Vol. 89, No. 3 DUKE MATHEMATICAL JOURNAL © 1997

DISCRETE SERIES CHARACTERS AND THE LEFSCHETZ FORMULA FOR HECKE OPERATORS

M. GORESKY, R. KOTTWITZ, AND R. MACPHERSON

This paper consists of three independent but related parts. In the first part (§§1–6), we give a combinatorial formula for the constants appearing in the "numerators" of characters of stable discrete series representations of real groups (see §3), as well as an analogous formula for individual discrete series representations (see §6). Moreover, we give an explicit formula (Theorem 5.1) for certain stable virtual characters on real groups; by Theorem 5.2, these include the stable discrete series characters, and thus we recover the results of §3 in a more natural way.

In the second part (§7), we use the character formula given in Theorem 5.1 to rewrite the Lefschetz formula of [GM1] (for the local contribution at a single fixed-point component to the trace of a Hecke operator on weighted cohomology) in the same spirit as that of Arthur's Lefschetz formula [A1] in terms of stable virtual characters on real groups (see Theorem 7.14.B). We then sum the contributions of the various fixed-point components and show that, in the case of middle-weighted cohomology, the resulting global Lefschetz fixed-point formula agrees with Arthur's Lefschetz formula. This gives a topological proof of Arthur's formula.

The third part of the paper (Appendices A and B) is purely combinatorial. In Appendix A, we develop the combinatorics of convex polyhedral cones on which our results on characters of real groups are based. The same combinatorics is used in Appendix B to prove a generalization of a combinatorial lemma of Langlands.

The formula for stable discrete series constants given in Theorem 3.1 is redundant, since it follows easily from Theorems 5.1 and 5.2. Nevertheless, the proof of Theorem 3.1 is instructive and should probably not be skipped by readers interested in the case of individual discrete series constants. Theorem 3.2 is not redundant, and in fact provides the link between our results on stable discrete series constants and individual discrete series constants. (We return to this point later in the introduction.) Because of the redundancy built into the paper, the reader who is mainly interested in the Lefschetz formula only needs to read §5, §7, and a little bit of Appendix A.

Let G be a connected reductive group over $\mathbb{Q}$, and let A_G denote the maximal $\mathbb{Q}$-split torus in the center of G. Let K_G be a maximal compact subgroup of

Received 27 March 1996. Revision received 13 August 1996.
Authors' research partially supported by National Science Foundation grant numbers DMS-9303550, DMS-9203380, DMS-9106522, respectively.

$G(\mathbb{R})$, and let X_G denote the homogeneous space

$$G(\mathbb{R})/(K_G \cdot A_G(\mathbb{R})^0).$$

Let K be a suitably small compact open subgroup of $G(\mathbb{A}_f)$. We denote by S_K the space

$$G(\mathbb{Q})\backslash[(G(\mathbb{A}_f)/K) \times X_G].$$

Let E be an irreducible representation of the algebraic group G on a finite-dimensional complex vector space. Then E gives rise to a local system $\mathbf{E}_K$ on S_K.

Let $P_0 = M_0N_0$ be a minimal parabolic subgroup of G, with Levi component M_0 and unipotent radical N_0. As usual, by a standard parabolic subgroup of G, we mean one that contains P_0. For any standard parabolic subgroup P, we write $P = MN$, where M is the unique Levi component of P containing M_0, and N is the unipotent radical of P. The reductive Borel-Serre compactification $\bar{S}_K$ of S_K is a stratified space whose strata are indexed by the standard parabolic subgroups of G. The stratum indexed by G is the space S_K. The stratum S_K^P indexed by the standard $P = MN$ is a finite union of spaces of the same type as S_K, but for the group M rather than G.

Let

$$j\colon S_K \hookrightarrow \bar{S}_K$$

denote the inclusion. Consider the object $\mathbf{R}j_*\mathbf{E}_K$ in the derived category of $\bar{S}_K$. The restriction to the stratum S_K^P of the ith cohomology sheaf of $\mathbf{R}j_*\mathbf{E}_K$ is the local system on S_K^P associated to the representation of M on

$$H^i(\mathrm{Lie}(N), E).$$

For any standard $P = MN$, we write $\mathfrak{A}_M$ for the real vector space

$$X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$$

and $\mathfrak{A}_M^*$ for its dual. Let $\nu \in \mathfrak{A}_{M_0}^*$, and suppose that the restriction of ν to $\mathfrak{A}_G$ coincides with the element of $X^*(A_G)$ by which A_G acts on E. For any standard parabolic subgroup $P = MN$, we write $\nu_P \in \mathfrak{A}_M^*$ for the restriction of ν to the subspace $\mathfrak{A}_M$ of $\mathfrak{A}_{M_0}$. Then ν determines a weight profile and hence a weighted cohomology complex (see [GHM]) $\bar{\mathbf{E}}_K$ on $\bar{S}_K$ (an object in the derived category of $\bar{S}_K$). The restriction to S_K^P of the ith cohomology sheaf of $\bar{\mathbf{E}}_K$ is the local system on S_K^P associated to the representation of M on

$$H^i(\mathrm{Lie}(N), E)_{\geqslant \nu_P},$$

the subspace of $H^i(\mathrm{Lie}(N), E)$ on which A_M acts by weights $\geqslant \nu_P$. (A weight $\mu \in X^*(A_M) \subset \mathfrak{A}_M^*$ is $\geqslant \nu_P$ if $\mu - \nu_P$ takes nonnegative values on the chamber in $\mathfrak{A}_M$ determined by P.)

The main result of [GM1] is an explicit version of the Lefschetz formula for the alternating sum of the traces of the self-maps on

$$H^i(\bar{S}_K, \bar{\mathbf{E}}_K)$$

induced by a Hecke correspondence. One of our goals in this paper (see Theorem 7.14.B) is to rewrite the Lefschetz formula of [GM1] in terms involving a stable virtual character Θ_ν on the group $G(\mathbb{R})$. If ν is so positive that $\bar{\mathbf{E}}_K$ coincides with the extension by zero of $\mathbf{E}_K$, then Θ_ν is just the character of the contragredient E^* of E (see 7.17). There is a similar (but more complicated) statement in the case when ν is sufficiently negative (see 7.18). In general, Θ_ν is given by

$$\Theta_\nu = \sum_P (-1)^{\dim(A_M/A_G)} i_M^G(\delta_P^{-1/2} \otimes (E_P^\nu)^*),$$

where E_P^ν is the virtual finite-dimensional representation of M

$$\sum_i (-1)^i H^i(\mathrm{Lie}(N), E)_{\geqslant \nu_P},$$

and δ_P is the usual modulus character

$$\delta_P(x) = |\det(x; \mathrm{Lie}(N))|$$

on $M(\mathbb{R})$; the sum is taken over all standard parabolic subgroups $P = MN$, and $i_M^G(\cdot)$ denotes normalized parabolic induction from $M(\mathbb{R})$ to $G(\mathbb{R})$.

Since there are simple formulas for characters that are parabolically induced from finite-dimensional representations of Levi subgroups, it is possible to determine the character Θ_ν explicitly (Theorem 5.1). In fact, Theorem 5.1 is just what is needed to rewrite the Lefschetz formula of [GM1] in terms of Θ_ν. (The numbers $L_Q^\nu(\gamma)$ that go into the definition of $L_M^\nu(\gamma)$ occur as factors in the local Lefschetz numbers.) If $G(\mathbb{R})$ has a discrete series and if ν is the "upper middle" weight profile ν_m of §5, then (see Theorem 5.2) Θ_ν agrees on all relevant maximal tori with

$$(-1)^{q(G)} \sum_{\pi \in \Pi} \Theta_\pi,$$

where Π is the L-packet of discrete series representations of $G(\mathbb{R})$ having the same infinitesimal and central characters as E^* (and in fact Θ_ν is equal to this virtual character if P_0 remains minimal over $\mathbb{R}$), and our formula essentially coincides with Arthur's formula for L^2-Lefschetz numbers of Hecke operators [A1] (see the remarks at the end of 7.19 for a detailed comparison with Arthur's formula). This provides evidence for the agreement of middle-weighted cohomology and L^2-cohomology in this degree of generality. (In the Hermitian symmetric case this agreement is known: middle-weighted cohomology agrees with intersection cohomology of the Baily-Borel compactification [GHM], and this in turn agrees with L^2-cohomology [L], [SS].)

This concludes our discussion of the global results in this paper. However, it remains to summarize the results on real groups. The two theorems just mentioned (Theorems 5.1 and 5.2) together give a simple formula for the stable discrete series character

$$\sum_{\pi \in \Pi} \Theta_\pi$$

on any maximal torus in G over $\mathbb{R}$. In particular, we obtain a simple formula (Theorem 3.1) for the constants $d(w)$ $(w \in W)$ appearing in stable discrete series character formulas (actually we prove Theorem 3.1 by a different method). Here, W is the Weyl group of a root system R such that $-1 \in W$, for which we have fixed a system of positive roots. The formula for $d(w)$ is expressed as a sum over W, each term in the sum being $1, -1$, or 0, and it bears no obvious relation to the formula of Herb [He1] for $d(w)$ in terms of 2-systems. The terms in the sum depend on the finite-dimensional representation E, although their sum does not, so that we in fact get finitely many different formulas, one for each cone in a certain decomposition of the positive Weyl chamber. In Theorem 3.2, we prove an unexpected symmetry for the function $d(w)$:

$$d(w^{-1}) = (-1)^{q(R)}\varepsilon(w)d(w),$$

where ε is the usual sign function on the Weyl group. This symmetry, together with the formula for $d(w^{-1})$ as a sum over W, gives a second formula for $d(w)$ as a sum over W. Our fixed system of positive roots determines a certain subgroup W_c of W (the "compact" Weyl group). If in the second formula for $d(w)$, we replace the sum over W by a sum over a coset of W_c in W, the resulting expression turns out to be a formula (see §6) for the constants appearing in the characters of *individual* discrete series representations. Of course, these constants were already known, implicitly by the work of Harish-Chandra and explicitly by the work of Hirai (or by combining formulas for the stable constants—Herb's or ours—with Shelstad's theory of endoscopy); what is perhaps interesting is the simplicity of our formula (again, we in fact get finitely many different formulas, all giving the same result).

More should be said about Theorem 5.2, which expresses stable discrete series characters as linear combinations of characters induced from finite-dimensional representations of Levi subgroups. J. Adams informs us that a result of this kind was known to G. Zuckerman when he wrote his 1974 Princeton thesis. (Certain examples are treated in the thesis, but a precise general statement is not given there.) We do not know if our result is the one Zuckerman had in mind, though it seems unlikely that there could be two essentially different formulas. What we do know is that an inversion procedure due to Langlands (a simple special case of his combinatorial lemma, often applied in Arthur's work on the trace formula) allows one to obtain from Theorems 5.2 and 5.3 an expression for the character of a finite-dimensional representation as a linear combination of standard characters, and it is not hard to see that this inverted formula coincides with the one due to Zuckerman [Z] (see also [V]). Langlands's inversion procedure works in both directions, so that one could also invert Zuckerman's theorem to obtain our Theorem 5.2. However, this would result in a more complicated proof. (Ours uses only elementary combinatorics and Harish-Chandra's characterization of discrete series characters.)

In this paper we use the following notation. For a finite set S, we write $|S|$ for the cardinality of S. For a subset A of a set S, we write ξ_A for the characteristic function of A. For a subgroup H of a group G, we write $N_G(H)$ (respectively, $\mathrm{Cent}_G(H)$) for the normalizer (respectively, centralizer) of H in G. (Sometimes we allow H to be a subgroup of a bigger group of which G is also a subgroup.) For a free abelian group X of finite rank, we write $X_{\mathbb{R}}$ for the real vector space $X \otimes_{\mathbb{Z}} \mathbb{R}$. Given an endomorphism A of a finite-dimensional vector space V, we write $\det(A; V)$ (respectively, $\mathrm{tr}(A; V)$) for the determinant (respectively, trace) of A whenever the name of A leaves doubt about which vector space V we have in mind.

Acknowledgements. We wish to bring to the reader's attention some related results of Franke [F], Harder [H2], Nair [N], and Stern [St]. We would like to thank G. Harder for useful discussions concerning topological trace formulas and J. Adams for his comments on the history of Theorem 5.2. The first author would like to thank the Institute for Advanced Study in Princeton for its hospitality and support, which was partially provided by National Science Foundation grant DMS-9304580.

1. The function $\psi_R(C_0, x, \lambda)$. In this section we consider a root system $(X, X^*, R, R^\vee)$. Here, X is a real vector space, X^* its dual vector space, $R \subset X^*$ a root system in X^* that spans X^*, and $R^\vee \subset X$ the coroot system in X. We write $W = W(R)$ for the Weyl group of the root system R. For any Weyl chamber C in X, we write $C^\vee$ for the corresponding Weyl chamber in X^*. In this section, we will define integers $\psi_R(C_0, x, \lambda)$; in §3 we will see that in case $-1 \in W$, these integers are (essentially) the ones appearing in the formulas for stable discrete series characters on real groups.

Let C_0 be a Weyl chamber in X. We write $\bar{C}_0$ for the closure of C_0. Let $\omega \in \bar{C}_0$ be a nonzero element in a 1-dimensional face of $\bar{C}_0$ (thus ω is, up to a positive scalar, a fundamental coweight for C_0). Put

$$R_\omega := \{\alpha \in R \mid \alpha(\omega) = 0\}.$$

For any chamber C in X relative to R, let $\tilde{C}$ denote the unique chamber in X relative to R_ω that contains C. For two chambers C_1, C_2 in X relative to R, we write $l_R(C_1, C_2)$ or just $l(C_1, C_2)$ for the number of root hyperplanes in X separating C_1 and C_2; thus $l(C_1, C_2)$ is the length with respect to C_1 of the unique element $w \in W$ such that $wC_1 = C_2$. We write R^+ for the set of roots in R that are positive on C_0. Finally we write R_ω^+ for $R_\omega \cap R^+$, the set of roots in R_ω that are positive on $\tilde{C}_0$.

