq i \leq s$$

PROPOSITION 5.10. *The map* $\lambda: U^s \times C_s \to A_s$ (which is given by $\lambda(u, c) = u.c.$) *is a* U^s *equivariant isomorphism.*

This follows from the general result.

LEMMA. *If* X *and* Y *are varieties with an action of a group* G, *and if* $\mu: X \to Y$ *is a* G *equivariant inclusion, and if* Y *is isomorphic (in a* G *equivariant way) to* $G \times Z$, *then* X *is isomorphic to* $G \times \mu^{-1}(Z)$ *under the map* $j(g, x) = gx$.

Proof. The inverse of j is given by $j^{-1}(x) = (g, g^{-1}x)$ where g is defined by $\mu(x) = (g, z)$.

COROLLARY. *The* G_s *orbits in* Z_s *intersect* A_s *in the sets* $U^s \times \mathcal{O}$, *where* $\mathcal{O}$ *is a* T *orbit in* C.

Proof. Every G_s orbit $\mathcal{O}$ meets A_s since every G_s orbit in $(\mathbb{P}^2)^s$ meets $(\mathbb{A}^2)^s$. Since A_s is stable under the Borel subgroup $B_s^- = U^s \cdot T$, it follows that $\mathcal{O} \cap A_s$ is a union of B_s^- orbits. But every B_s^- orbit in A_s is of the form $U^s \times \mathcal{O}'$, where $\mathcal{O}'$ is a T orbit in C. In order to prove the claim, it suffices to show that every B_s^- orbit in A_s is the intersection of A_s with a G_s orbit. This can be shown exactly as in [DP1]. Thus, we have proven:

THEOREM. *The* G_s *orbits of codimension* k *in* Z_s *are in one to one correspondence with the* k *dimensional cones of the R.P.D. of the "quadrant"* Q^s.

It will be useful to collect some further information on this picture. Consider $Z_{s-1} \subset Z_s$, A_{s-1}, A_s, C_{s-1}, C_s as before. Let σ be the element of G_s which is the identity except in the sth 2 by 2 block, where it is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then σ induces an automorphism of order 2 on T, and so also on the space V which is dual to the character group of T. One easily verifies that $C_{s-1} = C_s \cup \sigma(C_s)$, so

$$A_{s-1} = U^{s-1} \times C_{s-1} = (A_s \cap Z_{s-1}) \cup \sigma(A_s \cap Z_{s-1})$$

since $A_s \cap Z_{s-1} = U^{s-1} \times C_s$.

If we consider a boundary divisor D of Z_s, we can analyze its intersection with Z_{s-1}. Let v be the first lattice vector in the one dimensional cone of the R.P.D. of C_s which corresponds to D.

PROPOSITION. *If $\sigma(v) = v$, then $D \cap Z_{s-1}$ is an irreducible divisor corresponding to the vector $v \in Q_s \subset Q_{s-1}$. If $\sigma(v) \neq v$, then $D \cap Z_{s-1}$ is the union of two irreducible divisors corresponding to the two vectors v and $\sigma(v) \in Q_{s-1}$.*

Proof. To study $D \cap Z_{s-1}$ it is enough to analyze $D \cap C_{s-1}$ since every boundary divisor in Z_{s-1} meets C_{s-1} in a boundary divisor. Now,

$$D \cap C_{s-1} = D \cap (C_s \cap C_s^{\sigma})$$

and $D \cap C_s^{\sigma} = (D \cap C_s)^{\sigma}$ since $D = D^{\sigma}$. This proves the proposition, because of the one to one correspondence (under σ) between T orbits and cones.

5.11. Paving of M_s and Z_s by algebraic rational cells

§5.11.1. Introduction. The results of this paper would be much easier if X (or if Z_s) had a cell decomposition which was compatible with the action of G (or of G_s). Unfortunately the Bialynicki–Birula cells (which come from the torus action) do not decompose the G-orbits, so although they may be used to compute the Betti numbers of X ([Str]), the resulting formula is not immediately related to our formula (§2). Our solution to this problem is to find a paving by rational cells which are compatible with the map μ.

Recall that an (algebraic) rational cell C in a algebraic variety M is a locally closed algebraic subset such that $H_c^i(C; \mathbb{Q}) = 0$ for all but one value of i (which is called the dimension of the cell). A *paving* of M by rational cells is a decomposition

$$M = \coprod_{\alpha \in J} C_\alpha$$

into finitely many (algebraic) rational cells C_α, together with a total ordering of the index set J such that for each $\beta \in J$, the set

$$M_\beta = \bigcup_{\alpha \leq \beta} C_\alpha$$

is closed in M.

PROPOSITION. *The variety M_s has a paving by (algebraic) rational cells such that each $\partial_I M_s$ is a union of cells, the subvariety Z_s is a union of cells, each $\partial_I Z_s$ is a union of cells, and the map $\mu : Z_s \to \prod_{i=1}^{s} \mathbb{P}^2$ is a cellular map (with respect to the standard paving of $\mathbb{P}^2$ by rational cells: see below).*

Proof. The proof will take the rest of §5.11.

5.11.2. Lemma on rational cell decompositions. The proof of the following lemma is simple:

LEMMA. *Suppose that*

$$M = \coprod_{\lambda \in K} Z_\lambda$$

is a decomposition of an algebraic variety M into finitely many locally closed algebraic subsets Z_λ such that

(a) *each closure $\bar{Z}_\lambda$ is a union of Z's, i.e.*

$$\bar{Z}_\lambda = \bigcup_{\tau \in K'} Z_\tau$$

for some $K' \subset K$.

(b) *each Z_λ has a decomposition into rational algebraic cells.*

Then the induced decomposition of M into rational algebraic cells is a paving of M if and only if each of the decompositions of Z_λ into cells is a paving of Z_λ.

§5.11.3. Cell decompositions of toric varieties and their fibres

PROPOSITION. *Suppose $f: X \to Y$ is a morphism between nonsingular projective toric varieties. Let $p \in Y$ and let $\{\mathcal{O}_1, \ldots, \mathcal{O}_n\}$ be the set of torus orbits in X which have nonempty intersection with the fibre $f^{-1}(p)$. Then there is a partition $\{\Gamma_1, \ldots, \Gamma_m\}$ of the numbers $\{1, 2, \ldots, n\}$ such that each set*

$$e_i = \bigcup_{j \in \Gamma_i} (\mathcal{O}_j \cap f^{-1}(p))$$

is an affine algebraic cell in $f^{-1}(p)$, and the cells $\{e_1, \ldots, e_m\}$ form a paving of $f^{-1}(p)$.

Proof. This proof has 3 steps.

Step 1: *For the case Y is a single point.* Here we recall Danilov's ([D]) construction of the cell decomposition of X. Let $\Sigma \subset V$ denote the cone decomposition of the vectorspace V which corresponds to the variety X. Let $c_1, \ldots, c_m$ denote the *maximal* cones in Σ, and let $q \in c_1$ be a generic interior point (which we will call the Danilov point) in a maximal cone $c_1 \in \Sigma$. Then to each maximal cone c_j, Danilov associates certain faces $f \subset c_j$, according to the following rule: a face f is associated to a cone c_j if there is an interior point $q' \in c_j$ such that the line segment $\overline{qq'}$ has nonempty intersection with the face f. This association determines a partition $\{\Gamma_1, \ldots, \Gamma_m\}$ of the cones in Σ into groups, one for each maximal cone $c_j \in \Sigma$. Furthermore, if $\mathcal{O}(c)$ denotes the T-orbit (in X) which corresponds to a cone $c \in \Sigma$, then the set

$$e_i = \bigcup_{c \in \Gamma_i} \mathcal{O}(c)$$

is a locally closed algebraic subvariety of X.

THEOREM. ([D]) *If X is projective, then the subvarieties $\{e_1, \ldots, e_m\}$ are a paving of X (i.e. they can be ordered so that each $\bigcup_{i \leq q} e_i$ is closed in X), and if X is nonsingular then each e_i is an affine algebraic cell.*

Remark. Although $f^{-1}(p)$ is not a toric variety, it is a union of toric varieties which intersect along torus orbits. Although it is not true that any union of toric varieties has a paving by algebraic cells, we will show that the variety $f^{-1}(p)$ has such a decomposition.

Step 2: *For the case that* $p \in Y$ *is a fixed point.* Now suppose that $f: X \to Y$ is a morphism of nonsingular toric varieties. We obtain a cone decomposition Ω of a real vectorspace H corresponding to Y, and a linear projection $f_\#: V \to H$ which takes cones in Σ to cones in Ω. Let $p \in Y$ be fixed under the torus action. The fixed point p corresponds to a maximal cone $\omega \in \Omega$. Choose a Danilov point (for the variety Y), $a \in H$ to lie in the interior of ω, and choose a Danilov point (for the variety X), $q \in V$ to lie in the interior of an open cone $c \in \Sigma$ such that $f_\#(q) = a$ (and hence also $f_\#(c) \subset \omega$). It is easy to see that, having made these choices, the fibre $f^{-1}(p)$ will be a union of the cells in the cell decomposition of X which was defined in step 1. It follows from the "only if" part of 5.11.2 that this defines a paving of $f^{-1}(p)$ by affine algebraic cells.

Step 3: *Reduction to the case that* p *is a fixed point in* Y. The point $p \in Y$ lies in some orbit $\mathcal{O}$ in Y, and this corresponds to some $\omega \in \Omega$ of the cone decomposition of the vectorspace H associated to Y. Let $\langle \omega \rangle$ be the vector subspace of H which is spanned by ω. Then $\langle \omega \rangle \cap \Omega$ is a cone decomposition of $\langle \omega \rangle$, and $f^{-1}(\langle \omega \rangle) \cap \Sigma$ is a cone decomposition of $f_\#^{-1}(\langle \omega \rangle)$. Therefore these cone decompositions are associated to a morphism of toric varieties,

$$f': X' \to Y'$$

where $X' \subset X$ and $Y' \subset Y$, and where $p \in Y'$ is a fixed point of the torus action on Y'. Furthermore, $(f')^{-1}(p) = f^{-1}(p)$, so the cell decomposition of $(f')^{-1}(p)$ which is provided by step 2 above is the desired decomposition.

Remarks. Unfortunately, X' may be a singular toric variety. However, it will be nonsingular in a neighborhood of $(f')^{-1}(p)$, and Danilov's proof that the varieties e_i are cells uses only the nonsingularity near the e_i. The varieties X' and Y' can be easily described: The stabilizer $S = \mathrm{Stab}_T(p)$ acts on Y, and Y' is the closure of a generic orbit of S. The variety X' is the complete pre-image, $f^{-1}(Y')$, i.e. it is the closure of a generic orbit of the pre-image torus, $f_\#^{-1}(S)$.