LEMMA 1.1. (a) *The map $C \mapsto \tilde{C}$ yields a bijection from the set of chambers in X relative to R whose closures contain ω to the set of chambers in X relative to R_ω. If C_1, C_2 are chambers in X relative to R that contain ω, then*

$$l(C_1, C_2) = l_{R_\omega}(\tilde{C}_1, \tilde{C}_2).$$

(b) *The difference $|R^+| - |R_\omega^+|$ is odd if $\mathbb{R}\omega$ contains a coroot and is even otherwise.*

(c) *There exists a unique chamber C_0' in X such that $-\omega \in \bar{C}_0'$ and $\tilde{C}_0 = \tilde{C}_0'$. Moreover,*

$$\{\alpha \in R^+ \mid \ker(\alpha) \text{ separates } C_0, C_0'\} = R^+ \backslash R_\omega^+.$$

In particular

$$l(C_0, C_0') = |R^+| - |R_\omega^+|.$$

(d) *Suppose that there exists a positive scalar c such that $c\omega$ is a coroot $\alpha^\vee \in R^\vee$. Note that the corresponding root α belongs to R^+. Suppose that C'' is a chamber in X satisfying the following three conditions:*
(1) *α takes nonnegative values on C'';*
(2) *$\ker(\alpha)$ is a wall of C'';*
(3) *$\tilde{C}'' = \tilde{C}_0$.*
Then

$$l(C_0, C'') = (|R^+| - |R_\omega^+| - 1)/2.$$

(Note that by (b), the quantity on the right-hand side is an integer).

Proof. The assertion (a) is standard. We prove (b) by considering the action of the longest element w of $W(R_\omega)$ (longest with respect to R_ω^+) on the set $R/\pm$ obtained from R by taking the quotient by the action of the group $\{\pm 1\}$. Since $w^2 = 1$, it follows that $|R^+| = |R/\pm|$ has the same parity as the number of fixed points of w on $R/\pm$. Let $\beta \in R$. Since the $+1$ eigenspace for w is $\mathbb{R}\omega$, and the -1 eigenspace for w is the span of $R_\omega^\vee$, we see that $w\beta = \pm\beta$ if and only if $\beta \in R_\omega$ or $\beta^\vee \in \mathbb{R}\omega$. (Of course, these alternatives are mutually exclusive.) Therefore the number of fixed points of w on $R/\pm$ is $|R_\omega^+| + 1$ if $\mathbb{R}\omega$ contains a coroot, and is $|R_\omega^+|$ otherwise.

Now we consider (c). The existence and uniqueness of C_0' follow from the first statement in (a), applied to both ω and $-\omega$. Next, we prove the second statement in (c). Let $\alpha \in R^+$. Suppose first that α (strictly speaking, $\ker(\alpha)$) separates C_0, C_0'. Since $\tilde{C}_0 = \tilde{C}_0'$, it follows that $\alpha \notin R_\omega^+$. Conversely, suppose that $\alpha \notin R_\omega^+$. Then $\alpha(\omega) \neq 0$, so that α strictly separates $\omega, -\omega$. Since $\omega \in \bar{C}_0$ and $-\omega \in \bar{C}_0'$, it follows that α separates C_0, C_0'.

Finally we consider (d). We are interested in roots $\beta \in R$ that separate C_0, C''. Using (1) and (3), we see that any such β belongs to $R_0 := R \backslash (R_\omega \cup \{\pm\alpha\})$. To prove (d), we must show that exactly half of the elements of R_0 separate C_0, C''. Let $s \in W$ be the reflection in the root α. Then s preserves both R_ω and $\{\pm\alpha\}$, and hence preserves R_0 as well. Let $\beta \in R_0$. We will be done if we can show that β separates C_0, C'' if and only if $s\beta$ does not separate C_0, C''. Let C_0' be as in (c) and note that $C_0' = sC_0$. Of course, C_0, C'' are separated by β if and only if $C_0' = sC_0, sC''$ are separated by $s\beta$. By (2), sC'' and C'' are separated only by $\pm\alpha$; therefore, sC'', C'' are not separated by $s\beta$. Moreover, $s\beta$ does separate C_0, C_0' (use (c)). Therefore, C_0, C'' are separated by β if and only if C_0, C'' are not separated by $s\beta$, as we wished to show. The proof of the lemma is now complete.

Now we prepare to define the main object of study in this section. For any two chambers C_1 and C_2 in X, we write $\varepsilon(C_1, C_2)$ for $(-1)^{l(C_1, C_2)}$. For any chamber C in X, we define a function ψ_C on $X \times X^*$ as follows. Let $\alpha_1, \dots, \alpha_n$ ($n = \dim(X)$) be the simple roots in R relative to C, and let $\omega_1, \dots, \omega_n \in X$ be the basis for X dual to the basis $\alpha_1, \dots, \alpha_n$ for X^*. Let $x \in X$, $\lambda \in X^*$ and write

$$x = a_1\omega_1 + \cdots + a_n\omega_n$$

$$\lambda = b_1\alpha_1 + \cdots + b_n\alpha_n.$$

Let $I = \{1, \dots, n\}$ and define two subsets I_x, I_λ of I by

$$I_x = \{i \in I \mid a_i \geqslant 0\}$$

$$I_\lambda = \{i \in I \mid b_i \geqslant 0\}.$$

Then define $\psi_C(x, \lambda)$ by

$$\psi_C(x,\lambda) = \begin{cases} (-1)^{|I_\lambda|} & \text{if } I_\lambda = I \backslash I_x, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this function ψ_C coincides with the function denoted by $\psi_{\bar{C}}$ in Appendix A (see Lemma A.1).

Now let C_0 be a chamber in X. Define a function $\psi(C_0, \cdot, \cdot)$ on $X \times X^*$ by

$$\psi(C_0, x, \lambda) = \sum_C \varepsilon(C_0, C)\psi_C(x, \lambda),$$

where C runs through the set of chambers in X. When it is necessary to stress the root system R, we will write $\psi_R(C_0, x, \lambda)$ instead. We have the following obvious property:

$$\psi(wC_0, x, \lambda) = \varepsilon(w)\psi(C_0, x, \lambda) \tag{1.1}$$

$$= \psi(C_0, wx, w\lambda)$$

for any $w \in W$, where $\varepsilon(w)$ denotes the sign of w.

As usual, we say that an element $x \in X$ is *regular* if it lies on no root hyperplane. We say that an element $\lambda \in X^*$ is *R-regular* if it lies on no hyperplane of the form $\{\lambda \in X^* \mid \lambda(\omega) = 0\}$, where ω is a nonzero element of a 1-dimensional face of some closed Weyl chamber in X. Of course, this notion of regularity in X^* is in general different from the usual one, and we refer to the connected components in the set of R-regular elements in X^* as *R-chambers* in X^* to avoid confusion with the usual Weyl chambers in X^*.

Suppose that ω is a nonzero element in a 1-dimensional face of $\bar{C}_0$. We adopt the notation of Lemma 1.1 and the discussion preceding it (e.g., R_ω, R^+, R_ω^+, $\tilde{C}$, C_0'). Let Z denote the hyperplane $\{\lambda \in X^* \mid \lambda(\omega) = 0\}$ in X^*. We have the root system $(X/\mathbb{R}\omega, Z, R_\omega, R_\omega^\vee)$. Note that $\tilde{C}_0 = C_0 + \mathbb{R}\omega$ has the same image as C_0 in $X/\mathbb{R}\omega$; we denote this chamber in $X/\mathbb{R}\omega$ by C_0^ω.

LEMMA 1.2. *Fix $x \in X$. The function $\psi(C_0, x, \cdot)$ on X^* is constant on R-chambers. Suppose that λ, λ' are R-regular elements of X^* lying in adjacent R-chambers separated only by the hyperplane Z, and suppose further that $\lambda(\omega) > 0$, $\lambda'(\omega) < 0$. Then*

$$\psi(C_0, x, \lambda) - \psi(C_0, x, \lambda') = \begin{cases} -2\psi_{R_\omega}(C_0^\omega, \tilde{x}, \tilde{\lambda}) & \text{if } \mathbb{R}\omega \text{ contains a coroot}, \\ 0 & \text{otherwise}. \end{cases}$$

Here $\tilde{x}$ denotes the image of x in $X/\mathbb{R}\omega$ and $\tilde{\lambda}$ denotes the unique point of Z lying on the line segment joining λ and λ'. Moreover $\tilde{\lambda}$ is R_ω-regular, and if x is regular, then $\tilde{x}$ is regular relative to R_ω.

Proof. It is clear that $\psi(C_0, x, \cdot)$ is constant on R-chambers. The statement regarding the regularity of $\tilde{x}$ and $\tilde{\lambda}$ is easy and will be left to the reader. By Corollary A.3,

$$\psi(C_0, x, \lambda) - \psi(C_0, x, \lambda')$$

is equal to the sum over all chambers C such that $\bar{C}$ contains ω or $-\omega$ of terms

$$\pm\varepsilon(C_0, C)\psi_{\tilde{C}}(\tilde{x}, \tilde{\lambda}),$$

where the sign is $-$ if $\bar{C}$ contains ω and $+$ if $\bar{C}$ contains $-\omega$. We have abused notation slightly by writing $\tilde{C}$ when we mean its image in $X/\mathbb{R}\omega$; since $\bar{C}$ contains

ω or $-\omega$, this image coincides with the image of C in $X/\mathbb{R}\omega$. By Lemma 1.1(c), for each chamber C such that $\bar{C}$ contains ω, there exists a unique chamber C' such that $\bar{C}'$ contains $-\omega$ and $\tilde{C} = \tilde{C}'$. Combining the terms for C, C', we get

$$-\varepsilon(C_0, C)(1 - \varepsilon(C, C')\psi_{\tilde{C}}(\tilde{x}, \tilde{\lambda}).$$

From parts (b) and (c) of Lemma 1.1, we see that $\varepsilon(C, C')$ is -1 if $\mathbb{R}\omega$ contains a coroot, and is 1 otherwise. In the latter case, each of the combined terms is 0 and so is their sum, as desired. In the former case, the sum of the combined terms is

$$-2\sum_{C} \varepsilon(C_0, C)\psi_{\tilde{C}}(\tilde{x}, \tilde{\lambda}),$$

where C ranges through the set of chambers in X containing ω. It follows from Lemma 1.1(a) that this expression coincides with $-2\psi_{R_\omega}(C_0^\omega, \tilde{x}, \tilde{\lambda})$.

COROLLARY 1.3. *Suppose that $-1_X \notin W$. Then $\psi(C_0, x, \lambda) = 0$ for all regular $x \in X$ and all R-regular $\lambda \in X^*$.*

Proof. We prove this by induction on $\dim(X)$. If $\dim(X) = 0$, the statement is trivially true. Now assume that $\dim(X) > 0$. Fix a regular element $x \in X$. There exists R-regular $\lambda_0 \in X^*$ such that $\lambda_0(x) > 0$. By Proposition A.5, $\psi(C_0, x, \lambda_0) = 0$. Therefore, to prove the corollary it would be enough to show that

$$\psi(C_0, x, \lambda) - \psi(C_0, x, \lambda')$$

vanishes whenever λ, λ' lie in adjacent R-chambers separated by the hyperplane

$$Z = \{\lambda \in X^* \mid \lambda(\omega) = 0\}$$

determined by a nonzero element ω of some 1-dimensional face of the closure of some chamber in X; by (1.1) it is harmless to assume that $\omega \in \bar{C}_0$. By Lemma 1.2, this difference does vanish unless $\mathbb{R}\omega$ contains a coroot. So assume that $\mathbb{R}\omega$ contains a coroot $\alpha^\vee$. Put $\tilde{X} = X/\mathbb{R}\omega$. Then $-1_{\tilde{X}} \notin W(R_\omega)$ (since -1_X is the product of $-1_{\tilde{X}}$ and reflection in the root α), and therefore the difference again vanishes (by Lemma 1.2 and our induction hypothesis).

We need to introduce more notation. Let P (respectively, Q) denote the lattice of coweights in X (respectively, the lattice in X generated by the coroots). For any chamber C in X, we denote by $\delta_C \in P$ the half-sum of the coroots that are positive for C. Put

$$\hat{A}_{\mathrm{sc}} = \mathrm{Hom}(P, \mathbb{C}^\times)$$

$$\hat{A}_{\mathrm{ad}} = \mathrm{Hom}(Q, \mathbb{C}^\times).$$

The inclusion $Q \subset P$ induces a surjection

$$\hat{A}_{\mathrm{sc}} \to \hat{A}_{\mathrm{ad}}$$

of complex tori, whose kernel we denote by $Z^\vee$, so that we get an exact sequence

$$1 \to Z^\vee \to \hat{A}_{\mathrm{sc}} \to \hat{A}_{\mathrm{ad}} \to 1.$$

There are natural $\mathbb{C}^\times$-valued pairings $\langle \cdot, \cdot \rangle$ between P and $\hat{A}_{\mathrm{sc}}$ and between Q and $\hat{A}_{\mathrm{ad}}$. Let $s \in \hat{A}_{\mathrm{sc}}$ and suppose that $s^2 \in Z^\vee$. Define a root system R_s by

$$R_s^\vee = \{\alpha^\vee \in R^\vee \mid \langle \alpha^\vee, s \rangle = 1\}$$

$$R_s = \{\alpha \in R \mid \alpha^\vee \in R_s^\vee\}.$$

When we defined $\psi_R(C_0, \cdot, \cdot)$, we insisted that R generate X^*. Of course, this was just a matter of convenience. In the general case, the intersection of all the root hyperplanes in X is a linear subspace X_0 in X. Defining $\psi(C_0, \cdot, \cdot)$ as before, we find from (A.2) that $\psi(C_0, x, \lambda)$ is 0 unless λ vanishes on X_0, in which case

$$\psi(C_0, x, \lambda) = (-1)^{\dim(X_0)} \psi(\tilde{C}_0, \tilde{x}, \lambda),$$

where $\tilde{C}_0$ (respectively, $\tilde{x}$) denotes the image of C_0 (respectively, x) in X/X_0; note that on the right-hand side λ is regarded as an element of $(X/X_0)^*$. These remarks allow us to consider the function $\psi_{R_s}(\tilde{C}_0, x, \lambda)$ obtained from $(X, X^*, R_s, R_s^\vee)$, where $\tilde{C}_0$ denotes the unique chamber for R_s in X containing C_0.

LEMMA 1.4. *For all regular $x \in X$ and all $\lambda \in X^*$, there is an equality*

$$\sum_C \varepsilon(C_0, C)\langle \delta_C - \delta_{C_0}, s \rangle \psi_C(x, \lambda) = \psi_{R_s}(\tilde{C}_0, x, \lambda),$$

in which the sum runs over all chambers C in X. In particular, if $s^2 \neq 1$, then the left-hand side of this equality vanishes for regular x and R-regular λ.