§5.11.4. The standard paving of $\prod_{i=1}^s \mathbb{P}^2$. Define the standard paving of $\mathbb{P}^2$ to be the decomposition into cells

$$\mathbb{P}^2 = \{p_0\} \cup (\partial\mathbb{P}^2 - \{p_0\}) \cup (\mathbb{P}^2 - \partial\mathbb{P}^2)$$

where $p_0 = \partial\mathbb{P}^2 \cap \delta^1 = [0:0:1]$ (in the notation of §5.8). Note that $\mathbb{P}^2 - \partial\mathbb{P}^2$ is an algebraic rational cell (of dimension 4) since it deformation retracts to $\mathbb{R}\mathbb{P}^2$.

Define the standard paving of $\prod_{i=1}^s \mathbb{P}^2$ to be the product decomposition. The group G_s acts on $\prod_{i=1}^s \mathbb{P}^2$ with one orbit $\partial_I \prod_{i=1}^s \mathbb{P}^2$ for each subset $I \subset$

$\{1, 2, \ldots, s\}$:

$$\partial_I \prod_{i=1}^{s} \mathbb{P}^2 = \prod_{i=1}^{s} A_i$$

where

$$A_i = \begin{cases} \mathbb{P}^2 - \partial\mathbb{P}^2 & \text{if } i \notin I \\ \partial\mathbb{P}^2 & \text{if } i \in I \end{cases}$$

Each $\partial_I \prod_{i=1}^{s} \mathbb{P}^2$ is a union of rational cells in the product decomposition of $\prod_{i=1}^{s} \mathbb{P}^2$.

§5.11.5. Paving of Z_s by rational cells. In this section we show that Z_s has a paving by rational algebraic cells such that each $\partial_I Z_s$ is a union of cells.

By lemma 5.11.2, it suffices to find a paving of each $\partial_I Z_s$ by rational cells. Recall (§2.3) that

$$\partial_I Z_s = \mu^{-1}\left(\partial_I \prod_{i=1}^{s} \mathbb{P}^2\right)$$

It follows that the restriction

$$\mu : \partial_I Z_s \to \partial_I \prod_{i=1}^{s} \mathbb{P}^2$$

is a fibre bundle (since it is G_s equivariant and the base is G_s homogeneous). Let $p_I = (p_1, \ldots, p_s) \in \partial_I \prod_{i=1}^{s} \mathbb{P}^2$ denote the basepoint,

$$p_i = \begin{cases} [1:0:1] & \text{if } i \notin I \\ [0:0:1] & \text{if } i \in I \end{cases}$$

First we give a paving of $\mu^{-1}(p_I) \cap \partial_I Z_s$. But this is precisely the fibre (over p_I) of the map between toric varieties,

$$\bar{\pi} : \bar{T} \to \prod_{i=1}^{s} \mathbb{P}^1$$

(see §5.9), so it has a paving by affines $\{e_1, \ldots, e_m\}$, according to §5.11.3. (Note that the basepoint p_l was chosen so as to lie in $\prod_{i=1}^{s} \mathbb{P}^1$). We will now show that the subsets of the form

$$\tilde{e}_{jk} = G_s \cdot e_j \cap \mu^{-1}(c_k)$$

constitute a rational cell decomposition of $\partial_l Z_s$, where $G_s \cdot e_j$ is the G_s-saturation of the cell e_j, and where c_k is a rational cell in the standard (§5.11.4) decomposition of $\partial_l \prod_{i=1}^{s} \mathbb{P}^2$.

Note that $G_s \cdot e_j \cap \mu^{-1}(p_l) = e_j$ because each intersection

$$(G_s\text{-orbit in } Z_s) \cap \mu^{-1}(p_l) = (T\text{-orbit in } \bar{T}) \cap \mu^{-1}(p_l)$$

and the cell e_j is a union of such intersections (see §5.10). It follows that the cells $\tilde{e}_{jk}$ give a decomposition of $\partial_l Z_s$, and that they form a paving (using the lexicographic ordering induced from the ordering of the cells in $\mu^{-1}(p_l)$ and the cells in $\partial_l \prod_{i=1}^{s} \mathbb{P}^2$). We now show that $\tilde{e}_{jk}$ is a rational cell. The restriction

$$\mu : G_s \cdot e_j \to \partial_l \prod_{i=1}^{s} \mathbb{P}^2$$

is a G_s-equivariant fibration with fibre e_j. The restriction

$$\mu : \tilde{e}_{jk} = G_s \cdot e_j \cap \mu^{-1}(C_k) \to c_k \subset \partial_l \prod_{i=1}^{s} \mathbb{P}^2$$

is a fibre bundle over c_k with contractible fibre. Since c_k is a rational cell, it follows that $\tilde{e}_{jk}$ has no rational homology except in dimension 0. However $\tilde{e}_{jk}$ is a complex manifold and hence is oriented. By rational Poincaré duality, it follows that the cohomology with compact support of $\tilde{e}_{jk}$ vanishes except in the top dimension, so $\tilde{e}_{jk}$ is a rational cell.

§5.11.6. Paving of M_s by rational cells. The projection (§5.7) $M_s \to \mathcal{F}_s$ is an algebraic fibre bundle which is trivial over each of the Bruhat cells in $\mathcal{F}_s$. The fibre is Z_s. Therefore the pre-image of each Bruhat cell is paved with rational (algebraic) cells by the product paving. Lemma 5.11.2 implies that this gives a paving of M_s by rational cells.

Chapter 6. The cohomology ring of Z_s and M_s

§6.1. Statement of result

§6.1.1. Cohomology of Z_s. Our formula for $H^*(Z_s)$ involves the rational polyhedral decomposition described in §5.3 and §5.9, i.e.

$V = \{a \in \mathbb{R}^n \mid \Sigma a_i = 0\}$

Σ = the cone decomposition of V which is generated by hyperplanes $a_i = a_j$ (for $i \neq j$)

$H = \mathbb{R}^s$, with coordinates $\alpha_1, \alpha_3, \ldots, \alpha_{2s-1}$

$\alpha : V \to H$ the projection given by $\alpha_{2i-1}(a) = a_{2i} - a_{2i-1}$

Ω = the cone decomposition of H into quadrants, generated by the hyperplanes $\alpha_{2i-1} = 0$ (for $1 \leq i \leq s$).

We will be concerned with the one dimensional cones $\sigma \in \Sigma$ such that $\alpha_{2i-1}(\sigma) \geq 0$, for each i, $(1 \leq i \leq s)$. Let us denote the primitive generating vectors of these cones by $D_1, D_2, \ldots, D_r$. (See §5.10 for an analysis of these vectors: each vector D_i corresponds in a natural way to a codimension one orbit D_i of G_s on Z_s).

DEFINITION. Let $\mathbb{Q}[D_1, \ldots, D_r]$ be the polynomial ring generated by the (commuting) variables $[D_1], [D_2], \ldots, [D_r]$ of degree 2. Let I_1 be the ideal generated by the monomials $[D_{i_1}][D_{i_2}] \cdots [D_{i_k}]$ such that the vectors $D_{i_1}, D_{i_2}, \ldots, D_{i_k}$ do not form a cone in Σ.

For each j $(1 \leq j \leq s)$ consider the subset $\mathbf{j} \subset \{D_1, D_2, \ldots, D_r\}$ which is given by

$$\mathbf{j} = \{D \mid \alpha_{2j-1}(D) > 0\}$$

Let $[\mathbf{j}]$ denote the ideal generated by the (third degree) polynomials

$$\left\{ \left(\sum_{D \in \mathbf{j}} [D] \right)^2 [E] \;\middle|\; E \in \mathbf{j} \right\}.$$

Let I_2 be the ideal $[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + \cdots + [\mathbf{s}]$, i.e. the sum of the ideals $[\mathbf{j}]$. Let I_3 be the ideal generated by the linear forms

$$\sum_{i=1}^{r} f(D_i)[D_i]$$

where f is any one of the following functions:

$$\sum_{i=1}^{s} m_i(a_{2i-1} + a_{2i}) + \sum_{j=2s+1}^{n} n_j a_j$$

where the integers m_i and n_i satisfy

$$2\sum_{i=1}^{s} m_i + \sum_{j=2s+1}^{n} n_j = 0$$

THEOREM. *The cohomology ring $H^*(Z_s)$ is naturally isomorphic to*

$$\mathbb{Q}[D_1, D_2, \ldots, D_r]/(I_1 + I_2 + I_3)$$

Furthermore the isomorphism is induced by the map which assigns to each variable $[D_i]$ the cohomology class $[D_i] \in H^2(Z_s)$ which is dual to the divisor D_i.

The homomorphism $\Phi^*: H^*(Z_s) \to H^*(Z_{s-1})$ is easily described by its action on the generators $[D_i]$. Let $\sigma: Z_s \to Z_s$ denote the involution which is induced by exchanging the (labelling of the) coordinate axes, L_{2s} and L_{2s-1} (where $P_s = L_{2s} \oplus L_{2s-1}$). (This corresponds to an involution, which we also denote by σ, of $H = \mathbb{R}^s$, and which is given by multiplication by -1 in the last coordinate.

THEOREM (*continued*). *The image of the class $[D_i]$ under the homomorphism Φ^* is*

$$\Phi^*([D_i]) = \begin{cases} [D_i] & \textit{if } \sigma(D_i) = D_i \\ [D_i] + [\sigma(D_i)] & \textit{otherwise} \end{cases}$$

§6.1.2. Cohomology of M_s. The rational cohomology ring of M_s is an algebra over the cohomology $H^*(\mathcal{F}_s)$ of the flag manifold $\mathcal{F}_s$, and it is generated by the (degree 2, commuting) dual classes to the boundary divisors $D_1, D_2, \ldots, D_r$. In the notation of §6.1.1 above, we define ideals J_1, J_2, and J_3 in

$H^*(\mathcal{F}_s; \mathbb{Q})[D_1, \ldots, D_r]$ as follows:

(a) J_1 is the ideal generated by the monomials $[D_{i_1}][D_{i_2}] \cdots [D_{i_k}]$ such that the vectors $D_{i_1}, D_{i_2}, \ldots, D_{i_k}$ do not form a simplex in Σ.

(b) For each j $(1 \le j \le s)$ let $[\tilde{j}]$ denote the ideal generated by the polynomials

$$\left\{\left(\left(\sum_{D \in \mathbf{j}} [D]\right)^2 - 4c^1(P_j)^2 + 16c^2(P_j)\right) \cdot [E] \;\middle|\; [E] \in \mathbf{j}\right\}.$$

where $c^i(P_j)$ denotes (§6.11) the chern class of the tautological bundle P_j of 2-planes over $\mathcal{F}_s$. Let J_2 be the ideal

$$J_s = [\tilde{1}] + [\tilde{2}] + \cdots + [\tilde{s}]$$

(c) Let J_3 be the ideal generated by the first degree polynomials

$$\sum_{i=1}^{r} (f(D_i)[D_i]) - c^1(\Lambda^2 P_j)$$

where f is any one of the following functions:

$$a_{2j-1} + a_{2j} \quad (\text{where } 1 \le j \le s)$$

and by the first degree polynomials

$$\sum_{i=1}^{r} (f(D_i)[D_i]) - c^1(L_j)$$

where f is any one of the following functions:

$$a_j \quad (\text{where } 2s + 1 \le j \le n)$$

THEOREM. *The cohomology ring $H^*(M_s)$ is naturally isomorphic to*

$$H^*(\mathcal{F}_s)[D_1, \ldots, D_r]/[(J_1 + J_2 + J_3)$$

Furthermore the isomorphism is induced by the map which assigns to each variable $[D_i]$ the cohomology class $[\tilde{D}_i]$ of the divisor $\tilde{D}_i = K \times_{K_s} D_i \subset K \times_{K_s} Z_s = M_s$.

Remark. The cohomology ring $H^*(\mathscr{F}_s; \mathbb{Q})$ is the subring of $\mathbb{Q}[X_1, \ldots, X_n]/(S)$ which is invariant under the exchanges $X_{2i-1} \leftrightarrow X_{2i}$ (for $1 \le i \le s$), where (S) is the ideal generated by the elementary symmetric functions.

§6.1.3. Outline of proof. The proofs will occupy the rest of this chapter. First we give a different presentation for the cohomology of Z_s by introducing new divisors, δ_i^1 and δ_i^2 and by showing that Z_s has the cohomology of a toric variety with one codimension 1 orbit for each divisor in the collection $\{D_i, \delta_j^1, \delta_j^2\}$. This is done by generalizing (§6.2) Danilov's construction ([D]) for the cohomology of a toric variety, and then by verifying (§6.3, 6.4, 6.5, 6.6, 6.7) that Z_s satisfies the axioms of Danilov. Then (§6.10) we find a formula for the cohomology classes represented by the new divisors, δ_j^i in terms of the cohomology classes represented by the codimension 1 orbits D_i of the G_s action on Z_s. Substituting for these classes gives the formula of theorem 6.1.1. Finally, in §6.11 we "twist" the cohomology of Z_s over the cohomology of the flag manifold $\mathscr{F}_s$ to obtain theorem 6.1.2.

§6.2. The space Z_s has the cohomology of a toric variety

§6.2.0. Although Z_s is not a torus embedding, it can be associated to an R.P.D. of Euclidean space, and the method of Danilov can be used to present its cohomology. Since this construction works in a much more general setting than that of toric varieties, we will formulate the procedure in general.

§6.2.1. Definition of the ring $\mathbb{Q}[D]/(I+J)$. Suppose Z is a complete nonsingular complex algebraic variety, and $D_1, D_2, \ldots, D_r$ are irreducible divisors in Z. Let Γ be the group (under multiplication) of regular invertible functions on the space $Z^0 = Z - \bigcup D_i$, modulo the constant functions. Then Γ is a free abelian group of some rank, m. From this data we can form a ring and a homomorphism to $H^*(Z)$ as follows: Let $\mathbb{Q}[D]$ denote the polynomial ring in the formal variables $[D_1]$, $[D_2], \ldots, [D_r]$. Let $I \subset \mathbb{Q}[D]$ be the ideal generated by the monomials $[D_{i_1}][D_{i_2}] \cdots [D_{i_k}]$ for which the intersection $D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k}$ is empty. Let $J \subset \mathbb{Q}[D]$ be the ideal which is generated by the linear forms

$$\sum_{i=1}^{r} \operatorname{ord}_{D_i}(f)[D_i]$$

for each $f \in \Gamma$. Consider the ring $\mathbb{Q}[D]/(I+J)$.

PROPOSITION ([D]). *The association which assigns to each divisor D_i the cohomology class which is dual to its fundamental class, induces a ring homomorphism*

$$\mathbb{Q}[D]/(I+J) \to H^*(Z; \mathbb{Q}).$$

In the next 2 sections we will develop Danilov's criterion that this be an isomorphism.

§6.2.2. The cone decomposition of Hom$(\Gamma, \mathbb{R})$. Let $\tilde{V} = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{R})$. Each divisor D_i defines an (integral) vector $D_i \in \tilde{V}$ by

$$D_i(f) = \mathrm{ord}_{D_i}(f)$$

Fix $f \in \Gamma$. For any D_i, we have $\mathrm{ord}_{D_i}(f) \in \mathbb{Z}$. If this number is 0 for all D_i, then f is a constant function. Since Γ is contained in the lattice of integral functions on the set $\{D_1, \ldots, D_r\}$, it is free of rank $\leq r$.

For each collection $D_{i_1}, D_{i_2}, \ldots, D_{i_k}$ of divisors such that their total intersection is nonempty,

$$D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k} \neq \varnothing$$

we can define a cone of positive convex combinations of the corresponding vectors, i.e.

$$c(D_{i_1}, \ldots, D_{i_k}) = \{\Sigma a_j D_j \mid a_j > 0\}$$

§6.2.3. Proposition ([D]). *Suppose that*

(1) $\mathrm{rank}(\Gamma) = \dim_{\mathbb{C}}(Z)$

(2) *The cones $c(D_{i_1}, \ldots, D_{i_k})$ form a simplicial rational polyhedral decomposition $\tilde{\Sigma}$ of the vectorspace $\tilde{V}$.*

(3) *The number of cones $c(D_{i_1}, \ldots, D_{i_k})$ of maximal dimension is equal to the Euler characteristic of Z*

(4) *$H^*(Z; \mathbb{Q})$ is generated by the cohomology classes which are dual to the fundamental classes of the divisors $D_1, D_2, \ldots, D_r$.*

Then the homomorphism $\mathbb{Q}[D]/(I+J) \to H^*(Z; \mathbb{Q})$ is an isomorphism.

Remark. Although this result is not stated explicitly in [D], it is equivalent to this analysis for the cohomology of a torus embedding.

§6.3. Application to Z_s

§6.3.1. In the variety Z_s we consider the following list of divisors:

(a) $D_1, D_2, \ldots, D_r$, the (closures of the) codimension 1 orbits of the G_s action on Z_s

(b) $\delta_i^1 = \mu^{-1}(\tilde{\delta}_i^1)$

(c) $\delta_i^2 = \mu^{-1}(\tilde{\delta}_i^2)$

where $\mu : Z_s \to \prod_{j=1}^{s} \mathbb{P}^2$ is the map of §2.3 and §5.9, and where

$$\tilde{\delta}_i^j = \mathbb{P}^2 \times \mathbb{P}^2 \times \cdots \times \delta^j \times \cdots \times \mathbb{P}^2$$

(the δ^j appearing in the ith factor). (Recall from §5.8 that δ^1 is the closure in $\mathbb{P}^2$ of the set of symmetric matrices $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ such that $a = 0$, while δ^2 is the closure of the set of such matrices with $c = 0$).

§6.3.2. Proposition. *The collection of divisors* $D_1, \ldots, D_r, \delta_1^1, \ldots, \delta_s^1, \delta_2^1, \ldots, \delta_s^2$ (in the variety Z_s) *satisfy the hypotheses of Danilov's theorem* (§6.2.3), *so the cohomology ring of* Z_s *is isomorphic to* $\mathbb{Q}[D, \delta^1, \delta^2]/(I + J)$.

Proof. The proof of this proposition will cover sections 6.4, 6.5, 6.6, 6.7, 6.8.

§6.4. The vectorspace $\tilde{V}$ generated by Γ. Let us denote by Z_s^{00} the set

$$Z_s^{00} = Z_s - \bigcup D_i - \bigcup \delta_i^1 - \bigcup \delta_i^2$$

(to distinguish this from $Z_s^0 = Z_s - \bigcup D_i$).

Since Z_s^{00} consists entirely of nondegenerate quadrics, the elements can be represented (modulo multiples) by symmetric matrices with s 2×2 blocks

$$\begin{pmatrix} a_{2i-1} & b_i \\ b_i & a_{2i} \end{pmatrix} \qquad (1 \leq i \leq s)$$

and with $n - 2s$ 1×1 blocks, (a_j) (where $2s + 1 \leq j \leq n$). Since the quadric is nondegenerate, the determinant of each block is nonzero. We have removed the δ_1^i and the δ_2^i so the numbers a_i are nonzero.

PROPOSITION. *The group Γ of regular invertible functions on Z_s^{00} is the free* *abelian group generated by the following $n+s-1$ functions*:

$$A_i = a_{2i-1}/a_1 \qquad (2 \le i \le s)$$
$$F_i = (a_{2i-1}a_{2i} - b_i^2)/(a_{2i-1}^2) \qquad (1 \le i \le s)$$
$$G_i = a_{2i}/a_{2i-1} \qquad (1 \le i \le s)$$
$$E_j = a_j/a_1 \qquad (2s+1 \le j \le n)$$

Proof. This follows immediately from the following more general statement:

LEMMA. *Let $S_1, S_2, \ldots, S_t$ be irreducible hypersurfaces with irreducible equations $f_1, f_2, \ldots, f_t$ in an affine space $\mathbb{A}^m$. Then the group of invertible functions on $\mathbb{A}^m - \bigcup S_i$, modulo the constants, is the free abelian group generated by* (*the classes of the*) f_i.

Proof. The proof is clear by unique factorization of rational functions and the description of the regular functions on $\mathbb{A}^m - \bigcup S_i$.

Remark. We have chosen certain fractions as generators, in order to preserve U^s equivariance. This will become important in §6.6.

COROLLARY. *Property* (1) *of Danilov's lemma* (6.2.3) *is satisfied by the collection of divisors $\{D_i, \delta_j^1, \delta_j^2\}$, and the vectorspace $\tilde{V}^*$ has a basis given by the functions $\{A_i, F_i, G_i, E_j\}$.*

§6.5. Lemma on simplicial decompositions. The following lemma will be the main technical tool for verifying Danilov's property (2).