Proof. Since $s^2 \in Z^\vee$, the image of s^2 in $\hat{A}_{\mathrm{ad}}$ is 1, and therefore $\langle \alpha^\vee, s \rangle = \pm 1$ for every coroot $\alpha^\vee$. By the definition of R_s, we have $\langle \alpha^\vee, s \rangle = 1$ if $\alpha \in R_s$, and $\langle \alpha^\vee, s \rangle = -1$ if $\alpha \notin R_s$. Since $\delta_C - \delta_{C_0}$ is the sum of the coroots in $R^\vee$ that are positive on C and negative on C_0, we see that

$$\varepsilon(C_0, C)\langle \delta_C - \delta_{C_0}, s \rangle = \varepsilon_{R_s}(\tilde{C}_0, \tilde{C}),$$

where $\tilde{C}$ denotes the unique chamber of (X, R_s) containing C. Therefore, the left-hand side of the equality we are trying to prove is equal to the sum over chambers

D for (X, R_s) of $\varepsilon(\tilde{C}_0, D)$ times

$$\sum_C \psi_C(x, \lambda),$$

where C runs through the chambers for (X, R) contained in D. By Proposition A.4, the difference between

$$\sum_C \psi_C(x, \lambda)$$

and

$$\psi_D(x, \lambda)$$

is a sum of terms of the form $\pm\psi_F(x, \lambda)$, where F is a proper face of the closure of some chamber C of (X, R) contained in D. Here ψ_F is as in Appendix A. But $\psi_F(x, \lambda)$ vanishes unless $x \in \mathrm{span}(F)$ (see (A.2)). Therefore, for regular x, the left-hand side of the equality we are trying to prove is

$$\sum_D \varepsilon(\tilde{C}_0, D)\psi_D(x, \lambda),$$

which, by definition, is $\psi_{R_s}(\tilde{C}_0, x, \lambda)$.

It remains to prove the second statement of the lemma. If the rank of the root system R_s is smaller than that of R, then $\psi_{R_s}(\tilde{C}_0, x, \lambda)$ is 0 unless λ belongs to the proper linear subspace $\mathrm{span}(R_s)$ of X^*. But the left-hand side of the equality of the lemma is constant on R-chambers in X^*; therefore, it vanishes for regular x and R-regular λ. If $s^2 \neq 1$ and R_s has the same rank as R, then $-1_X \notin W(R_s)$ (since -1_X sends s to s^{-1} while all elements of $W(R_s)$ fix s), and therefore by Corollary 1.3, $\psi_{R_s}(\tilde{C}_0, x, \lambda)$ vanishes for all regular $x \in X$ (regular for R_s) and all R_s-regular $\lambda \in X^*$. Any $x \in X$ that is regular for R is regular for R_s, and again using that the left-hand side of the equality of the lemma is constant on R-chambers in X^*, we see that it vanishes for regular x and R-regular λ. This completes the proof of the lemma.

There is another result of this kind. With P, Q as before, now put

$$A_{\mathrm{sc}} = Q \otimes \mathbb{C}^\times$$

$$A_{\mathrm{ad}} = P \otimes \mathbb{C}^\times.$$

The inclusion $Q \subset P$ induces a surjection

$$A_{\mathrm{sc}} \to A_{\mathrm{ad}}$$

of complex tori, whose kernel we denote by Z, so that we get an exact sequence

$$1 \to Z \to A_{sc} \to A_{ad} \to 1.$$

There are natural $\mathbb{C}^\times$-valued pairings $\langle\cdot,\cdot\rangle$ between Q^* and A_{sc} and between P^* and A_{ad} (P^*, Q^* are the free abelian groups dual to P, Q, respectively). Note that Q^* is the lattice of weights in X^* and that P^* is the lattice in X^* generated by the roots. For any chamber C in X we write $\rho_C \in Q^*$ for the half-sum of the roots that are positive for C.

Let $a \in A_{sc}$ and suppose that $a^2 \in Z$. Define a root system R_a by

$$R_a = \{\alpha \in R \mid \langle \alpha, a\rangle = 1\}.$$

Let $\tilde{C}_0$ denote the unique chamber for R_a in X containing C_0.

LEMMA 1.5. *For all regular $x \in X$ and all $\lambda \in X^*$, there is an equality*

$$\sum_C \varepsilon(C_0, C)\langle \rho_C - \rho_{C_0}, a\rangle \psi_C(x, \lambda) = \psi_{R_a}(\tilde{C}_0, x, \lambda),$$

in which the sum runs over all chambers C in X. In particular, if $a^2 \neq 1$, then the left-hand side of this equality vanishes for regular x and R-regular λ.

The proof is essentially the same as that of Lemma 1.4.

2. The function $\psi_R(C_0, x, \lambda)$ in the case $-1 \in W$. We continue with $X, X^*, R, R^\vee, W$ as in §1. We still assume (for convenience) that R generates X^*, and we now add the assumption that $-1_X \in W$. Let α be a root and define a root system R_α by

$$R_\alpha^\vee = \{\beta^\vee \in R^\vee \mid \langle \alpha, \beta^\vee\rangle = 0\}$$

$$R_\alpha = \{\beta \in R \mid \beta^\vee \in R_\alpha^\vee\}.$$

Let Y denote the hyperplane $\{x \in X \mid \alpha(x) = 0\}$; then $R_\alpha^\vee \subset Y$. Let s_α be the reflection in the root α. Since -1_X belongs to W, so does $-s_\alpha$. But $-s_\alpha$ fixes α; hence it belongs to $W(R_\alpha)$. Since $-s_\alpha$ acts by -1 on Y, we conclude that $-1_Y \in W(R_\alpha)$. Therefore, $(Y, Y^*, R_\alpha, R_\alpha^\vee)$ satisfies the same conditions as $(X, X^*, R, R^\vee)$: R_α generates Y^* and $-1_Y \in W(R_\alpha)$. Note that $\alpha^\vee$ lies in the kernel of every root for R_α. Therefore $\alpha^\vee$ is a nonzero element in some 1-dimensional face of some chamber in X, and $\alpha^\vee$ can serve as the element ω considered in §1. Note that $R_\alpha = R_\omega$ with R_ω as in §1.

There are two notions of chambers in Y. Of course, we have the usual Weyl chambers D in Y coming from the root system R_α; these are determined by the hyperplanes $\beta = 0$ ($\beta \in R_\alpha$). There is a larger set of hyperplanes in Y, namely, those of the form $\beta = 0$ ($\beta \in R \backslash \{\pm\alpha\}$), and we will refer to the connected components E of the complement of this larger set of hyperplanes as *chambers in* Y *relative to* R.

Fix a chamber C_0 in X having Y as a wall. As in §1, we write $\tilde{C}$ for the unique chamber for X relative to $R_\alpha = R_\omega$ that contains C. It is easy to see that the map $C \mapsto \tilde{C} \cap Y$ is a bijection from the set of chambers C in X having Y as a wall and lying on the same side of Y as C_0 to the set of closed chambers in Y relative to R; note that the closure of $\tilde{C} \cap Y$ is equal to $\bar{C} \cap Y$. Using the chamber C_0, we obtain a function $\psi(C_0, \cdot, \cdot)$ on $X \times X^*$ as in §1.

Lemma 2.1. *Fix* $\lambda \in X^*$. *The function* $\psi(C_0, \cdot, \lambda)$ *on* X *is constant on the chambers in* X. *Suppose that* x, x' *are regular elements lying in adjacent chambers separated only by the hyperplane* Y, *and assume that* x, C_0 *lie on the same side of* Y *(so that* x', C_0 *lie on opposite sides of* Y*). Then*

$$\psi(C_0, x, \lambda) - \psi(C_0, x', \lambda) = 2\psi_{R_\alpha}(D_0, y, \lambda_Y),$$

where $\lambda_Y \in Y^*$ *denotes the restriction of* λ *to* Y, $y \in Y$ *is the unique point of* Y *lying on the line segment joining* x *and* x', *and* D_0 *is the chamber* $\tilde{C}_0 \cap Y$ *for* (Y, R_α). *Moreover,* y *is regular in* Y, *and if* λ *is* R*-regular, then* λ_Y *is* R_α*-regular in* Y^*.

Proof. It is clear that $\psi(C_0, \cdot, \lambda)$ is constant on chambers in X. By Lemma A.2

$$\psi(C_0, x, \lambda) - \psi(C_0, x', \lambda)$$

is equal to

$$2 \sum_C \varepsilon(C_0, C) \psi_{\tilde{C} \cap Y}(y, \lambda_Y),$$

where C runs over the chambers in X having Y as a wall and lying on the same side of Y as C_0 (and x). The factor 2 arises since we have combined the contributions of C and the unique chamber adjacent to C across the wall Y. We denote by $C^\#$ the unique chamber for (X, R) that is contained in $\tilde{C}$ and whose closure contains ω. Replacing α by $-\alpha$ if necessary, we may assume without loss of generality that α is nonnegative on C_0 and x, and hence on any C appearing in the sum above. Applying Lemma 1.1(d) to both C_0 and any such C (both satisfy conditions (1) and (2)), we see that

$$\varepsilon(C_0, C) = \varepsilon(C_0^\#, C^\#),$$

and then from Lemma 1.1(a), we see further that

$$\begin{aligned}\varepsilon(C_0, C) &= \varepsilon(\tilde{C}_0, \tilde{C}) \\ &= \varepsilon(D_0, D),\end{aligned}$$

where $D_0 = \tilde{C}_0 \cap Y$ and $D = \tilde{C} \cap Y$.

We have now shown that the left-hand side of the equality we are trying to prove is equal to

$$2\sum_D \varepsilon(D_0, D) \sum_E \psi_{\bar{E}}(y, \lambda_Y),$$

where D runs over the chambers of (Y, R_α) and E runs over the chambers in Y relative to R such that $E \subset D$. (The function $\psi_{\bar{E}}$ is the one attached in Appendix A to the closed convex polyhedral cone $\bar{E}$ in Y.) Each such D is the disjoint union of the corresponding E's together with the relative interiors of some closed convex polyhedral cones F of lower dimension, each of which is contained in some root hyperplane other than Y. It is clear that y lies on no root hyperplane of R other than Y; therefore $\psi_F(y, \lambda_Y)$ vanishes for all such F (see (A.2)). By Proposition A.4, the inner sum is equal to $\psi_D(y, \lambda_Y)$, and therefore the whole expression is equal to $2\psi_{R_\alpha}(D_0, y, \lambda_Y)$. This proves the lemma, except for the last statement, which we leave to the reader.

Again, let C_0 be a chamber in X and let R^+ be the set of roots in R that are positive for C_0. Let ε: $W \to \{\pm 1\}$ be the sign homomorphism. The longest element of W is -1_X. On the one hand,

$$\varepsilon(-1_X) = (-1)^{|R^+|}.$$

On the other hand,

$$\varepsilon(-1_X) = \det(-1_X) = (-1)^{\dim(X)}.$$

Therefore, $|R^+|$ and $\dim(X)$ have the same parity, and we can define an integer $q(R)$ by

$$q(R) := (|R^+| + \dim(X))/2.$$

To understand the significance of the integer $q(R)$, one should note that it is half the dimension of the symmetric space of the split semisimple real group G with root system R (a number that is traditionally denoted $q(G)$).

Let $C_0^\vee$ be the Weyl chamber in X^* corresponding to C_0. From $C_0^\vee$ and the coroot system $R^\vee$, we get a function $\psi_{R^\vee}(C_0^\vee, \cdot, \cdot)$ on $X^* \times X$. (The roles of X and

X^* are now reversed.) We have the notions of regularity (for $x \in X$) and R-regularity (for $\lambda \in X^*$) from before. Applying these definitions to $R^\vee$ rather than R, we have the notions of regularity for $\lambda \in X^*$ and $R^\vee$-regularity for $x \in X$; note that the set of regular elements in X^* is the union of the Weyl chambers in X^*. Since $-1_X \in W$, every coroot in X is a nonzero element in a 1-dimensional face of some chamber in X. (We saw this during the discussion at the beginning of this section.) Therefore, if $\lambda \in X^*$ is R-regular, it is automatically regular, and, similarly, if $x \in X$ is $R^\vee$-regular, it is automatically regular.

Lemma 2.2 *For any $R^\vee$-regular $x \in X$ and any R-regular $\lambda \in X^*$, there is an equality*

$$\psi_R(C_0, x, \lambda) = (-1)^{q(R)} \psi_{R^\vee}(C_0^\vee, \lambda, x).$$

Proof. We prove this by induction on $\dim(X)$. It is certainly true when $\dim(X) = 0$ (the empty root system). Now assume that $\dim(X) > 0$. Fix an $R^\vee$-regular element $x \in X$. There exists R-regular $\lambda_0 \in X^*$ such that $\lambda_0(x) > 0$, and the equality in the lemma holds for x, λ_0 since both sides of the equality vanish by Proposition A.5. Therefore, it is enough to show that

$$\psi_R(C_0, x, \lambda) - \psi_R(C_0, x, \lambda') = (-1)^{q(R)} (\psi_{R^\vee}(C_0^\vee, \lambda, x) - \psi_{R^\vee}(C_0^\vee, \lambda', x)) \tag{2.1}$$

whenever λ, λ' are R-regular elements of X^* lying in adjacent R-chambers. Let Z denote the unique hyperplane separating these two adjacent R-chambers. Thus, Z is of the form

$$Z = \{\lambda \in X^* \mid \lambda(\omega) = 0\}$$

for some nonzero ω lying in a 1-dimensional face of the closure of some Weyl chamber in X; we may assume without loss of generality that this Weyl chamber coincides with C_0. (By property (1.1) changing C_0 changes both sides of (2.1) by the same sign.) By switching λ, λ' if necessary, we may also assume that $\lambda(\omega) > 0$ and $\lambda'(\omega) < 0$.

First consider the case in which $\mathbb{R}\omega$ does not contain a coroot. Then the left-hand side of (2.1) vanishes by Lemma 1.2, while the right-hand side vanishes because $\psi_{R^\vee}(C_0^\vee, \cdot, x)$ is constant on Weyl chambers in X^*, not just on R-chambers. We are left with the case in which $\mathbb{R}\omega$ contains a coroot $\alpha^\vee$; replacing ω by a positive scalar multiple, we may as well assume that $\omega = \alpha^\vee$. Since $\alpha^\vee \in \bar{C}_0$, the root α is positive for C_0. By Lemma 1.2 the left-hand side of (2.1) is equal to

$$-2\psi_{R_\omega}(C_0^\omega, \tilde{x}, \tilde{\lambda})$$

(with notation as in that lemma).

Of course, we are going to use Lemma 2.1 to evaluate the right-hand side of (2.1). However, the hyperplane

$$Z = \{\lambda \in X^* \mid \lambda(\omega) = 0\} = \{\lambda \in X^* \mid \lambda(\alpha^\vee) = 0\}$$

need not be a wall of $C_0^\vee$, so that we need to introduce another chamber $(C^\vee)''$ in X^*, which is better suited to our purposes. We take $(C^\vee)''$ to be any chamber in X^* satisfying the following three conditions:

(1) $\alpha^\vee$ takes nonnegative values on $(C^\vee)''$;

(2) Z is a wall of $(C^\vee)''$;

(3) $((C^\vee)'')^\sim = (C_0^\vee)^\sim$.