Suppose $\tilde{\alpha}: \tilde{V} \to \tilde{H}$ is a surjective homomorphism between two real vectorspaces. Fix a simplicial R.P.D. $\tilde{\Omega}$ of $\tilde{H}$ and a single closed cone $\omega \in \tilde{\Omega}$. Suppose the remaining one dimensional cones in $\tilde{H} - \omega$ are given by the positive multiples of certain rational vectors $\{k_1, k_2, \ldots, k_r\}$. Suppose a collection $\tilde{\Sigma}$ of cones in $\tilde{V}$ is given, with a partition of the one dimensional cones into two disjoint collections, $\{\lambda_1, \ldots, \lambda_r\}$ and $\{\eta_1, \ldots, \eta_s\}$ which satisfy the following properties:

HYPOTHESIS 1. The set $\tilde{\alpha}^{-1}(\omega)$ is a union of cones in $\tilde{\Sigma}$, $\{\eta_1, \ldots, \eta_s\}$ are the one dimensional cones in $\tilde{\Sigma} \cap \tilde{\alpha}^{-1}(\omega)$, and $\tilde{\Sigma} \cap \tilde{\alpha}^{-1}(\omega)$ is a simplicial decomposition of $\tilde{\alpha}^{-1}(\omega)$.

HYPOTHESIS 2. The projection $\tilde{\alpha}$ defines a one to one correspondance between the one dimensional cones $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ in $\tilde{\Sigma}$ and the vectors $\{k_1, k_2, \ldots, k_r\}$, i.e. each λ_i contains a (unique) vector $v_i \in \lambda_i$ such that $\tilde{\alpha}(v_i) = k_i$.

HYPOTHESIS 3. Let $\{w_1, w_2, \ldots, w_s\}$ denote rational vectors which generate the one dimensional cones $\{\eta_1, \ldots, \eta_s\}$ in $\tilde{\Sigma} \cap \tilde{\alpha}^{-1}(\omega)$ and suppose that $I \subset \{1, 2, \ldots, r\}$ and $J \subset \{1, 2, \ldots, s\}$ are subsets. Then the collection of vectors $\{v_i, w_j \mid i \in I, j \in J\}$ span a cone in $\tilde{\Sigma}$ if and only if the following two conditions hold:

(a) there is a single closed cone in $\tilde{\Omega}$ which contains all the vectors

$$\{\tilde{\alpha}(v_i), \tilde{\alpha}(w_j) \mid i \in I, j \in J\}$$

(b) The positive combinations of the vectors $\{w_j\}$ form a cone in the given R.P.D. of $\tilde{\alpha}^{-1}(\omega)$.

LEMMA. *If a collection of cones $\tilde{\Sigma}$ in $\tilde{V}$ satisfy hypothesis* 1, 2, *and* 3 *above, then they form a simplicial* R.P.D. *of the vectorspace $\tilde{V}$.*

Proof. First we will show that the cones c form a decomposition of the space $\tilde{V}$, i.e. every nonzero point $p \in \tilde{V}$ has a unique representation as a convex combination of a collection of the v's and w's which satisfies the conditions (a) and (b) above. For any point $p \in \tilde{V}$, the image $\tilde{\alpha}(p)$ lies in a unique cone of the decomposition $\tilde{\Omega}$, so it has a representation

$$\tilde{\alpha}(p) = \sum_{i \in I} a_i \tilde{\alpha}(v_i) + \sum_k b_k y_k$$

(for some subset $I \subset \{1, 2, \ldots, r\}$ and vectors y_k in the one dimensional faces of ω), and the numbers a_i are uniquely determined. Therefore, $p - \Sigma a_i v_i \in \tilde{\alpha}^{-1}(\omega)$ so it has a unique representation,

$$p - \Sigma a_i v_i = \sum_{j \in J} c_j w_j$$

(for some subset $J \subset \{1, \ldots, s\}$) as a convex combination of a subset of the vectors $\{w_1, \ldots, w_s\}$ which span a cone. The vectors v_i, w_j, (for $i \in I, j \in J$) span a cone in $\tilde{\Sigma}$ because hypothesis 3 is satisfied.

Now we will show that this cone decomposition $\tilde{\Sigma}$ of $\tilde{V}$ is simplicial, i.e. if the vectors $\{v_{i_1}, \ldots, v_{i_m}, w_{j_1}, \ldots, w_{j_n})$ satisfy conditions (a) and (b), then they are linearly independent. But suppose some linear combination vanishes,

$$\Sigma a_i v_i + \Sigma b_j w_j = 0$$

By applying $\tilde{\alpha}$ we see that $a_i = 0$ (since the vectors $\{\tilde{\alpha}(v_i)\}$ are linearly independent modulo the $\tilde{\alpha}(w_j)$). However the vectors $\{w_j\}$ span a simplicial cone in $\tilde{V}$, so the $b_j = 0$ also.

In the same way we see that the cones in $\tilde{\Sigma}$ are disjoint.

§6.6. Four cone decompositions. We wish to apply lemma 6.5 to the cones in the set $\tilde{\Sigma}$ in $\tilde{V}$ (§6.2.2, §6.3.2) to see that it is a simplicial cone decomposition. For this we will associate vectorspaces and cone decompositions to each of the varieties in the following diagram (which was studied in §5.9.1),

$$\begin{array}{ccccc} C_s = \bar{\pi}^{-1}(\mathbb{A}^1)^s & \subset & \bar{T} & \longrightarrow & Z_s \\ \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \mu} \\ (\mathbb{A}^1)^s & \subset & \prod_{i=1}^{s} \mathbb{P}^1 & \longrightarrow & \prod_{i=1}^{s} \mathbb{P}^2 \end{array}$$

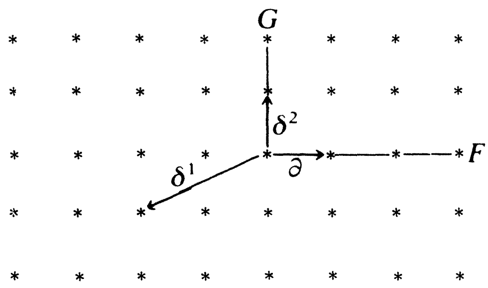
We will describe a combinatorial relationship between the corresponding cone decompositions. First we describe a simplicial cone decomposition $\tilde{\Omega}$ of the space $\tilde{H} = \prod_{i=1}^{s} \mathbb{R}^2$ which corresponds to the variety $\prod_{i=1}^{s} \mathbb{P}^2$.

The quadrics which lie in the open subset $\mathbb{P}^{00} = \mathbb{P}^2 - \delta^1 - \delta^2 - \partial\mathbb{P}^2$ are parametrized by the (multiples of the) symmetric matrices $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with $a \neq 0$, $c \neq 0$ and $ac - b^2 \neq 0$. The functions $F = (ac - b^2)/a^2$ and $G = c/a$ form a basis of the group of regular invertible functions on $\mathbb{P}^{00}$ (modulo the constant functions), and the orders of their zeroes and poles on the above divisors are:

$$\begin{array}{ll} \operatorname{ord}_{\delta^1}(F) = -2 & \operatorname{ord}_{\delta^1}(G) = -1 \\ \operatorname{ord}_{\delta^2}(F) = 0 & \operatorname{ord}_{\delta^2}(G) = 1 \\ \operatorname{ord}_{\partial}(F) = 1 & \operatorname{ord}_{\partial}(G) = 0 \end{array}$$

Thus the variety $\mathbb{P}^2$ of complete quadrics in $\mathbb{P}^1$ is associated to the following

R.P.D. of $\mathbb{R}^2$



DIAGRAM

We will denote the vector $(-2, -1)$ by δ^1, the vector $(0, 1)$ by δ^2, and the vector $(1, 0)$ by ∂.

DEFINITION. Let $\tilde{\Omega}$ be the product cone decomposition of the space $\tilde{H} = \prod_{i=1}^{s} \mathbb{R}^2$ and let ω be the s dimensional cone in $\tilde{\Omega}$ which is spanned by the standard basis vectors $(1, 0)$ in each copy of $\mathbb{R}^2$, i.e.

$$\omega = \{\Sigma a_i(\vec{\partial}_i) \mid a_i \geq 0\}$$

where $\vec{\partial}_i = (0, 0, \ldots, \partial_i, 0, \ldots, 0) \in (\mathbb{R}^2)^s$.

Recall that in §6.2.2 we have associated a cone decomposition $\tilde{\Sigma}$ of the vectorspace $\tilde{V}$ corresponding to the variety Z_s. In §5.3 we have associated a cone decomposition Σ of the vectorspace V corresponding to the toric variety $\bar{T}$, together with a cone preserving homomorphism $\alpha: V \to H = \mathbb{R}^s$, where H is decomposed (by a collection of cones Ω) into quadrants. (Unfortunately the coordinates of α are denoted $\alpha_1, \alpha_3, \ldots, \alpha_{2s-1}$). Notice that ω is a union of cones in Ω. We define Σ^1 to be the set of cones in Σ which are mapped by α into the closed quadrant ω. Notice that the vectors which span the cones in Σ^1 are the ones associated to the divisors in the open set $C_s = (\bar{\pi})^{-1}(\mathbb{A}^1)^s$. By §5.10, these vectors are in natural one to one correspondence with the boundary divisors D_i in Z_s. The following facts are easy to verify: (notation as in §6.4).

Fact 1: The subgroup of Γ (the regular invertible functions on Z^{00}) which is generated by F_i and G_i is the pullback (by μ) of the regular invertible functions on $\prod_{i=1}^{s} \mathbb{P}^{00}$ (i.e. they form a basis for the vectorspace $\tilde{H}$).

Fact 2: The elements A_i, F_i, and E_j in Γ are invariant under the action of the subgroups

$$\prod_{i=1}^{s}\begin{pmatrix}1 & 0\\ * & 1\end{pmatrix}$$

in the various s blocks of G_s. These invariant functions, when restricted to the diagonal torus $T \subset Z^0$ generate the group of characters of T (i.e. they form a basis for the vectorspace V^*).