The notation $(\cdot)^\sim$ used in (3) has the following meaning: for any chamber $C^\vee$ in X^*, we write $(C^\vee)^\sim$ for the unique chamber in X^* relative to $R_\omega^\vee = (R^\vee)_{\alpha^\vee}$ that contains $C^\vee$. It is easy to see that $(C^\vee)''$ exists: pick any chamber $E^\vee$ in Z relative to $R^\vee$ contained in $(C_0^\vee)^\sim \cap Z$ and take for $(C^\vee)''$ the unique chamber in X^* satisfying (1) and (2) and having the property that the closure of $E^\vee$ is equal to the intersection of Z with the closure of $(C^\vee)''$.

By (1.1) and Lemma 2.1, the right-hand side of (2.1) is equal to

$$2(-1)^{q(R)}\varepsilon(C_0^\vee, (C^\vee)'')\psi_{R_\omega^\vee}(D_0^\vee, \tilde{\lambda}, \tilde{x}),$$

with $\tilde{\lambda}, \tilde{x}$ as before and $D_0^\vee$ the chamber $((C^\vee)'')^\sim \cap Z$ in Z for the root system $R_\omega^\vee$. It follows from Lemma 1.1(d) that

$$\varepsilon(C_0^\vee, (C^\vee)'') = (-1)^{(|R^+|-|R_\omega^+|-1)/2};$$

since

$$q(R) - q(R_\omega) = (|R^+| - |R_\omega^+| + 1)/2,$$

we conclude that

$$(-1)^{q(R)}\varepsilon(C_0^\vee, (C^\vee)'') = -(-1)^{q(R_\omega)}.$$

By our induction hypothesis,

$$(-1)^{q(R_\omega)}\psi_{R_\omega^\vee}(D_0^\vee, \tilde{\lambda}, \tilde{x}) = \psi_{R_\omega}(C_0^\omega, \tilde{x}, \tilde{\lambda}).$$

Of course, we used that $D_0^\vee = (C_0^\omega)^\vee$ and that $\tilde{x}, \tilde{\lambda}$ are suitably regular. Therefore, the right-hand side of (2.1) equals

$$-2\psi_{R_\omega}(C_0^\omega, \tilde{x}, \tilde{\lambda}),$$

which coincides with the expression we found for the left-hand side. This concludes the proof of the lemma.

3. Stable discrete series constants $\bar{c}_R$. In the theory of stable discrete series characters on real groups, which we will review briefly in §4, there appear integer-valued functions (see [K], [He1], [He2])

$$\bar{c}_R : X_{\text{reg}} \times X^*_{\text{reg}} \to \mathbb{Z}$$

for every root system $(X, X^*, R, R^\vee)$ satisfying the two conditions of §2. (R generates X^* and $-1_X \in W$, where $W = W(R)$ denotes the Weyl group of R.) Here, X_{reg} and X^*_{reg} denote the sets of regular elements in X and X^*, respectively (regular in the usual sense, so that X_{reg}, X^*_{reg} can also be described as the unions of the Weyl chambers in X, X^*, respectively). The functions $\bar{c}_R$ satisfy the following five properties:

(1) $\bar{c}_R(0,0) = 1$ if R is empty;

(2) $\bar{c}_R(x,\lambda)$ depends only on the chamber in X in which x lies and the chamber in X^* in which λ lies;

(3) $\bar{c}_R(x,\lambda) = 0$ unless $\lambda(x) \leqslant 0$;

(4) if $x, x' \in X$ lie in adjacent chambers, separated only by the root hyperplane Y, then

$$\bar{c}_R(x,\lambda) + \bar{c}_R(x',\lambda) = 2\bar{c}_{R_Y}(y, \lambda_Y),$$

where $R_Y \subset Y^*$ is the root system whose set of coroots is $R^\vee \cap Y$, λ_Y is the restriction of λ to Y, and y is the unique point of Y lying on the line segment joining x and x';

(5) $\bar{c}_R(wx, w\lambda) = \bar{c}_R(x,\lambda)$ for all $w \in W(R)$.

It is well known that the collection of functions $\bar{c}_R$ is characterized uniquely by properties (1), (3), and (4). (This follows easily from an induction on $\dim(X)$ as in the proofs of Lemma 2.2 and Corollary 1.3.) Of course, these properties are reminiscent of ones enjoyed by the functions $\psi_R(C_0, x, \lambda)$ studied in §2. For $x \in X_{\text{reg}}$ denote by C_x the unique chamber in X containing x. We now define an integer-valued function m_R on $X_{\text{reg}} \times X^*_{\text{reg}}$ by

$$m_R(x,\lambda) = \psi_R(C_x, x, \lambda).$$

THEOREM 3.1. *The functions $\bar{c}_R$ and m_R are equal.*

Proof. We need only show that m_R satisfies properties (1), (3), and (4) above. Property (1) is trivial. Property (3) follows from Proposition A.5. Property (4) follows from (1.1) and Lemma 2.1.

There is a more efficient way to encode the information in the function $\bar{c}_R$. Fix a Weyl chamber C_0 in X, and let $C_0^\vee$ be the corresponding Weyl chamber in X^*.

Then define an integer-valued function d on $W = W(R)$ by putting

$$d(w) := \bar{c}_R(x_0, w\lambda_0) \qquad (w \in W),$$

where x_0 (respectively, λ_0) is any point in C_0 (respectively, $C_0^\vee$). Of course, d depends (in a simple way) on the choice of C_0. Applying this construction to the root system $R^\vee$ (and the chamber $C_0^\vee$), we get a function $d^\vee$ on $W = W(R^\vee) = W(R)$.

THEOREM 3.2. *For all $w \in W$ there are equalities*
(1) $d^\vee(w) = d(w)$
(2) $d(w^{-1}) = (-1)^{q(R)}\varepsilon(w),\ d(w)$,
where $e(w)$ denotes the sign of w.

Proof. Pick a W-equivariant isomorphism $j: X \to X^*$. Let $x_0 \in C_0$; then $\lambda_0 := j(x_0) \in C_0^\vee$. Since ψ_R depends only on the root hyperplanes and not on the roots themselves, it is clear that

$$\psi_{R^\vee}(C_0^\vee, \lambda_0, wx_0) = \psi_R(C_0, x_0, w\lambda_0), \tag{3.1}$$

and by Theorem 3.1, this just says that

$$d^\vee(w) = d(w).$$

Equality (2) follows from Theorem 3.1 and Lemma 2.2 (use (1.1) and (3.1) as well).

4. Background material on stable characters. In this section we review some of the theory of characters of irreducible representations of real groups. Let G be a connected reductive group over $\mathbb{R}$. Let E be an irreducible finite-dimensional complex representation of the algebraic group G. We are interested in irreducible representations π of $G(\mathbb{R})$ (irreducible Harish-Chandra modules) having the same infinitesimal character as E. Harish-Chandra associated to any such π its character Θ_π, a real-analytic function on $G_{\text{reg}}(\mathbb{R})$, the set of regular semisimple elements in $G(\mathbb{R})$.

Let T be a maximal torus in G. Let $\mathscr{B}(T)$ denote the set of Borel subgroups of G over $\mathbb{C}$ containing T. Let R be the set of roots of T in G. For $B \in \mathscr{B}(T)$, denote by $\lambda_B \in X^*(T)$ the highest weight of E relative to B, denote by $\rho_B \in X^*(T)_{\mathbb{R}}$ half the sum of the roots in R that are positive for B, and denote by Δ_B the Weyl denominator

$$\Delta_B = \prod_{\alpha>0}(1 - \alpha^{-1})$$

for T relative to B. (The index set is the subset of R consisting of roots that are positive for B.)

The character of E on $T_{\text{reg}}(\mathbb{R}) := T(\mathbb{R}) \cap G_{\text{reg}}(\mathbb{R})$ is given by

$$\operatorname{tr}(\gamma; E) = \sum_{B \in \mathscr{B}(T)} \lambda_B(\gamma) \cdot \Delta_B(\gamma)^{-1} \qquad (\gamma \in T_{\text{reg}}(\mathbb{R})).$$

The character Θ_π of π on $T_{\text{reg}}(\mathbb{R})$ is given by a similar expression

$$\Theta_\pi(\gamma) = \sum_{B \in \mathscr{B}(T)} n(\gamma, B) \lambda_B(\gamma) \Delta_B(\gamma)^{-1} \tag{4.1}$$

for certain integers $n(\gamma, B)$ depending on (γ, B). Of course, the invariance of Θ_π under conjugation by $G(\mathbb{R})$ implies that

$$n(\gamma, B) = n(w\gamma w^{-1}, wBw^{-1}) \qquad \text{for all } w \in \Omega(T(\mathbb{R}), G(\mathbb{R})), \tag{4.2}$$

where $\Omega(T(\mathbb{R}), G(\mathbb{R}))$ denotes the real Weyl group

$$N_{G(\mathbb{R})}(T)/T(\mathbb{R}).$$

For $\gamma \in T_{\text{reg}}(\mathbb{R})$, define subsets R_γ and R_γ^+ of R by

$$R_\gamma := \{\alpha \in R \mid \alpha \text{ is real and } \alpha(\gamma) > 0\}$$

$$R_\gamma^+ := \{\alpha \in R \mid \alpha \text{ is real and } \alpha(\gamma) > 1\}.$$

Note that R_γ is a root system and that R_γ^+ is a positive system in R_γ. Moreover, R_γ depends only on the connected component Γ of $T(\mathbb{R})$ in which γ lies; thus we sometimes write R_Γ instead of R_γ. Harish-Chandra [HC, Lemma 25] showed that

$$n(\gamma_1, B) = n(\gamma_2, B) \qquad \text{if } \Gamma_1 = \Gamma_2 \quad \text{and} \quad R_{\gamma_1}^+ = R_{\gamma_2}^+, \tag{4.3}$$

where Γ_i denotes the connected component of $T(\mathbb{R})$ in which γ_i lies.

Of course, any finite $\mathbb{Z}$-linear combination Θ of characters Θ_π as above can also be expressed in the form (4.1) for integers $n(\gamma, B)$ satisfying (4.2) and (4.3). (We refer to Θ as a virtual character on $G(\mathbb{R})$.) We are particularly interested in virtual characters Θ on $G(\mathbb{R})$ that are *stable* in the sense that

$$\Theta(\gamma) = \Theta(\gamma')$$

whenever $\gamma, \gamma' \in G_{\text{reg}}(\mathbb{R})$ are stably conjugate. A virtual character Θ is stable if and only if the integers $n(\gamma, B)$ satisfy the following strengthening of (4.2) (for all T):

$$n(\gamma, B) = n(w\gamma w^{-1}, wBw^{-1}) \qquad \text{for all } w \in W(\mathbb{R}), \tag{4.4}$$

where W is the Weyl group of $T_{\mathbb{C}}$ in $G_{\mathbb{C}}$ and $W(\mathbb{R})$ is the subgroup of W consisting of all elements that are fixed by complex conjugation. (Of course, $W(\mathbb{R})$ contains $\Omega(T(\mathbb{R}), G(\mathbb{R})$.)

Let A be the maximal split subtorus of T, and let M be the centralizer of A in G, a Levi subgroup of G. As usual for $\gamma \in M(\mathbb{R})$, we define a real number $D_M^G(\gamma)$ by

$$D_M^G(\gamma) = \det(1 - \mathrm{Ad}(\gamma); \mathrm{Lie}(G)/\mathrm{Lie}(M)).$$

We will need the following result of Arthur [A1] and Shelstad.

LEMMA 4.1. *For any stable virtual character Θ on $G(\mathbb{R})$, the function*

$$\gamma \mapsto |D_M^G(\gamma)|^{1/2}\Theta(\gamma)$$

on $T_{\mathrm{reg}}(\mathbb{R})$ extends continuously to $T(\mathbb{R})$.

Proof. Let Γ be a connected component of $T(\mathbb{R})$, and let Γ_{reg} denote its intersection with $T_{\mathrm{reg}}(\mathbb{R})$. To prove the lemma, we must show that

$$|D_M^G(\gamma)|^{1/2}\Theta(\gamma)$$

extends continuously from Γ_{reg} to Γ. Pick an element $a \in \Gamma$ such that $a^2 = 1$. (It is easy to see that such an element exists.) The root system R_Γ defined above is equal to the set of real roots $\alpha \in R$ such that $\alpha(a) = 1$; thus a lies in the center of the connected reductive subgroup of G containing T with root system R_Γ, and we conclude that a is fixed by the Weyl group $W(R_\Gamma)$ of R_Γ. Thus Γ is fixed by the subgroup $W(R_\Gamma)$ of $\Omega(T(\mathbb{R}), G(\mathbb{R}))$, and since both $|D_M^G(\gamma)|^{1/2}$, $\Theta(\gamma)$ are invariant under $\Omega(T(\mathbb{R}), G(\mathbb{R}))$ (which normalizes M), it follows that the function

$$|D_M^G(\gamma)|^{1/2}\Theta(\gamma)$$

on Γ_{reg} is invariant under $W(R_\Gamma)$. Let T_c denote the maximal anisotropic subtorus of T. Then

$$\Gamma = a \cdot T_c(\mathbb{R}) \cdot \exp(\mathfrak{A}),$$

where

$$\mathfrak{A} = X_*(A)_{\mathbb{R}} = \mathrm{Lie}(A(\mathbb{R})).$$

The Weyl group $W(R_\Gamma)$ fixes $T_c(\mathbb{R})$ as well as a. Fix a positive system R_Γ^+ in R_Γ, and let $\bar{C}$ be the corresponding closed chamber in $\mathfrak{A}$. Then $\bar{C}$ is a closed fundamental domain for the action of $W(R_\Gamma)$ on $\mathfrak{A}$, and therefore a $W(R_\Gamma)$-invariant function on Γ is continuous if and only if its restriction to

$$\Gamma^+ := a \cdot T_c(\mathbb{R}) \cdot \exp(\bar{C})$$

is continuous. Therefore, it is enough to show that

$$|D_M^G(\gamma)|^{1/2}|\Theta(\gamma)$$

extends continuously to Γ^+. For any regular element $\gamma \in \Gamma^+$ we have $R_\gamma^+ = R_\Gamma^+$, and thus there are integers $m(B)$ ($B \in \mathscr{B}(T)$) such that for all regular $\gamma \in \Gamma^+$,

$$\Theta(\gamma) = \sum_{B \in \mathscr{B}(T)} m(B)\lambda_B(\gamma)_B(\gamma)^{-1}.$$