Remark. The diagonal conics in Z_s are the intersection $T = \bar{T} \cap Z_s^{\times}$, and so the diagonal quadrics in $\prod_{i=1}^{s} \mathbb{P}^2$ are the intersection

$$\bar{T} = \left(\prod_{i=1}^{s} \mathbb{P}^2\right)^{(00)} \cap \prod_{i=1}^{s} \mathbb{P}^1$$

These fit in the following diagram

$$\begin{tikzcd} T \arrow[r, "\Phi"] \arrow[d] & Z_s^{(00)} \arrow[d] \\ \bar{T} \arrow[r, "\Psi"] & \left(\prod_{i=1}^{s} \mathbb{P}^2\right)^{(00)} \end{tikzcd}$$

Thus we have four lattices of regular invertible functions modulo constants,

$$\begin{tikzcd} \Gamma(\bar{T}) & \Gamma(Z_s^{(00)}) \arrow[l, "\Phi^*"'] \\ \Gamma(T) \arrow[u] & \Gamma\left(\prod_{i=1}^{s} \mathbb{P}^2\right)^{(00)} \arrow[l, "\Psi^*"'] \arrow[u] \end{tikzcd}$$

where the vertical arrows are injective and the horizontal arrows are surjective, by fact (2) above.

The one dimensional cones in $\bar{\Sigma}$ According to §6.2.2 and §6.4.2, the one dimensional cones in $\bar{\Sigma}$ are given by the positive multiples of the (three kinds of) vectors,

$$\delta_i^1(f) = \mathrm{ord}_{\delta_i^1}(f) \qquad (1 \le i \le s)$$

$$\delta_i^2(f) = \mathrm{ord}_{\delta_i^2}(f) \qquad (1 \le i \le s)$$

$$D_j(f) = \mathrm{ord}_{D_j}(f) \qquad (1 \le j \le r)$$

as f varies over the basis elements (§6.4) for the group Γ of regular invertible functions on Z_s^{00}. These orders are computed in the following matrix:

	D_k	$\delta_k^1\,(k \ne 1)$	δ_k^2	δ_1^1	
A_i	as for $\bar{T}$	$1 \cdot \delta(i, k)$	0	-1	$(2 \le i \le s)$
F_i	≥ 0 as for $\bar{T}$	$-2 \cdot \delta(i, k)$	0	$-2\delta(i, 1)$	$(1 \le i \le s)$
E_i	as for $\bar{T}$	0	0	-1	$(2s + 1 \le i \le n)$
G_i	0	$-1 \cdot \delta(i, k)$	$1 \cdot \delta(i, k)$	$-\delta(i, 1)$	$(1 \le i \le s)$

(where $\delta(i, k) = 1$ if $i = k$ and is 0 if $i \ne k$).

Proof. By the geometric description of the open set $A_s = U^s \times C_s$, which was given in §5.10, and the fact that the functions A_i, F_i, E_i are U^s invariant, it follows that for any function f on this list, we have

$$\mathrm{ord}_{D_i}(f) = \mathrm{ord}_{D_i \cap C_s}(f \mid C_s)$$

which gives most of the first column in the above table. The fact that G_i vanishes on D_k follows from the fact that G_i is invertible outside $\bar{\delta}_i^1 \cup \bar{\delta}_i^2$ and that every divisor D_k meets this complement. The computation of columns 2 and 3 is easily performed by restricting to the open set Z_s^0 which meets the divisors $\bar{\delta}_i^j$. In this open set the coordinates a_i, b_i, are homogeneous coordinates and δ_i^1 has the homogeneous equation $a_{2i-1} = 0$, while δ_i^2 has the homogeneous equation $a_{2i} = 0$. An inspection of the functions gives the above calculation.

Consider the combinatorial correspondence (§5.10) between the T-stable divisors $D_i \cap C_s$ in C_s and the G_s-stable divisors D_i in Z_s. Corresponding to each divisor we have a vector in $\tilde{V}$ and in V (respectively). This correspondence extends to a unique linear embedding $V \to \tilde{V}$ because of the above calculation. It identifies V with the subspace of $\tilde{V}$ which is generated by the vectors D_i

$$\begin{array}{ccc} \{T\text{-stable divisors in } C_s\} & \leftrightarrow & \{G_s\text{-stable divisors in } Z_s\} \\ \downarrow & & \downarrow \\ V & \longrightarrow & \tilde{V} \end{array}$$

The map $\tilde{\alpha}: \tilde{V} \to \tilde{H}$ takes V to the subspace $H \subset \tilde{H}$ which is spanned by the $\tilde{\partial}_i$. We obtain a diagram of vectorspaces

$$\begin{array}{ccc} V & \xrightarrow{\varphi_*} & \tilde{V} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\tilde{\alpha}} \\ H & \xrightarrow{\psi_*} & \tilde{H} \end{array} \quad \text{and cones} \quad \begin{array}{ccc} \Sigma^1 & \longrightarrow & \tilde{\Sigma} \\ \downarrow & & \downarrow \\ \omega & \longrightarrow & \tilde{\Omega} \end{array}$$

§6.7. Proof of property (2) of Danilov

PROPOSITION. *The cones* $\tilde{\alpha}: \tilde{\Sigma} \to \tilde{\Omega}$ *of* §6.6.2 *satisfy the hypotheses of lemma* 6.5 *and therefore they form a simplicial decomposition of* $\tilde{V}$ (*where the special cone* $\omega \in \tilde{\Omega}$ *is the s dimensional cone which is spanned by the unit vectors* (1, 0) *in each copy of* $\mathbb{R}^2$).

Proof. We will show that

(1) $\tilde{\alpha}^{-1}(\omega)$ is a union of cones in $\tilde{\Sigma}$, and that $\tilde{\Sigma} \cap \tilde{\alpha}^{-1}(\omega)$ is a simplicial decomposition

(2) For each one dimensional cone in $\tilde{H} - \omega$, there is a unique one dimensional cone in $\tilde{\Sigma}$ which lies over it

(3) For any choice of subsets $I \subset \{1, 2, \ldots, s\}$, $J \subset \{1, 2, \ldots, s\}$, $K \subset \{1, 2, \ldots, r\}$ the collection of divisors

$$\{\delta_i^2, \delta_j^2, D_k \mid i \in I, j \in J, k \in K\}$$

have nonempty intersection in Z_s (i.e. they span a cone in $\tilde{\Sigma}$) if and only if

(a) the divisors $\{\bar{\delta}_i^1, \bar{\delta}_j^2, \mu(D_k) \mid i \in I, j \in J, k \in K\}$ have nonempty intersection in $\prod_{i=1}^s \mathbb{P}^2$ and

(b) the divisors $\{\bar{T} \cap D_k \mid k \in K\}$ have nonempty intersection in $\bar{T}$

Proof of (1). It is easy to see that $\tilde{\alpha}^{-1}(\omega)$ is a union of cones in $\tilde{\Sigma}$: these are precisely the cones in $\Sigma \subset \tilde{\Sigma}$ which lie over the positive quadrant $\varphi_*^{-1}(\omega) \subset H$. In particular, they form a simplicial decomposition of $\tilde{\alpha}^{-1}(\omega)$.

Proof of (2). This follows directly from the computation of the matrix (§6.6).

Proof of (3). This is straightforward, since the divisor D_k projects (in each $\mathbb{P}^2$ factor) either to $\partial\mathbb{P}^2$ or to all of $\mathbb{P}^2$, and the pattern of intersections of the divisors $\partial\mathbb{P}^1$, δ^1, and δ^2 is given by the diagram in §5.8.

§6.8. Proof of property (3) of Danilov. We must count the number of maximal cones in $\tilde{\Sigma}$.

Notational Remark: Since the s projections to the coordinate axes, $V \to \mathbb{R}$ are labelled by the roots $\alpha_1, \alpha_3, \ldots, \alpha_{2s-1}$, we will denote the corresponding projections $\tilde{V} \to \mathbb{R}^2$ by $\tilde{\alpha}_1, \tilde{\alpha}_3, \ldots, \tilde{\alpha}_{2s-1}$.

Each maximal cone in $\tilde{\Sigma}$ projects to a maximal cone in $\tilde{\Omega}$. These project to maximal cones in each $\mathbb{R}^2$ factor, and there are 3 such maximal cones: span (δ^1, δ^2), span (∂, δ^1), span (∂, δ^2). Consider the set $\tilde{\alpha}^{-1}(\omega)$ in $\tilde{V}$ which we call the ω-quadrant. Since the ω-quadrant is in the image of Φ_* we may identify it with a (generalized) quadrant of the vectorspace V. For any subset $I \subset \{1, 2, \ldots, 1\}$ we define the corresponding (closed) corner of the ω-quadrant to be

$$\{p \in \tilde{\alpha}^{-1}(\omega) \mid \alpha_{2i-1}(p) = 0 \text{ for all } i \in I\}$$

$$= \{p \in \tilde{\alpha}^{-1}(\omega) \mid \tilde{\alpha}_{2i-1}(p) = 0 \text{ for all } i \in I\}$$

Now suppose that $c \in \tilde{\Sigma}$ is a maximal cone. We denote by c' the unique largest cone in $\bar{c} \cap \tilde{\alpha}^{-1}(\omega)$ (where $\bar{c}$ denotes the closure of c). This lies in some smallest corner F of the ω quadrant, which in turn corresponds to some subset $I \subset \{1, 2, \ldots, s\}$. For each $i \in I$ we have,

$$\tilde{\alpha}_{2i-1}(c) = \text{span}\,(\delta^1, \delta^2)$$

because $\tilde{\alpha}_{2i-1}(c)$ is a maximal cone in $\mathbb{R}^2$ which does not contain ∂ as a face. On the other hand, if $i \notin I$, then

$$\tilde{\alpha}_{2i-1}(c) = \text{either} \begin{cases} \text{span}\,(\partial, \delta^1) \\ \text{span}\,(\partial, \delta^2) \end{cases}$$

We conclude that the number of maximal cones $c \in \tilde{\Sigma}$ such that c' is the maximal cone in $\bar{c} \cap \tilde{\alpha}^{-1}(\omega)$ comes to $2^{s-|I|}$. Furthermore, the corner F has $(n - |I|)!/(2^{s-|I|})$ maximal cones c'. This is because the vector subspace of V spanned by F is decomposed into cones according to the cone decomposition of the toric variety $\bar{T}_{|I|}$ which has $(n - |I|)!$ maximal cones, divided equally among $2^{s-|I|}$ generalized quadrants isomorphic to F. Also, there are $\binom{n}{|I|}$ corners F of codimension $|I|$. In summary, the number of maximal cones in $\tilde{\Sigma}$ is

$$\sum_{h=0}^{s} \binom{n}{h}(n-h)! = \mathscr{X}(Z_d)$$

as desired.

§6.9. Proof of property (4) of Danilov. We have seen (§5.11) that Z_s has a paving by even dimensional (algebraic) rational cells such that the closure of each rational cell is a union of proper intersections of some collection of divisors in the list

$$\{D_1, \ldots, D_r, \delta_1^1, \ldots, \delta_s^1, \delta_1^2, \ldots, \delta_s^2\}$$

But the rational cells form a basis for the rational cohomology of Z_s, so the cohomology classes of the above divisors must generate the cohomology of Z_s.

§6.10. Proof of theorem 6.1.1. In this section we will solve for the cohomology classes $[\delta_j^i]$ in terms of the cohomology classes $[D_p]$, and then substitute into the equations (§6.2.2) for the relations in $H^*(Z_s)$.

§6.10.1. Lemma. *In $H^*(Z_s; \mathbb{Q})$ the following relations hold*:

$$[\delta_i^1] = [\delta_i^2] = \tfrac{1}{2}\Sigma[D_p]$$

where the sum is taken over all codimension 1 *orbits D_p of G_s on Z_s such that for each i $(1 \le i \le s)$ we have*

$$\mu_i(D_p) = \partial$$

i.e.,

$$\alpha_{2i-1}(D_p) > 0$$

Proof of Lemma. In the cohomology $H^*(\mathbb{P}^2)$ we have the following relation:

$$[\delta^1] = [\delta^2] = \tfrac{1}{2}[\partial]$$

This can be seen from the R.P.D. (§6.6) of $\mathbb{R}^2$ which is associated to $\mathbb{P}^2$. Each of the regular invertible functions on $\mathbb{P}^2 - \delta^1 - \delta^2 - \partial$ gives a relation in $\mathbb{P}^2$, which (for the functions F and G) read

$$F(\partial)[\partial] + F(\delta^1)[\delta^1] + F(\delta^2)[\delta^2] = 0$$

$$G(\partial)[\partial] + G(\delta^1)[\delta^1] + G(\delta^2)[\delta^2] = 0$$

or

$$[\partial] = 2[\delta^1]$$

$$[\delta^1] = [\delta^2]$$

It follows that, in $H^*(Z_s)$ we have the relations

$$\mu_i^*([\delta^1]) = \mu_i^*([\delta^2]) = \tfrac{1}{2}\mu_i^*([\partial])$$

where $\mu_i : Z_s \to \mathbb{P}^2$ is the composition of μ with the projection to the ith factor. Now compute $\mu_i^*([\partial])$: For each codimension one G_s orbit D on Z_s we have

$$\mu_i(D) = \partial \Leftrightarrow \alpha_{2i-1}(D) > 0$$

and therefore there are numbers r_j such that

$$\mu_i^*([\partial]) = \Sigma r_j[D_j]$$

where the sum is taken over all D_j such that $\alpha_{2i-1}(D_j) > 0$, where r_j is given by

$$r_j \partial = \alpha_{2i-1}(D_j)$$

However, it turns out that the numbers r_j are all equal to 1 since the vectors D_j are the minimal lattice vectors such that $\alpha_{2i-1}(D_j) \in \mathbb{Z}$ and $\alpha_{2i-1}(D_j) \geq 0$. (see §5.3)

§6.10.2. Completion of the proof. The substitution (§6.10.1) gives rise to a surjective ring homomorphism,

$$\mathbb{Q}[D_1, \ldots, D_r, \delta_1^1, \ldots, \delta_s^1, \delta_1^2, \ldots, \delta_s^2] \xrightarrow{\Phi} \mathbb{Q}[D_1, \ldots, D_r]$$

and we will show that $\Phi(I+J)=(I_1+I_2+I_3)$. First we show that $\Phi(I)=I_1+I_2$: If a collection of vectors $\{D_\alpha, \delta^1_\beta, \delta^2_\gamma\}_{\alpha\in A,\beta\in B,\gamma\in C}$ do not form a cone in $\bar{\Sigma}$ (and hence give a relation in I), then there are two possibilities:

(1) the vectors $\{D_\alpha\}_{\alpha\in A}$ do not form a cone in Σ

(2) there is an index i (where $1\le i\le s$) such that

(a) both δ^1_i and δ^2_i are present in the collection and

(b) some D_α in the collection projects (under μ_i) to $\partial\mathbb{P}^2$.

(see §6.7 and §5.10 for this combinatorics). In case (1) we have one of the generating functions in the ideal I_1. In case (2) we have one of the generating functions in I_2 for the following reason: If $\mu_i(D_\alpha)=\partial\mathbb{P}^2$ then $[\delta^1_i][\delta^2_i][D_\alpha]=0$ (since $\delta^1\cap\delta^2\cap\partial\mathbb{P}^2=\varnothing$). Substituting from §6.10.1 for $[\delta^1_i]$ and $[\delta^2_i]$ gives

$$(\tfrac{1}{2}\Sigma[D_p])^2[D_\alpha]=0$$

(where the sum is taken over those D_p such that $\alpha_{2i-1}(\vec{D}_p)>0$), which is one of the generating elements of I_2.

Now consider the linear relations $\Phi(J)$. Each of the functions A_i, F_i, G_i, E_j gives a relation in J. The G_i have already been considered in §6.10 and give

$$\delta^1_i=\delta^2_i=\tfrac{1}{2}\Sigma[D_p]$$

The remaining 3 types of relations can be read from the matrix of §6.6: we only need to substitute $\delta^k_1=\frac{1}{2}\Sigma[D_p]$. Since the vectors D_i are elements of V, we have

$$\operatorname{ord}_{D_k}F_i=\operatorname{ord}_{D_k\cap\bar{T}}(F_i\cap\bar{T})$$

according to the table in §6.6. In other words, in order to compute $\operatorname{ord}_{D_\alpha}$ for the functions A_i, F_i, or E_j, we can restrict these functions to the torus embedding. This gives rise to the following relations:

$$A_i:\ \sum_\alpha \operatorname{ord}_{D_\alpha}\left(\frac{a_{2i-1}}{a_1}\right)[D_\alpha]+\delta^1_i-\delta^1_1=0 \qquad (1\le i\le s)$$

$$F_i:\ \sum_\alpha \operatorname{ord}_{D_\alpha}\left(\frac{a_{2i}}{a_{2i-1}}\right)[D_\alpha]-2\delta^1_1=0 \qquad (1\le i\le s)$$

$$E_j:\ \sum_\alpha \operatorname{ord}_{D_\alpha}\left(\frac{a_j}{a_1}\right)[D_\alpha]-\delta^1_1=0 \qquad (2s+1\le j\le n)$$

adding equations (F_i) to twice (A_i) gives

$$F_i': \sum_\alpha \operatorname{ord}_{D_\alpha}\left(\frac{a_{2i}a_{2i-1}}{a_1a_1}\right)[D_\alpha] - \delta_1^1 = 0 \qquad (1 \le i \le s)$$

The relations (F_i') and (E_j) are now equivalent to the following: For any collection of integers m_i $(1 \le i \le s)$ and n_j $(2s+1 \le j \le n)$ such that

$$0 = 2\sum_i m_i + \sum_j n_j$$

we have

$$\begin{aligned}&\sum_{i=1}^{s} m_i(F_i') + \sum_{j=2s+1}^{n} (E_j)\\ &\quad = \sum_\alpha \left(\sum_{i=1}^{s} m_i \operatorname{ord}_{D_\alpha}\left(\frac{a_{2i}a_{2i-1}}{a_1a_1}\right) - m_1 2\delta_1^2 + \sum_{j=2s+1}^{n} n_j \operatorname{ord}_{D_\alpha}\left(\frac{a_j}{a_1}\right) - n_j\delta_1^1\right)[D_\alpha] = 0\end{aligned}$$

So

$$\sum_\alpha \operatorname{ord}_{D_\alpha}\left(\prod_{i=1}^{s} (a_{2i-1}a_{2i})^{m_i} \cdot \prod_{j=2s+1}^{n} (a_j)^{n_j}\right)[D_\alpha] = 0$$

Taking logarithms to pass to Lie algebra notation, we have $\sum_\alpha f(\vec{D}_\alpha)[D_\alpha] = 0$ whenever

$$f = \sum_{i=1}^{s} m_i(a_{2i} + a_{2i-1}) + \sum_{j=2s+1}^{n} n_j a_j$$

provided

$$2\sum_i m_i + \sum_j n_j = 0$$

and these are the generating functions of the ideal I_3.

§6.11. Proof of theorem 6.1.2

§6.11.1. We will apply the theorem of Leray–Hirsch ([Bo1], [Bo2]) to the fibre bundle $\pi: M_s \to \mathcal{F}_s$, using our computation (§6.1.1) of the cohomology of the fibre, Z_s. Recall that the flag manifold $\mathcal{F}_s$ consists of all orthogonal direct sum decompositions,

$$\mathbb{C}^n = P_1 \oplus P_2 \oplus \cdots \oplus P_s \oplus L_{2s+1} \oplus \cdots \oplus L_n$$

and therefore the planes P_i and the lines L_j are tautological vectorbundles over $\mathcal{F}_s$. The cohomology classes $[D_\alpha] \in H^2(Z_s)$ are restrictions of classes $[\tilde{D}_\alpha] \in H^2(M_s)$, where

$$\tilde{D}_\alpha = K \times_{K_s} D_\alpha \subset K \times_{K_s} Z_s = M_s$$

It follows that the cohomology of M_s is generated (over $H^*(\mathcal{F}_s)$) by the classes $\{[\tilde{D}_\alpha]\}$, and we have surjections

$$\begin{tikzcd} H^*(\mathcal{F}_s)[\tilde{D}_1, \ldots, \tilde{D}_r] \arrow[r, "\kappa"] \arrow[d, "\sigma"'] & H^*(M_s) \arrow[d] \\ \mathbb{Q}[D_1, \ldots, D_r] \arrow[r] & H^*(Z_s) \end{tikzcd}$$

It follows that the kernel of κ is $\tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3$, where $\tilde{I}_j$ is the unique ideal such that $\sigma(\tilde{I}_j) = I_j$ and $\kappa(\tilde{I}_j) = 0$.

§6.11.2. The ideal J_1. The vectors $\{D_{i_1}, D_{i_2}, \ldots, D_{i_k}\}$ do not form a cone in Σ if and only if the divisors $\{D_{i_1}, D_{i_2}, \ldots, D_{i_k}\}$ have empty intersection in Z_s, if and only if the divisors $\{\tilde{D}_{i_1}, \tilde{D}_{i_2}, \ldots, \tilde{D}_{i_k}\}$ have empty intersection in M_s. Therefore the ideal $\tilde{I}_1$ is J_1.