The Weyl group W_M of $T_{\mathbb{C}}$ in $M_{\mathbb{C}}$ is a subgroup of $W(\mathbb{R})$, and this subgroup fixes A pointwise, and hence preserves Γ and R_Γ^+. Therefore, it follows from (4.4) that

$$\text{(4.5)} \qquad m(B) = m(wBw^{-1}) \qquad \text{for all } w \in W_M.$$

Choose a parabolic subgroup $P = MN$ having M as the Levi component and having the property that every element of R_Γ that appears in Lie(N) is nonnegative on $\bar{C}$. (Here N denotes the unipotent radical of P.) Put

$$\Delta_P = \prod_\alpha (1 - \alpha^{-1}),$$

where α runs through the roots of T in Lie(N). We claim that $\Delta_P(\gamma)$ is nonnegative for all $\gamma \in \Gamma^+$. Indeed, complex conjugation preserves the set of roots of T in Lie(N). If the complex conjugate $\bar{\alpha}$ is different from α, then the contribution of $\alpha, \bar{\alpha}$ to Δ_P is $(1 - \alpha(\gamma)^{-1})$ times its complex conjugate; this contribution is certainly nonnegative. If α is real, then $\alpha(a) = \pm 1$. If $\alpha(a) = -1$, then $\alpha(\gamma)^{-1}$ is negative and therefore $1 - \alpha(\gamma)^{-1}$ is positive. If $\alpha(a) = 1$, then $\alpha \in R_\Gamma$, and by our choice of P, we have $\alpha(\gamma) \geqslant 1$, so that $1 - \alpha(\gamma)^{-1} \geqslant 0$. It follows from the claim that

$$|D_M^G(\gamma)|^{1/2} = \Delta_P(\gamma) \cdot \delta_P^{1/2}(\gamma),$$

where δ_P denotes the modulus character

$$\delta_P(x) := |\det(x; \mathrm{Lie}(N))|$$

on $M(\mathbb{R})$. Therefore it is enough to show that

$$\sum_{B \in \mathscr{B}(T)} m(B) \cdot \Delta_P(\gamma) \cdot \lambda_B(\gamma) \cdot \Delta_B(\gamma)^{-1}$$

extends continuously to Γ^+. But it follows immediately from (4.5) that this last expression is a linear combination of characters of irreducible finite-dimensional representations of M, and of course such a linear combination extends continuously to Γ^+ (and even to all of $T(\mathbb{R})$). This completes the proof of the lemma.

For any stable virtual character Θ on $G(\mathbb{R})$, we denote by $\Phi_M^G(\gamma, \Theta)$ the (unique) continuous extension of

$$|D_M^G(\gamma)|^{1/2}\Theta(\gamma)$$

to $T(\mathbb{R})$ whose existence is asserted in the lemma we just proved. Sometimes it is convenient to extend $\Phi_M^G(\gamma, \Theta)$ to a function on the set of all elliptic elements in $M(\mathbb{R})$ (in other words, the set of $M(\mathbb{R})$-conjugates of elements in $T(\mathbb{R})$) by taking the unique extension that is invariant under conjugation by $M(\mathbb{R})$.

The functions $\Phi_M^G(\cdot, \Theta)$ behave simply under induction. Let $Q = LU$ be a parabolic subgroup of G with Levi subgroup L and unipotent radical U. Let Θ_L be a stable virtual character on $L(\mathbb{R})$ and let $\Theta = i_L^G(\Theta_L)$ be the virtual character on $G(\mathbb{R})$ obtained from Θ_L by the usual normalized parabolic induction. Let $T \subset M \subset G$ be as above. Then Θ is stable, and for all $\gamma \in T(\mathbb{R})$,

$$\Phi_M^G(\gamma, \Theta) = \sum_{gL(\mathbb{R})} \Phi_M^{gLg^{-1}}(\gamma, \Theta_{gLg^{-1}}), \tag{4.6}$$

where the sum runs over the set of cosets $gL(\mathbb{R})$ of $L(\mathbb{R})$ in $G(\mathbb{R})$ such that $gLg^{-1} \supset M$, and where $\Theta_{gLg^{-1}}$ denotes the virtual character on $gL(\mathbb{R})g^{-1}$ obtained from Θ_L on $L(\mathbb{R})$ via the isomorphism

$$\mathrm{Int}(g) \colon L \to gLg^{-1}$$

($\mathrm{Int}(g)(x) := gxg^{-1}$). For regular γ in $T(\mathbb{R})$, the formula (4.6) is just the usual formula for the character of a parabolically induced representation, and by continuity, the formula remains valid on all of $T(\mathbb{R})$.

We finish this section by discussing stable discrete series characters. We now assume that there exists an elliptic maximal torus T_e in G. (Elliptic means that T_e/Z is anisotropic, where Z denotes the center of G.) As usual we let $q(G)$ denote half the dimension of the symmetric space associated to the adjoint group of G. (Our hypothesis on G guarantees that this dimension is even.) Let Π be the L-packet consisting of all (isomorphism classes of) discrete series representations of $G(\mathbb{R})$ having the same infinitesimal and central characters as the finite-dimensional representation E. Put

$$\Theta^E = (-1)^{q(G)} \sum_{\pi \in \Pi} \Theta_\pi,$$

where Θ_π denotes the character of π; then the virtual character Θ^E is stable [HC, Lemma 61], [S]. Any discrete series representation π of $G(\mathbb{R})$ is obtained by induction from a discrete series representation of the normal subgroup

$$Z(\mathbb{R})\ \mathrm{im}[G_{sc}(\mathbb{R}) \to G(\mathbb{R})],$$

where G_{sc} denotes the simply connected cover of the derived group G_{der} of G. Therefore, Θ^E is supported on this normal subgroup (of finite index).

Let $A \subset T \subset M \subset G$ be as above. Let $\gamma \in T_{reg}(\mathbb{R})$, and define R_γ as above. The character value $\Theta^E(\gamma)$ is given by (4.1) for certain integers $n(\gamma, B)$. We will now review how these integers are related to the stable discrete series constants discussed in §3. Let T_c denote the maximal anisotropic subtorus in T; note that $T = AT_c$. Let L denote the centralizer of T_c in G; then $L_{\mathbb{C}}$ is a Levi subgroup of $G_{\mathbb{C}}$, and L contains T. Note that the roots of T in L are precisely the real roots of T.

Let $T(\mathbb{R})_1$ denote the maximal compact subgroup of $T(\mathbb{R})$. Then $T_c(\mathbb{R})$ is the identity component of $T(\mathbb{R})_1$, and there is a direct product decomposition

$$T(\mathbb{R}) = A(\mathbb{R})^0 \times T(\mathbb{R})_1. \tag{4.7}$$

We decompose our regular element $\gamma \in T(\mathbb{R})$ according to the decomposition (4.7):

$$\gamma = \exp(x) \cdot \gamma_1$$

for uniquely determined elements x in $X_*(A)_{\mathbb{R}} = \mathrm{Lie}(A)$ and γ_1 in $T(\mathbb{R})_1$. Let J denote the identity component of the centralizer of γ_1 in L. The root system of T in J is precisely R_γ.

Of course we may as well assume that γ belongs to

$$Z(\mathbb{R})\ \mathrm{im}[G_{sc}(\mathbb{R}) \to G(\mathbb{R})];$$

otherwise $n(\gamma, B) = 0$ for all $B \in \mathscr{B}(T)$. In this case, we claim that -1 belongs to the Weyl group of R_γ. In proving the claim we may as well assume that γ lies in the image of $G_{sc}(\mathbb{R})$, and therefore we may as well assume that $G_{sc} = G$. Replacing T_e by a conjugate, we may assume that T_c is contained in T_e; then T_e is contained in L. Therefore the connected center of L is equal to T_c. (It contains T_c and is contained in both T and T_e.) Moreover the maximal compact subgroups of $L(\mathbb{R})$ are connected since the derived group L_{der} of L is simply connected and L/L_{der} is anisotropic. It follows that by conjugating T in L, we may assume that γ_1 belongs to T_e and hence that T_e is contained in J. Therefore the connected center of J is also equal to T_c. The maximal torus T in J is split modulo the connected center T_c of J, and therefore its split component A is a split maximal torus in J_{der}. But J contains an anisotropic maximal torus, namely, T_e, and therefore $-1 \in W(R_\gamma)$.

Thus the root system R_γ in $X^*(A/A_G)_{\mathbb{R}}$ is of the type considered in §3, and from this root system we obtain an integer-valued function $\bar{c}$ on

$$(X_*(A/A_G)_{\mathbb{R}})_{\text{reg}} \times (X^*(A/A_G)_{\mathbb{R}})_{\text{reg}}$$

(see §3). The integer $n(\gamma, B)$ is given by

$$n(\gamma, B) = \bar{c}(x, p(\lambda_B + \rho_B - \lambda_0))$$

where

$$p\colon X^*(T)_{\mathbb{R}} \to X^*(A)_{\mathbb{R}}$$

is the natural restriction map and $\lambda_0 \in X^*(T)_{\mathbb{R}}$ is obtained from the character $\lambda_0 \in X^*(A_G)$ by which A_G acts on E by viewing $X^*(A_G)_{\mathbb{R}}$ as a direct summand of $X^*(T)_{\mathbb{R}}$ in the usual way.

5. The stable virtual characters Θ_γ. Let F be a subfield of $\mathbb{R}$. (The two examples we have in mind are $\mathbb{Q}$ and $\mathbb{R}$.) Let G be a connected reductive group over F. By a parabolic subgroup of G, we mean a parabolic subgroup of G defined over F, and by a Levi subgroup of G, we mean a Levi component defined over F of some parabolic subgroup of G.

Let M be a Levi subgroup of G. We write $\mathscr{F}^G(M)$ for the set of parabolic subgroups of G containing M, and we write $\mathscr{P}^G(M)$ for the subset of $\mathscr{F}^G(M)$ consisting of those parabolic subgroups for which M is a Levi component; we often abbreviate $\mathscr{F}^G(M)$, $\mathscr{P}^G(M)$ as $\mathscr{F}(M)$, $\mathscr{P}(M)$. Let A_M denote the maximal F-split torus in the center of M, and write $\mathfrak{A}_M$ for the real vector space

$$\mathfrak{A}_M := X_*(A_M)_{\mathbb{R}},$$

where the subscript $\mathbb{R}$ indicates that we have tensored $X_*(A_M)$ over $\mathbb{Z}$ with $\mathbb{R}$. By a root of A_M we mean a nonzero weight of A_M in Lie(G). Any $P \in \mathscr{P}(M)$ determines a chamber C_P in $\mathfrak{A}_M$ consisting of the points $x \in \mathfrak{A}_M$ such that

$$\langle x, \alpha \rangle > 0$$

for every root of A_M in Lie(N), where N denotes the unipotent radical of P. The map $P \mapsto C_P$ is a bijection from $\mathscr{P}(M)$ to the set of chambers in $\mathfrak{A}_M$. (A chamber in $\mathfrak{A}_M$ is a connected component of the complement in $\mathfrak{A}_M$ of the union of the root hyperplanes in $\mathfrak{A}_M$.) Let $Q \in \mathscr{F}(M)$, and let L denote the unique Levi component of Q containing M. Then $\mathfrak{A}_L$ is a subspace of $\mathfrak{A}_M$, so that C_Q can be regarded as a cone in $\mathfrak{A}_M$. Moreover, $\mathfrak{A}_M$ is equal to the disjoint union

$$\mathfrak{A}_M = \coprod_{Q \in \mathscr{F}(M)} C_Q.$$

We fix a minimal parabolic subgroup P_0 of G and fix a Levi component M_0 of P_0 over F. We say that a parabolic subgroup P of G is *standard* if it contains P_0, and we say that P is *semistandard* if it contains M_0. Thus, $\mathscr{F}(M_0)$ is the set of semistandard parabolic subgroups. Given a semistandard parabolic subgroup P of G, we write N_P for the unipotent radical of P and M_P for the unique Levi component of P containing M_0. We will often write A_P, $\mathfrak{A}_P$ rather than A_{M_P}, $\mathfrak{A}_{M_P}$. In fact, we will often abbreviate M_P, N_P as M, N, so that

$$P = MN.$$

When we use Q to denote a semistandard parabolic subgroup, we will often write L, U instead of M_Q, N_Q, so that

$$Q = LU.$$

Let E be an irreducible representation of the algebraic group G on a finite-dimensional complex vector space, and let ν be an element in $(\mathfrak{A}_{P_0})^*$, the real vector space dual to $\mathfrak{A}_{P_0}$, such that the restriction of ν to $\mathfrak{A}_G$ coincides with the character by which A_G acts on E. (This character is an element of $X^*(A_G)$, a lattice in $\mathfrak{A}_G^*$.)