§6.11.3. The ideal J_2. Fix j with $1 \le j \le s$. We will show that if $\alpha_{2j-1}(D_p) > 0$, then

$$(\Sigma[\tilde{D}_k])^2[\tilde{D}_p] = 4((c^1)^2 - 4c^2)[\tilde{D}_p]$$

where the sum is taken over all $\tilde{D}_k$ such that $\alpha_{2j-1}(\tilde{D}_k) > 0$, and where c^1 and c^2 are the first and the second Chern classes of the tautological two dimensional vectorbundle, $P_j \to \mathcal{F}_s$. Consider the space $K \times_{K_s} \mathbb{P}^2$ of all pairs $((P, L), Q)$ where (P, L) is an orthogonal direct sum decomposition of $\mathbb{C}^n$ as above, and where Q is a complete quadric in the plane P_j. This contains the submanifold $K \times_{K_s} \partial\mathbb{P}^2$ of all direct sum decompositions

$$\mathbb{C}^n = P_1 \oplus \cdots \oplus L_{2j-1} \oplus L_{2j} \oplus \cdots \oplus P_s \oplus L_{2s+1} \oplus \cdots \oplus L_n$$

(which is the same as the manifold $\mathcal{F}_s$ except that we have broken the two dimensional space P_j into two one dimensional spaces). Let $i: K \times_{K_s} \partial\mathbb{P}^2 \to K \times_{K_s} \mathbb{P}^2$ denote the inclusion and let ν denote the normal bundle of this inclusion. The bundles P_j on $\mathcal{F}_s$ pull back to bundles (which we also denote by P_j) on $K \times_{K_s} \mathbb{P}^2$.

SUBLEMMA. $c^1(\nu)^2 = 4i^*((c^1(P_j)^2 - 4c^2(P_j))$.

Remark. The advantage of this sublemma is that it gives $c^1(\nu)$ in terms of a vectorbundle which is pulled back from $K \times_{K_s} \mathbb{P}^2$.

Proof of sublemma. First we show that the normal bundle ν is $T \otimes T$, where T is the bundle of tangents to the fibre of the projection $\pi \circ i: K \times_{K_s} \partial\mathbb{P}^2 \to \mathcal{F}_s$. Note first that the normal bundle of $\partial\mathbb{P}^2$ in $\mathbb{P}^2$ is $T_1 \otimes T_1$ where T_1 is the tangent bundle of $\partial\mathbb{P}^2$ because

(a) $\partial\mathbb{P}^2$ is the image of the diagonal Δ under the quotient mapping

$$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2 = \mathbb{P}^1 \times \mathbb{P}^1/\tau$$

where the involution τ switches factors,

(b) The normal bundle of $\partial\mathbb{P}^2$ in $\mathbb{P}^2$ is thus the quotient T_1/τ where

$$\begin{aligned} T_1 &\cong \text{normal bundle of } \Delta \text{ in } \mathbb{P}^1 \times \mathbb{P}^1 \cong \text{tangent bundle of } \Delta \\ &\cong \text{tangent bundle of } \partial\mathbb{P}^2 \end{aligned}$$

(c) For any line bundle L, if $\tau: L \to L$ denotes multiplication by -1, then the quotient L/τ is isomorphic to $L \otimes L$.

The same argument applies to the fibres of the projection $K \times_{K_s} \partial\mathbb{P}^2 \to \mathcal{F}_s$

It is easily seen that

$$T = \operatorname{Hom}(\xi, P_j/\xi) \cong \xi^* \otimes P_j/\xi$$

where ξ is the tautological line bundle on $K \times_{K_s} \partial\mathbb{P}^2$ which is associated to the fibres of $\pi \circ i$. The relation

$$\xi \oplus P_j/\xi \cong P_j$$

gives

$$c^1(P_j) = c^1(\xi) + c^1(P_j/\xi)$$

$$c^2(P_j) = c^1(\xi) \cdot c^1(P_j/\xi)$$

so

$$\begin{aligned}(c^1(\nu))^2 &= 4(-c^1(\xi)+c^1(P_j/\xi))^2\\ &= 4(c^1(\xi)^2+c^1(P_j/\xi)^2-2c^1(\xi)\cdot c^1(P_j/\xi))\\ &= 4(c^1(P_j)^2-4c^2(P_j))\end{aligned}$$

which proves the sublemma. Now consider the following fibre square:

$$\begin{tikzcd}
\cup\{\tilde{D}_\alpha\} \arrow[r, "\bar{i}"] \arrow[d, "\bar{\mu}_j"'] & M_s = K\times_{K_s} Z_s \arrow[d, "\tilde{\mu}_j"] \\
K\times_{K_s}\partial\mathbb{P}^2 \arrow[r] & K\times_{K_s}\mathbb{P}^2 \arrow[d, "\pi"] \\
 & \mathscr{F}_s
\end{tikzcd}$$

where the union $\bigcup\{\tilde{D}_\alpha\}$ is taken over all $\tilde{D}_\alpha$ such that $\tilde{\mu}_j(\tilde{D}_\alpha)=K\times_{K_s}\partial\mathbb{P}^2$ (i.e. $\alpha_{2j-1}(D_\alpha)>0$). We will use $\{\tilde{D}_p\}$ to denote the homology class in $H_{\dim(\tilde{D}_p)}(\bigcup\tilde{D}_\alpha)$ represented by $\tilde{D}_p$, and $[\tilde{D}]$ to denote the cohomology class represented by $\tilde{D}$ in $H^2(M_s)$. We shall also use the notation $\{K\times_{K_s}\partial\mathbb{P}^2\}$ to denote the fundamental homology class in $H_*(K\times_{K_s}\partial\mathbb{P}^2)$. We use "dual" to denote the Poincaré duality isomorphism. Now compute

$$\begin{aligned}\operatorname{dual}\Big(\sum_\alpha[\tilde{D}_\alpha])^2\cdot[\tilde{D}_p]\Big) &= \Big(\sum_\alpha[\tilde{D}_\alpha]\Big)^2\cdot\bar{i}_*(\{\tilde{D}_p\})\\ &= \bar{i}_*\Big(\bar{i}^*\Big(\sum_\alpha[\tilde{D}_\alpha]\Big)^2\cdot\{\tilde{D}_p\}\Big)\\ &= \bar{i}_*(\bar{i}^*\tilde{\mu}_j^*\operatorname{dual}(i_*\{K\times_{K_s}\partial\mathbb{P}^2\})^2\cdot\{\tilde{D}_p\})\\ &= \bar{i}_*(\bar{\mu}_j^*(i^*\operatorname{dual}i_*\{K\times_{K_s}\partial\mathbb{P}^2\})^2\cdot\{\tilde{D}_p\})\\ &= \bar{i}_*(\bar{\mu}_j^*(c^1(\nu))^2\cdot\{\tilde{D}_p\})\\ &= \bar{i}_*(\bar{\mu}_j^*i^*(4c^1(P_j)^2-16c^2(P_j))\cdot\{\tilde{D}_p\})\\ &= \bar{i}_*(\bar{i}^*\tilde{\mu}_j^*(4c^1(P_j)^2-16c^2(P_j))\cdot\{\tilde{D}_p\})\\ &= \tilde{\mu}_j^*(4c^1(P_j)^2-16c^2(P_j))\cdot\bar{i}_*(\{\tilde{D}_p\})\\ &= \operatorname{dual}(4c^1(P_j)^2-16c^2(P_j))\cdot[\tilde{D}_p]\end{aligned}$$

as desired.

6.11.4. The ideal J_3. (This section is parallel to that in [DP3] so we will only sketch the calculation). The character $f = a_{2j-1} + a_{2j}$ $(1 \leq j \leq s)$ of K_s induces a line bundle $\Lambda^2(P_j)$ on $\mathcal{F}_s = K/K_s$. For each of these characters we have a relation

$$\sum_{\alpha} \operatorname{ord}_{\tilde{D}_\alpha}(f)[\tilde{D}_\alpha] = c^1(\Lambda^2(P_j))$$

because the function f gives a section of $\Lambda^2(P_j)$ whose zeroes and poles are contained in the divisors $\tilde{D}_\alpha$ (and $c^1(\Lambda^2 P_j) = c^1(P_j)$). Similarly the character $g = a_j$ $(2s+1 \leq j \leq n)$ induces a section of the line bundle L_j on $\mathcal{F}_s$ and this gives a relation

$$\sum_{\alpha} \operatorname{ord}_{\tilde{D}_\alpha}(g)[\tilde{D}_\alpha] = c^1(L_j)$$

These are the generators of the ideal J_3.

§6.12. Proof of theorem 6.1.2. Assume the integers m_i and n_j satisfy $2\Sigma m_i + \Sigma n_j = 0$. Then the character

$$f = \sum_{i=1}^{s} (a_{2i} + a_{2i-1}) + \sum_{j=2s+1}^{n} n_j a_j$$

induces a line bundle

$$L_f = \bigotimes_{i=1}^{s} (\Lambda^2 P_i)^{m_i} \otimes \bigotimes_{j=2s+1}^{n} L_j$$

on $\mathcal{F}_s = K/K_s$. For each of these characters we have a relation

$$\sum_{\alpha} \operatorname{ord}_{\tilde{D}_\alpha}(f)[\tilde{D}_\alpha] = c^1(L_f)$$

because the function f gives a section of L_f whose zeroes and poles are contained in the divisors $\tilde{D}_\alpha$. These are the generators of an ideal which coincides with J_3 because of the following lemma:

LEMMA. *In $V^* = \operatorname{Hom}(V, \mathbb{R})$ the following two subspaces are equal*:

$W_1 = \operatorname{span}\{a_{2i} + a_{2i-1}, a_j \mid 1 \leq i \leq s, 2s+1 \leq j \leq n\}$

$$W_2 = \left\{ \sum_{i=1}^{s} m_i(a_{2i} + a_{2i-1}) + \sum_{j=2s+1}^{n} n_j a_j \,\middle|\, \sum_{i=1}^{s} 2m_i + \sum_{j=2s+1}^{n} n_j = 0 \right\}$$

Proof of lemma. In V^* we have the relation $\sum_{i=1}^{n} a_i = 0$, so $\dim(W_1) = n - s - 1$. Clearly $W_2 \subset W_1$. In fact, it is the image of the map $\Phi: \mathbb{R}^{n-s-1} \to W_1$ which is the restriction to the subspace $\sum_i 2m_j + \sum_j n_j = 0$ of the map $\Phi: \mathbb{R}^{n-s} \to W_1$ which is given by

$$\Phi(m_1, m_2, \ldots, m_s, n_{2s+1}, \ldots, n_n) = \sum_{i=1}^{s} m_i(a_{2i} + a_{2i-1}) + \sum_{j=2s+1}^{n} n_j a_j$$

We claim Φ is injective. But

$$\ker(\Phi) = \{(\vec{m}, \vec{n}) \mid m_1 = m_2 = \cdots = m_s = n_{2s+1} = \cdots = n_n\}$$

because the only linear combination of the a's which is 0 in V^* is $\sum a_i$. If we let b denote the value of $m_i = n_j$ then (since $2\sum m_i + \sum n_j = 0$) we have $(n-s)b = 0$, i.e. $b = 0$.