We are going to use E, ν to define a virtual representation of the real group $G(\mathbb{R})$, the character of which we will denote by Θ_ν. The first step is to define an element $\nu_P \in \mathfrak{A}_P^*$ for each semistandard parabolic subgroup $P = MN$. There is a unique standard parabolic subgroup $P' = M'N'$ conjugate to P under $G(F)$. There exists $g \in G(F)$, unique up to right multiplication by $M(F)$, such that $gPg^{-1} = P'$ and $gMg^{-1} = M'$, and the inner automorphism $x \mapsto gxg^{-1}$ of G induces isomorphisms

$$A_P \simeq A_{P'}$$

$$\mathfrak{A}_P \simeq \mathfrak{A}_{P'}$$

independent of the choice of g. Let $\nu' \in \mathfrak{A}_{P'}^*$ be the restriction of the linear form ν to the subspace $\mathfrak{A}_{P'}$ of $\mathfrak{A}_{P_0}$, and then use the isomorphism $\mathfrak{A}_P \simeq \mathfrak{A}_{P'}$ to transport ν' over to an element $\nu_P \in \mathfrak{A}_P^*$. This completes the definition of ν_P. It is easy to see that if $P, Q \in \mathscr{F}^G(M_0)$ and $P \subset Q$, then ν_Q is the image of ν_P under the natural restriction map

$$\mathfrak{A}_P^* \to \mathfrak{A}_Q^*.$$

The next step is to use E, ν to define a virtual finite-dimensional complex representation E_P^ν of M for any semistandard parabolic subgroup $P = MN$. We begin by considering the Lie algebra cohomology groups

$$H^i(\mathrm{Lie}(N), E);$$

these are finite-dimensional complex representations of M. (We use the usual *left* action of M.) Of course, the action of the split torus A_P on $H^i(\mathrm{Lie}(N), E)$ decomposes this space as a direct sum of weight subspaces

$$H^i(\mathrm{Lie}(N), E)_\mu,$$

where μ runs through $X^*(A_P)$ (a lattice in $\mathfrak{A}_P^*$). We write C_P^* for the closed convex cone in $\mathfrak{A}_P^*$ dual to C_P; thus, C_P^* consists of all $\mu \in \mathfrak{A}_P^*$ such that $\langle x, \mu \rangle \geqslant 0$ for all $x \in C_P$. We write

$$H^i(\mathrm{Lie}(N), E)_{\geqslant \nu_P}$$

for the subspace of $H^i(\mathrm{Lie}(N), E)$ obtained by taking the direct sum of all the weight spaces $H^i(\mathrm{Lie}(N), E)_\mu$ for $\mu \in X^*(A_P)$ such that $\mu - \nu_P \in C_P^*$; of course,

$$H^i(\mathrm{Lie}(N), E)_{\geqslant \nu_P}$$

is stable under the action of M. We write E_P^ν for the virtual M-module

$$\sum_i (-1)^i H^i(\mathrm{Lie}(N), E)_{\geqslant \nu_P}.$$

We now use a theorem of Kostant [Ko] to express E_P^ν in terms of irreducible representations of M. Let T be a maximal torus of M over $\mathbb{C}$ and let B_M be a Borel subgroup of M over $\mathbb{C}$ containing T. For any B_M-dominant weight $\mu \in X^*(T)$, we let V_μ^M be an irreducible finite-dimensional complex representation of M with highest weight μ. The set of Borel subgroups B of G over $\mathbb{C}$ containing T and contained in P is in natural bijection with the set of Borel subgroups of M over $\mathbb{C}$ containing T. Thus our choice of B_M determines a Borel subgroup B of G over $\mathbb{C}$ containing T and contained in P, characterized by the equality

$$B_M = B \cap M.$$

Let R (respectively, R_M) be the set of roots of T in G (respectively, M). The Borel subgroups B, B_M determine positive systems R^+, R_M^+ in R, R_M, and, of course,

$$R_M^+ = R_M \cap R^+.$$

Let $\lambda_B \in X^*(T)$ denote the highest weight (with respect to B) of the irreducible representation E of G, and let $\rho_B \in X^*(T)_{\mathbb{R}}$ denote half the sum of the roots in R^+. Let W (respectively, W_M) denote the Weyl group of $T_{\mathbb{C}}$ in $G_{\mathbb{C}}$ (respectively, $M_{\mathbb{C}}$). Let W' denote the set of Kostant representatives for the cosets $W_M \backslash W$;

thus W' consists of the elements $w \in W$ such that

$$w^{-1}(R_M^+) \subset R^+.$$

(Obviously W' depends on the choice of B_M.) Kostant's theorem on Lie(N)-cohomology states that as an M-module $H^i(\mathrm{Lie}(N), E)$ is isomorphic to

$$\bigoplus_w V^M_{w(\lambda_B+\rho_B)-\rho_B},$$

where w runs through the set of Kostant representatives of length i. (We use the length function on W determined by B.) Note that the weight $w(\lambda_B + \rho_B) - \rho_B$ is indeed B_M-dominant for any Kostant representative w.

Let

$$\varepsilon: W \to \{\pm 1\}$$

be the usual sign function on W ($\varepsilon(w)$ is $(-1)^{l(w)}$, where $l(w)$ denotes the length of w). We see from Kostant's theorem that the virtual representation E_P^ν is given by

$$\sum_{w \in W'} \varepsilon(w) \cdot \xi_{C_P^*}(p_M(w(\lambda_B + \rho_B) - \rho_B) - \nu_P) \cdot V^M_{w(\lambda_B+\rho_B)-\rho_B}$$

where $\xi_{C_P^*}$ denotes the characteristic function of the subset C_P^* of $\mathfrak{A}_M^*$ and p_M denotes the restriction map

$$X^*(T)_{\mathbb{R}} = (X_*(T)_{\mathbb{R}})^* \to \mathfrak{A}_M^*$$

induced by the inclusion of A_M in T.

Now we are ready to define the virtual character Θ_ν on the real group $G(\mathbb{R})$. For any semistandard parabolic subgroup $P = MN$, we write δ_P for the modulus quasi-character on $M(\mathbb{R})$ given by

$$\delta_P(x) = |\det(x; \mathrm{Lie}(N))|$$

for $x \in M(\mathbb{R})$. We write $(E_P^\nu)^*$ for the contragredient of the virtual representation E_P^ν. Then

$$\delta_P^{-1/2} \otimes (E_P^\nu)^*$$

is a virtual representation of the real group $M(\mathbb{R})$, which we may induce from $P(\mathbb{R})$ to $G(\mathbb{R})$ to obtain a virtual representation

$$i_P^G(\delta_P^{-1/2} \otimes (E_P^\nu)^*)$$

of $G(\mathbb{R})$. We are using the usual *normalized* parabolic induction, which builds in a factor of $\delta_P^{1/2}$; if we used unnormalized induction, we would simply be inducing $(E_P^\nu)^*$ from $P(\mathbb{R})$ to $G(\mathbb{R})$. We write Θ_P^ν for the (Harish-Chandra) character of

$$i_P^G(\delta_P^{-1/2} \otimes (E_P^\nu)^*)$$

and define a virtual character Θ_ν on $G(\mathbb{R})$ by putting

$$\Theta_\nu := \sum_P (-1)^{\dim(A_P/A_G)} \Theta_P^\nu,$$

where P runs over the set of standard parabolic subgroups of G. Note that Θ_ν is stable. Indeed, the character of E_P^ν is obviously stable on $M(\mathbb{R})$, and stability is preserved by parabolic induction.

Now we fix a Levi subgroup M of G containing M_0, and we assume that $M_{\mathbb{R}}$ contains a maximal torus T over $\mathbb{R}$ such that T/A_M is anisotropic over $\mathbb{R}$. It follows that A_M coincides with the maximal $\mathbb{R}$-split torus in the center of M, and this in turn implies that any parabolic subgroup of G over $\mathbb{R}$ containing M is automatically defined over F. Note that A_M is the maximal $\mathbb{R}$-split torus in T and that T is elliptic in $M_{\mathbb{R}}$. The discussion following Lemma 4.1 applies to the stable character Θ_ν, and thus we obtain a continuous function $\Phi_M^G(\gamma, \Theta_\nu)$ on $T(\mathbb{R})$. Sometimes we abbreviate $\Phi_M^G(\gamma, \Theta_\nu)$ as $\Phi_M(\gamma, \Theta_\nu)$.

We are now going to use E, ν to define another function $L_M^\nu(\gamma)$ on $T(\mathbb{R})$ (with M and T as above); we will see in §7 that this function arises naturally in the Lefschetz trace formula for Hecke operators. Once the definition is complete, our goal will be to show that $\Phi_M(\gamma, \Theta_\nu)$ is in fact equal to $L_M^\nu(\gamma)$. Let $\gamma \in T(\mathbb{R})$. There is a direct product decomposition

$$T(\mathbb{R}) = A_M(\mathbb{R})^0 \times T(\mathbb{R})_1,$$

where $T(\mathbb{R})_1$ denotes the maximal compact subgroup of $T(\mathbb{R})$. Therefore, we can write γ as

$$\gamma = \exp(x) \cdot \gamma_1$$

for unique elements $x \in \mathfrak{A}_M$ and $\gamma_1 \in T(\mathbb{R})_1$. The complex number $L_M^\nu(\gamma)$ that we are in the process of defining has the form

$$L_M^\nu(\gamma) := \sum_Q (-1)^{\dim(A_L/A_G)} \cdot |D_M^L(\gamma)|^{1/2} \cdot \delta_Q^{-1/2}(\gamma) \cdot L_Q^\nu(\gamma),$$

where the sum runs over $Q = LU$ in $\mathscr{F}(M)$ such that x is contained in the subspace $\mathfrak{A}_L$ of $\mathfrak{A}_M$, and where $L_Q^\nu(\gamma)$ is a complex number we have yet to define. The factor $D_M^L(\gamma)$ was defined in §4, just before Lemma 4.1.

In order to define $L_Q^\nu(\gamma)$, we choose a Borel subgroup B of G over $\mathbb{C}$ containing T and contained in Q, and we put $B_L := B \cap L$, a Borel subgroup of L over $\mathbb{C}$ containing T; it turns out that $L_Q^\nu(\gamma)$ is independent of this choice. We now use the same notational system as we used when discussing Kostant's theorem (though we are now using Q, L instead of P, M). In particular, we have the set W' of Kostant representatives for the cosets $W_L \backslash W$, the irreducible representations

$$V^L_{w(\lambda_B+\rho_B)-\rho_B}$$

of L, and the restriction map

$$p_L \colon X^*(T)_{\mathbb{R}} \to \mathfrak{A}_L^*.$$

The *open* convex polyhedral cone C_Q in $\mathfrak{A}_L$ determines a function

$$\varphi_{C_Q}(\cdot,\cdot)$$

on $\mathfrak{A}_L \times \mathfrak{A}_L^*$, as in the last part of Appendix A, and we will denote this function simply by $\varphi_Q(\cdot,\cdot)$. We define $L_Q^\nu(\gamma)$ by

$$L_Q^\nu(\gamma) := (-1)^{\dim(A_L)} \sum_{w \in W'} \varepsilon(w) \cdot \operatorname{tr}(\gamma^{-1}; V^L_{w(\lambda_B+\rho_B)-\rho_B}) \cdot \varphi_Q(-x, p_L(w(\lambda_B+\rho_B)-\rho_B) - \nu_Q).$$

THEOREM 5.1. *The two functions $\Phi_M(\gamma, \Theta_\nu)$ and $L_M^\nu(\gamma)$ on $T(\mathbb{R})$ are equal.*

Proof. By definition, $\Phi_M(\gamma, \Theta_\nu)$ is given by

$$\sum_Q (-1)^{\dim(A_L/A_G)} \cdot \Phi_M(\gamma, \Theta_Q^\nu), \tag{5.1}$$

where $Q = LU$ runs over the set of standard parabolic subgroups of G. Applying equation (4.6) to the induced character Θ_Q^ν of $G(\mathbb{R})$, we see that $\Phi_M(\gamma, \Theta_\nu)$ is equal to

$$\sum_Q (-1)^{\dim(A_L/A_G)} \sum_{Q'} \Phi_M^{L'}(\gamma, \delta_{Q'}^{-1/2} \otimes (E_{Q'}^\nu)^*), \tag{5.2}$$

where the index set for the first sum is the same as before and the index set for the second sum is the set of parabolic subgroups Q' over $\mathbb{R}$ containing M such that Q' is conjugate under $G(\mathbb{R})$ to Q. Since, as we remarked earlier, every parabolic subgroup of G over $\mathbb{R}$ containing M is automatically defined over F, we see that

$\Phi_M(\gamma, \Theta_\nu)$ is equal to

$$\sum_{Q \in \mathscr{F}(M)} (-1)^{\dim(A_L/A_G)} \Phi_M^L(\gamma, \delta_Q^{-1/2} \otimes (E_Q^\nu)^*) \tag{5.3}$$

(as usual $Q = LU$). Recalling the expression for E_Q^ν that we found using Kostant's theorem, we see that $\Phi_M(\gamma, \Theta_\nu)$ is equal to

$$\sum_{Q \in \mathscr{F}(M)} (-1)^{\dim(A_L/A_G)} \cdot |D_M^L(\gamma)|^{1/2} \cdot \delta_Q^{-1/2}(\gamma) \cdot \sum_{w \in W'} \varepsilon(w) \\ \cdot \operatorname{tr}(\gamma^{-1}; V^L_{w(\lambda_B+\rho_B)-\rho_B}) \cdot \xi_{C_Q^*}(p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q). \tag{5.4}$$

The notation here is the same as that used during our discussion of Kostant's theorem. In particular, given $Q \in \mathscr{F}(M)$, we must choose a Borel subgroup B of G over $\mathbb{C}$ containing T and contained in Q in order to define W', the set of Kostant representatives.

Let $Q = LU$ be a parabolic subgroup in $\mathscr{F}(M)$. As usual $\mathfrak{A}_M$ is a disjoint union of convex cones $C_{Q'}$, one for each $Q' \in \mathscr{F}(M)$. But M is also a Levi subgroup of L, and therefore $\mathfrak{A}_M$ is also a disjoint union of convex cones $C_{Q''}$, one for each $Q'' \in \mathscr{F}^L(M)$. For any $Q' \in \mathscr{F}(M)$ such that $Q' \subset Q$, we put $Q'' := Q' \cap L$, an element of $\mathscr{F}^L(M)$. The map $Q' \mapsto Q''$ sets up a bijection

$$\{Q' \in \mathscr{F}(M) \mid Q' \subset Q\} \simeq \mathscr{F}^L(M),$$

and the convex cones $C_{Q'}$, $C_{Q''}$ in $\mathfrak{A}_M$ are related by the equality

$$C_{Q''} = C_{Q'} + \mathfrak{A}_L.$$

Recall that we have written γ as

$$\gamma = \exp(x) \cdot \gamma_1$$

for uniquely determined $x \in \mathfrak{A}_M$ and $\gamma_1 \in T(\mathbb{R})_1$. For each parabolic subgroup $Q \in \mathscr{F}(M)$, we denote by $Q' = L'U'$ the unique element of $\mathscr{F}(M)$ such that

(1) $Q' \subset Q$, and
(2) $-x \in C_{Q'} + \mathfrak{A}_L$.

It follows from the second condition that x belongs to $\mathfrak{A}_{L'}$.