§7. Larger Compactifications

In [DP1] a family of "larger" compactifications of X^0 (the symmetric variety of nondegenerate quadrics) is defined. In this chapter we give the cohomology of these larger compactifications. The proofs of the results here are exactly parallel to the proofs of the analogous statements for X, so we will omit them.

§7.1. Identifying the larger compactifications. Let $T \subset X^0$ denote the subset of completely diagonal nondegenerate quadrics (i.e. the nonsingular symmetric matrices, modulo multiples of the identity). The torus embedding $\bar{T}$ is a compactification of T which is associated to the rational polyhedral decomposition Σ of the vectorspace V, as in §5.3. The R.P.D. Σ is *nonsingular* in that each closed cone σ is simplicial and the one dimensional cones in the faces of σ form a basis (over $\mathbb{Z}$) for the lattice of one parameter subgroups of T which lie in the plane of σ. It is also Σ_n*-invariant*, i.e. it is invariant under the reflection through the hyperplanes $a_i = a_j$ (for $i \neq j$). For any Σ_n-invariant simplicial nonsingular R.P.D. Σ' which refines the R.P.D. Σ, the associated toric variety is a "wonderful" T-equivariant compactification $\bar{T}'$ of T, and there is a canonical T-equivariant morphism $\bar{\Psi}: \bar{T}' \to \bar{T}$.

THEOREM. [DP1] *Given Σ' as above, there is a wonderful G-equivariant compactification X' of X^0, such that $\bar{T}'$ is the closure in X' of T, and there is a G-equivariant surjection*

$$\psi : X' \to X$$

(where X is the variety of complete quadrics) whose restriction to $\bar{T}'$ is the morphism $\bar{\Psi}$.

§7.2. The spaces Z_s and M_s

DEFINITION. Given such a compactification, and an integer s (with $1 \le s \le [n/2]$) let

$$Z'_s = \bar{\Psi}^{-1}(Z_s)$$
$$M'_s = K \times_{K_s} Z'_s$$

PROPOSITION. *The variety Z'_s is the closure in X' of the set of nondegenerate quadrics which are diagonal with respect to the orthogonal direct sum decomposition into coordinate planes and lines,*

$$\mathbb{C}^n = P_1 \oplus \cdots \oplus P_s \oplus L_{2s+1} \oplus \cdots \oplus L_n$$

This proposition gives rise to natural maps $Z'_s \to Z'_{s-1}$ and therefore to a tower of spaces

$$X' \to M'_m \leftarrow M'_{m-1} \leftarrow \cdots \leftarrow M'_0$$

(where $m = [n/2]$) with an action of $\Gamma_s = \Sigma_s \times \Sigma_{n-2s}$ on M'_s. We also obtain by composition a map

$$\mu' : Z'_s \to \prod_{i=1}^{s} \mathbb{P}^2$$

from which we define

$$\partial_I Z'_s = (\mu')^{-1}\left(\partial_I \prod_{i=1}^{s} \mathbb{P}^2\right)$$

for any subset $I \subset \{1, 2, \ldots, s\}$. Define the toric variety

$$\bar{T}'_s = (\mu')^{-1}(p)$$

where $p = \mu'(Q_0)$ denotes the basepoint of $\prod_{i=1}^{s} \mathbb{P}^2$ (as in §2.5).

§7.3. Statement of results

THEOREM. *The homomorphism*

$$H^*(X'; \mathbb{Q}) \to H^*(M'_m; \mathbb{Q})$$

is injective. The image is precisely those cohomology classes which, for each s, $(0 \le s \le m)$ pull back to Γ_s-invariant classes in $H^*(M'_s; \mathbb{Q})$. The ideals

$$I^i_s = \ker (H^i(X') \to H^i(M'_s))$$

filter $H^i(X')$, and there are canonical isomorphisms

$$I^i_{s-1}/I^i_s \cong \bigoplus_{a+b=i} H^a(\mathcal{F}_s) \otimes H^{b-4a}(\bar{T}'_s)$$

§7.4. Cohomology ring structure of M_s. The cohomology ring of M'_s is also described as in §6.1, to which we now refer for the notation to be used in this section. The linear map $\alpha: V \to H \cong \mathbb{R}^s$ takes cones in Σ' to cones in Ω, since Σ' is a refinement of the cone decomposition Σ. (The components of α are called $\alpha_1, \alpha_3, \ldots, \alpha_{2s-1}$). Let $D'_i, \ldots, D'_r$ denote the primitive generating vectors of the one dimensional cones in Σ' such that, for each i (with $1 \le i \le s$) we have

$$\alpha_{2i-1}(D'_j) \ge 0$$

There is a one to one correspondance between the vectors D'_j and the codimension one orbits D'_j of G_s on Z'_s.

THEOREM. *The cohomology ring $H^*(M'_s; \mathbb{Q})$ is naturally isomorphic to*

$$H^*(\mathcal{F}_s; \mathbb{Q})[D'_1, \ldots, D'_r]/(J'_1 + J'_r + J'_3)$$

where

(a) J'_1 *is the ideal generated by the monomials* $[D'_{i_1}][D'_{i_2}]\cdots[D'_{i_k}]$ *such that the vectors* $D'_{i_1},\ldots,D'_{i_k}$ *do not form a cone in* Σ'

(b) J'_2 *is the ideal generated by the polynomials*

$$\left\{\left(\sum_{D'\in j'}[D']\right)^2 - 4c^1(P_j)^2 + 16c^2(P_j)\cdot[E'] \,\middle|\, [E']\in \mathbf{j}'\right\}.$$

where

$$\mathbf{j}' = \{D' \mid D' \text{ is a codimension } 1 \text{ orbit and } \alpha_{2j-1}(D') > 0\}$$

and where j *is allowed to vary over the numbers* $1, 2, \ldots, s$.

(c) J'_3 *is the ideal generated by the first degree polynomials*

$$\left(\sum_{i=1}^{r} f(D'_i)\cdot[D'_i]\right) - c^1(\Lambda^2 P_j)$$

where f *is any one of the following functions*:

$$a_{2i-1} + a_{2i} \quad (\textit{where } 1 \le i \le s)$$

and by the first degree polynomials

$$\left(\sum_{i=1}^{r} f(D'_i)\cdot[D'_i]\right) - c^1(L_j)$$

where f *is any one of the following functions*:

$$a_j \quad (\textit{where } 2s+1 \le j \le n)$$

Furthermore, the isomorphism is induced by the map which assigns to each variable $[D'_i]$ *the cohomology class dual to the divisor* $K\times_{K_s} Z'_s$.

Chapter 8. Problems and conjectures

§8.1. Intersection homology. There is a vector bundle E over X and a map $m: E \to \mathrm{pgl}\,(n)$ to the Lie algebra of $\mathrm{PGL}_n\,(\mathbb{C})$ called the "moment map". The space E is the closure in $X \times \mathrm{pgl}(n)$ of X^0 which is embedded by sending a quadric Q to (Q, L) where L is the (Killing) orthocomplement to the stabilizer algebra of Q. The map m is projection on the second factor. We conjecture that the Decomposition Theorem [BBD] applied to the map m gives the same decomposition of $H^*(X; \mathbb{Q}) = H^*(E; \mathbb{Q})$ as in theorem 1.4. We calculated this for $n \leq 4$, and the effort to explain the surprising result led to the research of this paper.

§8.2. Perfect filtration of X. A rationally perfect filtration of X is a filtration by closed subsets $\phi = Y_{-1} \subset Y_0 \subset Y_1 \subset \cdots Y_k = X$ such that $H^*(X, \mathbb{Q}) \cong \bigoplus_{i=0}^{k} H^*(Y_i, Y_{i-1}; \mathbb{Q})$ (i.e. such that the spectral sequence from the right side to the left degenerates). For each stratum X_I of X, we define the height to be $\Sigma\, [(g_i + 1)/2]$ where the integers g_i are the lengths of the gaps in I (a gap is a string of consecutive integers j with $1 \leq j \leq n$ that are not in I). We conjecture that if Y_i is the union of all strata of X of height $\leq i$, then the filtration is rationally perfect, and the resulting decomposition of the cohomology of X coincides with that of Theorem 1.4. Most filtrations by closed unions of strata are not perfect. This conjecture (and our first main theorem) would be trivial if $Y_i - Y_{i-1}$ had a paving by affines, but this is not the case.

§8.3. The varieties Z_s and toric varieties. In §6.2, we prove that the variety Z_s of diagonal complete quadrics has the same rational homology as a specific toric variety $\bar{T}$. We conjecture that Z_s has the same rational homotopy type as $\bar{T}$, and that $\bar{T}$ is a specialization of Z_s as an algebraic variety.

§8.4. Other symmetric varieties. We conjecture that the rational cohomology of every complete symmetric variety has a direct sum decomposition like that of Theorem 1.4. As remarked in the introduction, there are such decompositions for the completions of the adjoint groups [DP3]; in this case there is only one summand. In general, there should be a summand for every class of associated parabolic subgroups corresponding to strata on which the maximal torux acts with fixed points. These strata are classified in [DS].

§8.5. Cells. The variety X has a decomposition into complex affine spaces, or cells, by the theory of Bialynicki-Birula. It would be interesting to compute their dual cohomology classes explicitly in our formalism.

§8.6. Schubert calculus. In [DP2] is defined a universal ring for the Schubert calculus of X^0. It is the limit of the cohomology rings of the compactifications X' of §7. This ring can be defined intrinsically on X^0 by a certain equivalence of cycles. It would be interesting to exhibit cycles in X^0 which represent basis elements in this ring, and to determine which cohomology classes can be represented by positive cycles in X^0.

§8.7. Contact formulae. One might analyze conditions of osculation (as one does for tangency in the contact formula) and study the corresponding cohomology classes in the appropriate compactifications.

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