Now let $Q_1 = L_1U_1$ be an element of $\mathscr{F}(M)$ such that $x \in \mathfrak{A}_{L_1}$. Pick a Borel subgroup B in G over $\mathbb{C}$ containing T and contained in Q_1. We are interested in the terms in (5.4) indexed by parabolic subgroups $Q \in \mathscr{F}(M)$ such that $Q' = Q_1$. We have inclusions

$$T \subset B \subset Q_1 \subset Q,$$

so that we can (and do) use B to define the set W' of Kostant representatives for the cosets $W_L \backslash W_G$. As before we write Q'' for the element $Q_1 \cap L$ of $\mathscr{F}^L(M)$. Define a function $\Delta^L_{Q''}$ on $T(\mathbb{R})$ (a partial Weyl denominator for the group L) by

$$\Delta^L_{Q''}(\gamma) = \prod_\alpha (1 - \alpha(\gamma)^{-1}),$$

where α runs through the set of roots of T in $\mathrm{Lie}(N'')$. (N'' denotes the unipotent radical of Q''.) We claim that $\Delta^L_{Q''}(\gamma^{-1})$ is a nonnegative real number (for γ, Q, Q_1, Q'' as in the discussion preceding the definition of $\Delta^L_{Q''}$). Since Q'' is defined over $\mathbb{R}$, the set of roots α of T in $\mathrm{Lie}(N'')$ is stable under complex conjugation. Complex conjugate pairs $\bar{\alpha}, \alpha$ with $\bar{\alpha} \neq \alpha$ make a nonnegative contribution to $\Delta^L_{Q''}$, since $1 - \bar{\alpha}(\gamma)$ is complex conjugate to $1 - \alpha(\gamma)$. Let α be a root of T in $\mathrm{Lie}(N'')$ such that $\bar{\alpha} = \alpha$. It is enough to show that $1 - \alpha(\gamma)$ is nonnegative. Since $Q' = Q_1$, the element $-x$ belongs to $C_{Q''}$, which implies that

$$\langle x, \alpha \rangle \leqslant 0.$$

Since $\alpha = \bar{\alpha}$, the value of α on any element of $T(\mathbb{R})_1$ is ± 1. Therefore, $\alpha(\gamma_1) = \pm 1$ and

$$\alpha(\gamma) = \exp(\langle x, \alpha \rangle) \cdot \alpha(\gamma_1) \leqslant 1,$$

as desired.

It follows from the claim that

$$|D^L_{L_1}(\gamma)|^{1/2} \cdot \delta_Q^{-1/2}(\gamma) = \delta_{Q_1}^{-1/2}(\gamma) \cdot \Delta^L_{Q''}(\gamma^{-1}).$$

Moreover, applying Kostant's theorem to L and its parabolic subgroup Q'', it is easy to see that for $w \in W'$,

$$\Delta^L_{Q''}(\gamma^{-1}) \cdot \mathrm{tr}(\gamma^{-1}; V^L_{w(\lambda_B+\rho_B)-\rho_B})$$

is equal to

$$\sum_{u \in W'_L} \varepsilon(u) \cdot \mathrm{tr}(\gamma^{-1}; V^{L_1}_{uw(\lambda_B+\rho_B)-\rho_B}),$$

where W'_L is the set of Kostant representatives for the cosets $W_{L_1} \backslash W_L$ (relative to the Borel subgroup $B \cap L$ of L). It is also easy to see that

$$W'_L W'$$

is the set W'' of Kostant representatives for the cosets $W_{L_1}\backslash W$ (relative to the Borel subgroup B of G), and that for $w \in W'$, $u \in W'_L$,

$$p_L(uw(\lambda_B + \rho_B)) = p_L(w(\lambda_B + \rho_B)).$$

Therefore, the contribution of such a Q to (5.4) is

$$(-1)^{\dim(A_L/A_G)} \cdot |D_M^{L_1}(\gamma)|^{1/2} \cdot \delta_{Q_1}^{-1/2}(\gamma) \cdot \sum_{w \in W''} \varepsilon(w) \cdot \operatorname{tr}(\gamma^{-1}; V^{L_1}_{w(\lambda_B+\rho_B)-\rho_B}) \cdot \xi_{C_Q^*}(p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q).$$

Since all factors in this expression except for the first and last depend on Q only through Q_1, we see that $\Phi_M(\gamma, \Theta_\nu)$ is equal to

$$(5.5) \quad \sum_{Q_1}(-1)^{\dim(A_{L_1}/A_G)} \cdot |D_M^{L_1}(\gamma)|^{1/2} \cdot \delta_{Q_1}^{-1/2}(\gamma) \cdot \sum_{w \in W''} \varepsilon(w) \cdot \operatorname{tr}(\gamma^{-1}; V^{L_1}_{w(\lambda_B+\rho_B)-\rho_B}) \cdot \sum_{Q}(-1)^{\dim(A_{L_1}/A_L)} \cdot \xi_{C_Q^*}(p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q),$$

where the index set for the first sum is the set of $Q_1 = L_1U_1 \in \mathscr{F}(M)$ such that $x \in \mathfrak{A}_{L_1}$, and the index set for the second sum is the set of $Q \in \mathscr{F}(M)$ such that $Q' = Q_1$.

Comparing (5.5) with the definition of $L_M^\nu(\gamma)$, we see that in order to prove that $\Phi_M(\gamma, \Theta_\nu)$ is equal to $L_M^\nu(\gamma)$, it is enough to prove the equality

$$(5.6) \quad \sum_{Q}(-1)^{\dim(A_{L_1}/A_L)} \cdot \xi_{C_Q^*}(p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q) = (-1)^{\dim(A_{L_1})} \cdot \varphi_{Q_1}(-x, p_{L_1}(w(\lambda_B + \rho_B) - \rho_B) - \nu_{Q_1}).$$

The sum in (5.6) is taken over the set of $Q \in \mathscr{F}(M)$ such that $Q' = Q_1$, or, in other words, the set of $Q \in \mathscr{F}(M)$ such that $Q \supset Q_1$ and $-x \in C_{Q_1} + \mathfrak{A}_L$. Therefore, the left-hand side of (5.6) is equal to

$$\sum_{Q}(-1)^{\dim(A_{L_1}/A_L)} \cdot \xi_{C_{Q_1}+\mathfrak{A}_L}(-x) \cdot \xi_{C_Q^*}(p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q),$$

where the sum is now taken over all $Q \in \mathscr{F}(M)$ such that $Q \supset Q_1$, and this is indeed equal to the right-hand side of (5.6), as one easily sees from Lemma A.6 and the fact that the restriction of

$$p_{L_1}(w(\lambda_B + \rho_B) - \rho_B) - \nu_{Q_1}$$

to the subspace $\mathfrak{A}_L$ of $\mathfrak{A}_{L_1}$ is equal to

$$p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q.$$

This concludes the proof of Theorem 5.1.

We will now make a particular choice for ν, and we will show (Theorem 5.2) that with this choice the virtual character Θ_ν becomes especially simple. Denote by N_0 the unipotent radical of our chosen minimal parabolic subgroup P_0. Let $\rho_0 \in \mathfrak{A}^*_{P_0}$ denote half the sum of the roots (counted with multiplicity) of A_{P_0} in Lie(N_0). As usual, we regard $\mathfrak{A}^*_G$ as a direct summand of $\mathfrak{A}^*_{P_0}$. Define $\nu_m \in \mathfrak{A}^*_{P_0}$ by

$$\nu_m := -\rho_0 + \lambda_0,$$

where $\lambda_0 \in X^*(A_G) \subset \mathfrak{A}^*_G$ is the character by which A_G acts in the representation E. From ν_m we obtain the virtual character Θ_{ν_m} on $G(\mathbb{R})$.

Suppose first that there exists an elliptic maximal torus T_e in G over $\mathbb{R}$, so that $G(\mathbb{R})$ has a discrete series. As in §4 we put

$$\Theta^E = (-1)^{q(G)} \sum_{\pi \in \Pi} \Theta_\pi,$$

where Π is the L-packet of discrete series representations of $G(\mathbb{R})$ having the same infinitesimal and central characters as the finite-dimensional representation E. Note that the contragredient of Θ^E is equal to Θ^{E^*}, where E^* denotes the contragredient of E.

THEOREM 5.2. *The virtual characters Θ_{ν_m} and Θ^{E^*} agree on $T_{\mathrm{reg}}(\mathbb{R})$ for any maximal torus T in G over $\mathbb{R}$ whose $\mathbb{R}$-split component is both defined and split over F.*

Proof. It is enough to prove the theorem when $F = \mathbb{R}$, in which case we must show that Θ_{ν_m} is equal to Θ^{E^*}. We appeal to the characterization of Θ^{E^*} provided by Theorem 3 of [HC]. Clearly Θ_{ν_m} and Θ^{E^*} are invariant distributions with the same infinitesimal and central characters. Moreover it is obvious from the definition of Θ_{ν_m} that Θ_{ν_m} agrees with Θ^{E^*} on $T_e(\mathbb{R}) \cap G_{\mathrm{reg}}(\mathbb{R})$. (On $T_e(\mathbb{R}) \cap G_{\mathrm{reg}}(\mathbb{R})$ the virtual character Θ^{E^*} coincides with that of the finite-dimensional representation E^*.) Thus the only nontrivial point is to check the validity of the second condition in Harish-Chandra's theorem:

$$\sup_{\gamma \in G_{\mathrm{reg}}(\mathbb{R})} |D(\gamma)|^{1/2} |\Theta_{\nu_m}(\gamma)| \omega(\gamma) < \infty,$$

where ω is the unique homomorphism from $G(\mathbb{R})$ to the group of positive real numbers whose restriction to $A_G(\mathbb{R})$ is equal to the absolute value of the quasi character by which $A_G(\mathbb{R})$ acts in E, and where

$$D(\gamma) = \det(1 - \mathrm{Ad}(\gamma); \mathrm{Lie}(G)/\mathrm{Lie}(T)),$$

where T denotes the unique maximal torus of G containing γ.

It is enough to check that for every maximal torus T of G

$$\Phi_M(\gamma, \Theta_{\nu_m}) \cdot \omega(\gamma) \tag{5.7}$$

is bounded on $T_{\mathrm{reg}}(\mathbb{R})$. Here M is (as usual) the centralizer of the split component of T; we used that roots of T in M are imaginary and hence that

$$|\det(1 - \mathrm{Ad}(\gamma); \mathrm{Lie}(M)/\mathrm{Lie}(T))|$$

is bounded on $T(\mathbb{R})$. But the boundedness of (5.7) follows directly from Theorem 5.1. Indeed, it is enough to check that for all $P = MN \in \mathscr{P}(M)$

$$\mathrm{tr}(\gamma^{-1}; V^M_{w(\lambda_B+\rho_B)-\rho_B}) \cdot \omega(\gamma) \cdot \delta_P^{-1/2}(\gamma) \tag{5.8}$$

is bounded on $T(\mathbb{R})$ whenever $w \in W'$ is such that

$$\varphi_P(-x, p_M(w(\lambda_B + \rho_B) - \lambda_0)) \neq 0. \tag{5.9}$$

By Proposition A.5, the condition (5.9) implies that

$$\langle x, p_M(w(\lambda_B + \rho_B) - \lambda_0)\rangle \geqslant 0. \tag{5.10}$$

Since

$$T(\mathbb{R}) = A_M(\mathbb{R})^0 \times T(\mathbb{R})_1$$

and $T(\mathbb{R})_1$ is compact, only the character by which $A_M(\mathbb{R})^0$ acts in $V^M_{w(\lambda_B+\rho_B)-\rho_B}$ is relevant to the boundedness of (5.8). In fact, the function (5.8) of γ transforms under $A_M(\mathbb{R})^0$ by the element

$$p_M(-w(\lambda_B + \rho_B) + \rho_B + \lambda_0 - \rho_N) = p_M(-w(\lambda_B + \rho_B) + \lambda_0) \in \mathfrak{A}^*_M,$$

where ρ_N is half the sum of the roots of T in $\mathrm{Lie}(N)$, and thus (5.10) does imply that (5.8) is bounded on $T(\mathbb{R})$. This completes the proof.

Now we drop the assumption that G has an elliptic maximal torus over $\mathbb{R}$. First, for arbitrary G and suitably regular γ, we will rewrite $\Phi_M(\gamma, \Theta_{\nu_m})$ in terms of the functions ψ_R of §1. Second, for G having no elliptic maximal torus, we will use this expression for $\Phi_M(\gamma, \Theta_{\nu_m})$ to show that it vanishes under a certain regularity hypothesis on the highest weight of E.

Let $T \subset M \subset G$ be as usual. Let $\gamma \in T(\mathbb{R})$ and write

$$\gamma = \exp(x) \cdot \gamma_1$$

with $x \in \mathfrak{A}_M$ and $\gamma_1 \in T(\mathbb{R})_1$. Assume that x is regular in $\mathfrak{A}_M$, in the sense that no root of A_M vanishes on x. Then by (A.2) (or rather its analog for the function φ_Q), $L_M^\nu(\gamma)$ is given by

$$L_M^\nu(\gamma) = (-1)^{\dim(A_G)} \sum_{P \in \mathscr{P}(M)} \delta_P^{-1/2}(\gamma) \cdot \sum_{w \in W'} \varepsilon(w) \cdot \operatorname{tr}(\gamma^{-1}; V^M_{w(\lambda_B+\rho_B)-\rho_B})$$
$$\cdot \varphi_P(-x, p_M(w(\lambda_B + \rho_B) - \rho_B) - \nu_P).$$

Fix a Borel subgroup B_M of M over $\mathbb{C}$ containing T. For each $P \in \mathscr{P}(M)$, we let $B(P)$ denote the unique Borel subgroup of G over $\mathbb{C}$ such that

$$T \subset B(P) \subset P$$

and $B(P) \cap M = B_M$. Write $P_x = MN_x$ for the unique element of $\mathscr{P}(M)$ whose chamber in $\mathfrak{A}_M$ contains $-x$, and write $B(x)$ for $B(P_x)$. Let $\mathscr{B}(T)$ denote the set of Borel subgroups of G over $\mathbb{C}$ containing T, and let $\mathscr{B}(T)'$ denote the subset of $\mathscr{B}(T)$ consisting of Borel subgroups B such that $B \cap M = B_M$. Then

$$L_M^\nu(\gamma) = (-1)^{\dim(A_G)} \sum_{B \in \mathscr{B}(T)'} \sum_{P=MN \in \mathscr{P}(M)} \varepsilon(B, B(P))$$
$$\cdot \delta_P^{-1/2}(\gamma) \cdot \operatorname{tr}(\gamma^{-1}; V^M_{\lambda_B+\rho_B-\rho_{B(P)}})$$
$$\cdot \varphi_P(-x, p_M(\lambda_B + \rho_B - \rho_N) - \nu_P),$$

where $\varepsilon(B, B(P)) = \varepsilon(w)$ for the unique element $w \in W$ such that $wB(P)w^{-1} = B$ and where ρ_N is half the sum of the roots of T in $\operatorname{Lie}(N)$. Of course,

$$\rho_{B(P)} = \rho_M + \rho_N,$$

where ρ_M denotes half the sum of the roots of T in M that are positive for B_M. Thus

$$\rho_{B(P)} - \rho_{B(x)} = \rho_N - \rho_{N_x} \in X^*(T)$$

is trivial on the intersection of T with the derived group of M, and therefore it defines a homomorphism

$$M \to \mathbb{G}_m.$$

It follows that

$$\operatorname{tr}(\gamma^{-1}; V^M_{\lambda_B+\rho_B-\rho_{B(P)}}) = \operatorname{tr}(\gamma^{-1}; V^M_{\lambda_B+\rho_B-\rho_{B(x)}}) \cdot \langle \gamma^{-1}, \rho_{B(x)} - \rho_{B(P)} \rangle,$$

and this shows that

$$L^\nu_M(\gamma) = (-1)^{\dim(A_G)} \sum_{B \in \mathscr{B}(T)'} \operatorname{tr}(\gamma^{-1}; V^M_{\lambda_B+\rho_B-\rho_{B(x)}}) \cdot \delta^{-1/2}_{P_x}(\gamma) \cdot \varepsilon(B, B(x)) \cdot a_B,$$

where

$$a_B = \sum_{P=MN \in \mathscr{P}(M)} \langle \gamma^{-1}, \rho_{B(x)} - \rho_{B(P)} \rangle \cdot \delta^{1/2}_{P_x}(\gamma) \cdot \delta^{-1/2}_{P}(\gamma) \cdot \varepsilon(B(P), B(x)) \cdot \varphi_P(-x, p_M(\lambda_B + \rho_B - \rho_N) - \nu_P).$$

Let $\bar{P} = M\bar{N}$ be the element of $\mathscr{P}(M)$ opposite P. Note that

$$\langle \gamma^{-1}, \rho_{B(x)} - \rho_{B(P)} \rangle \delta^{1/2}_{P_x}(\gamma) \delta^{-1/2}_{P}(\gamma)$$

is equal to the sign of the real number

$$\langle \gamma^{-1}, \rho_{B(x)} - \rho_{B(P)} \rangle = \langle \gamma^{-1}, \rho_{N_x} - \rho_N \rangle.$$

But $\rho_{N_x} - \rho_N$ is the sum of all roots α of T in $N_x \cap \bar{N}$, and since this set of roots is preserved by complex conjugation, the sign in question is

$$\prod_\alpha \operatorname{sgn} \alpha(\gamma^{-1}),$$

where α runs through the real roots of T in $N_x \cap \bar{N}$. For such a real root,

$$\operatorname{sgn} \alpha(\gamma^{-1}) = \operatorname{sgn} \alpha(\gamma_1).$$

Of course, the sign $\varepsilon(B(P), B(x))$ is -1 raised to the number of roots of T in $N_x \cap \bar{N}$, and again, since this set of roots is stable under complex conjugation, this sign is -1 raised to the number of real roots of T in $N_x \cap \bar{N}$. Let R_γ be the set of real roots α of T in G such that $\alpha(\gamma_1) = 1$ (or, equivalently, such that $\alpha(\gamma) > 0$). Of course, R_γ is a root system in $(\mathfrak{A}_M/\mathfrak{A}_G)^*$ (though it need not span that space). Let $\mathscr{C}$ be the set of Weyl chambers in $\mathfrak{A}_M$ for the root system R_γ. For $C_1, C_2 \in \mathscr{C}$, let $\varepsilon(C_1, C_2)$ be -1 raised to the number of root hyperplanes (for roots in R_γ) separating C_1 and C_2. Let C_0 be the unique element in $\mathscr{C}$ that contains $-x$. For $P \in \mathscr{P}(M)$, let C_P denote the unique element of $\mathscr{C}$ that contains the chamber in $\mathfrak{A}_M$

determined by P. Then it follows from the discussion above that

$$a_B = \sum_{P=MN\in\mathscr{P}(M)} \varepsilon(C_0, C_P)\varphi_P(-x, p_M(\lambda_B + \rho_B - \rho_N) - \nu_P).$$

Now suppose that $\nu = \nu_m$. Then

$$p_M(\lambda_B + \rho_B - \rho_N) - \nu_P = p_M(\lambda_B + \rho_B - \lambda_0)$$

is independent of P. Since x is regular, it follows from (A.13) and Lemma A.4 that a_B is equal to

$$\sum_{C\in\mathscr{C}} \varepsilon(C_0, C)\psi_{\bar{C}}(-x, p_M(\lambda_B + \rho_B - \lambda_0)). \tag{5.11}$$

This is nothing but

$$(-1)^{\dim(A_G)}\psi_{R_\gamma}(C_0, -x, p_M(\lambda_B + \rho_B - \lambda_0)).$$

(Strictly speaking, we only defined the functions ψ_R for root systems spanning the vector space in which they lie, but, of course, the definition extends immediately to the general case.) Note that by (A.2) each term of (5.11) vanishes unless $p_M(\lambda_B + \rho_B - \lambda_0)$ belongs to the span of R_γ.

Now suppose that the highest weight of E satisfies the following property: for every proper Levi subgroup M of G, for every maximal torus T of M over $\mathbb{C}$ and for every $B \in \mathscr{B}(T)$, the element

$$\lambda_B + \rho_B - \lambda_0$$

of $X^*(T)$ is nontrivial on A_M.

THEOREM 5.3. *Assume that G contains no elliptic maximal torus over $\mathbb{R}$. Then, under the hypothesis above on the highest weight of E, the complex number $\Phi_M(\gamma, \Theta_{\nu_m})$ is 0, and, consequently, if F is $\mathbb{R}$, the virtual character Θ_{ν_m} is 0.*

Proof. By Theorem 5.1 we must show that L_M^ν is 0. By continuity it is enough to show this for $\gamma \in T(\mathbb{R})$ such that x is regular in $\mathfrak{A}_M$. Then, by the discussion above, it is enough to show that (5.11) vanishes. As in §4, let T_c denote the maximal anisotropic subtorus in T, and let J denote the centralizer in G of T_c and γ_1. Then R_γ is the root system R_J of T in J. Clearly T_c is central in J and J/T_c is a split group with split maximal torus T/T_c.

Since by hypothesis G/A_G has no anisotropic maximal torus over $\mathbb{R}$, the same is true of J/A_G and J/T_cA_G. Since J/T_cA_G is a split group, either A_J is strictly bigger than A_G or $A_J = A_G$ (in which case R_J spans $(\mathfrak{A}_M/\mathfrak{A}_G)^*$) and $-1_{\mathfrak{A}_M/\mathfrak{A}_G}$

does not belong to the Weyl group of R_J. In the first case $\mathfrak{A}_J$ is of the form $\mathfrak{A}_L$ for some proper Levi subgroup L of G containing M, and therefore our hypothesis on the highest weight of E implies that every term of (5.11) vanishes. In the second case, Corollary 1.3 (applied to the root system R_γ in $\mathfrak{A}_M/A_G$) implies that (5.11) is 0, since our hypothesis on the highest weight of E ensures that $p_M(\lambda_B + \rho_B - \lambda_0)$ is R_γ-regular. (Any intersection of root hyperplanes in $\mathfrak{A}_M/\mathfrak{A}_G$ for roots in R_γ is of the form $\mathfrak{A}_L/\mathfrak{A}_G$ for some Levi subgroup L of G containing M, and R_γ-regularity is equivalent to nonvanishing on every such nonzero intersection.)

6. Discrete series constants. In the beginning of §4 (see (4.1)–(4.3)), we reviewed the form taken by the character of an irreducible representation of a real reductive group. In this section we are concerned with the case of discrete series representations. We then refer to the integers $n(\gamma, B)$ appearing in (4.1) as *discrete series constants.* (However, we will no longer use the notation $n(\gamma, B)$.) In this section we give a simple formula for the discrete series constants. Because of a descent property satisfied by the constants (see [K, 13.4]), it is enough to give the formula in the following special case.

Let G be a split semisimple simply connected group over $\mathbb{R}$, and assume that G contains an anisotropic maximal torus T_e. Let A be a split maximal torus in G. We choose an isomorphism $A \simeq T_e$ over $\mathbb{C}$ that is induced by an inner automorphism of G over $\mathbb{C}$ and use it to identify the character groups of A and T_e. We put

$$X^* := X^*(T_e)_{\mathbb{R}} \simeq X^*(A)_{\mathbb{R}}$$

and

$$X := X_*(T_e)_{\mathbb{R}} \simeq X_*(A)_{\mathbb{R}}.$$

The roots and coroots of T_e in G give us a root system $(X, X^*, R, R^\vee)$. Of course the set R spans X^*, and -1 belongs to the Weyl group $W = W(R)$.

Let τ be a regular element in $X^*(T_e)$. Associated to τ is a discrete series representation $\pi(\tau)$ of $G(\mathbb{R})$ having infinitesimal character τ and having the same central character as the finite-dimensional representation having infinitesimal character τ. We are interested in the constants needed to express the values of the character of $\pi(\tau)$ at regular elements in the identity component $A(\mathbb{R})^0$ of $A(\mathbb{R})$; this is the special case alluded to above.

We need a little preparation before we can state our formula for these constants. There is a unique maximal compact subgroup K of $G(\mathbb{R})$ containing $T_e(\mathbb{R})$, and the roots of T_e in K form a subset R_c of R (such roots are said to be *compact*). We write W_c for the Weyl group of R_c and identify it with a subgroup of W. It is not hard to see that the normalizer $\widetilde{W}_c$ of W_c in W is given by

$$\widetilde{W}_c = \{w \in W \mid w(R_c) = R_c\}.$$

Let C be a chamber in X. The chamber C determines a subset R_C of R in the following way. Let $\delta_C \in X$ denote the half-sum of the coroots that are positive for C, and put

$$R_C = \{\alpha \in R \mid \alpha(\delta_C) \in 2\mathbb{Z}\};$$

note that no simple root (for C) belongs to R_C. We denote by W_C the Weyl group of R_C. We identify W_C with a subgroup of W, and let $\widetilde{W}_C$ denote its normalizer in W. As before we write $C^\vee$ for the Weyl chamber in X^* corresponding to C.

It is known (see [AV, 6.24(f)]) that there exists a chamber C such that R_C equals R_c. For such a chamber C we have $W_C = W_c$ and $\widetilde{W}_C = \widetilde{W}_c$. The $\widetilde{W}_c$-orbit of C is uniquely determined by the condition that R_C equal R_c.

As before, let τ be a regular element of $X^*(T_e)$. (Actually, in the definition we are about to make, we could just as well let τ be any regular element in $X^*(T_e)_{\mathbb{R}} = X^*$.) Let C be a chamber in X. (For the time being we do *not* assume that $R_C = R_c$.) Let x be an $R^\vee$-regular element of X, and let λ be an element of X^* lying in the W-orbit $W \cdot \tau$ of τ (see §1 for the definition of $R^\vee$-regularity). We define an integer $b_R(\tau, C; x, \lambda)$ by

$$b_R(\tau, C; x, \lambda) = (-1)^{q(R)} \sum_{w \in W(\tau, C, \lambda)} \varepsilon(x, wC)\psi_{wC^\vee}(\lambda, x). \tag{6.1}$$

In the case $R_C = R_c$ these constants (for $\lambda \in W \cdot \tau$) are the ones needed to express the value of the character of $\pi(\tau)$ at the point $a \in A(\mathbb{R})^0$ obtained from $x \in X$ via the exponential map. (We have identified $X_*(A)$ with $X_*(T_e)$, and thus we may view X as the Lie algebra of $A(\mathbb{R})$.) However, for technical reasons it is best to define $b_R(\tau, C; x, \lambda)$ for *any* chamber C.

The expression (6.1) requires some explanation. The integer

$$q(R) = [|R^+| + \dim(X)]/2$$

was used already in §2, and its interpretation in terms of G was also given there. The index set for the sum is the coset

$$W(\tau, C, \lambda) := \{w \in W \mid w^{-1}\lambda \in W_C \cdot \tau\}$$

of W_C in W. (Of course, $W_C \cdot \tau$ denotes the orbit of τ under W_C.) As in §1, for any two chambers C_1, C_2 in X we write $\varepsilon(C_1, C_2)$ for the sign of the Weyl group element w such that $wC_1 = C_2$. The sign $\varepsilon(x, wC)$ appearing in (6.1) is by definition $\varepsilon(C_x, wC)$, where C_x denotes the unique chamber in X containing the (regular) element $x \in X$. Finally $\psi_{wC^\vee}(\lambda, x)$ is the function of $(\lambda, x) \in X^* \times X$ defined in §1; it is obtained from the coroot system $R^\vee$ and the Weyl chamber $wC^\vee$ in X^*.

Of course, the integer $b_R(\tau, C; x, \lambda)$ depends only on the $R^\vee$-chamber of X in which x lies. But in fact we claim that $b_R(\tau, C; x, \lambda)$ depends only on the Weyl chamber of X in which x lies (x is still assumed to be $R^\vee$-regular). Indeed it follows from Lemma 1.5 (applied to $R^\vee$) that for all $s \in \hat{A}_{sc}$ such that $s^2 \in Z^\vee$

$$\sum_D \varepsilon(D_0^\vee, D^\vee)\langle \delta_D - \delta_{D_0}, s\rangle \psi_{D^\vee}(\lambda, x) \tag{6.2}$$

depends only on the Weyl chamber in which x lies. Here we are using the notation δ_D, $\hat{A}_{sc}$, s, $Z^\vee$ of Lemma 1.4 (since we are applying Lemma 1.5 to $R^\vee$ rather than R, we need the notation of Lemma 1.4), and we have applied Lemma 1.2 to the root system $R_s^\vee$ of Lemma 1.4. Summing (6.2) over all $s \in \hat{A}_{sc}$ such that $s^2 \in Z^\vee$, we find that

$$\sum_D \varepsilon(D_0^\vee, D^\vee)\psi_{D^\vee}(\lambda, x) \tag{6.3}$$

depends only on the Weyl chamber in which x lies, where the sum is now taken over all chambers D such that the element $\delta_D - \delta_{D_0} \in Q$ lies in $2Q$. (Here $Q \subset X$ denotes the lattice generated by $R^\vee$.) This set of chambers can also be described as the set of chambers wD_0 where w ranges through the stabilizer in W of the element $\delta_{D_0} \in ((1/2)Q)/2Q$. We can think of $((1/2)Q)/2Q$ as consisting of 4-torsion elements in a maximal torus A_{sc} in the semisimple simply connected complex group G_{sc} with root system R, and by a theorem of Steinberg, this stabilizer is the Weyl group of the root system of the centralizer of $\delta_{D_0} \in A_{sc}$ in G_{sc}, namely,

$$\{\alpha \in R \mid \langle \delta_{D_0}, \alpha\rangle \in 2\mathbb{Z}\} = R_{D_0}.$$

Therefore the sum in (6.3) is over wD_0 ($w \in W_{D_0}$), and our claim has been proved. (Take $D_0 = w_0 C$ for any $w_0 \in W(\tau, C, \lambda)$.)

In order to prove that the integers $b_R(\tau, C; x, \lambda)$ are the ones appearing in the character formula for $\pi(\tau)$ on $A(\mathbb{R})^0$, we must show that they satisfy various properties. The first is that

$$b_R(\tau, C; x, \lambda) = b_R(\tau, wC; x, \lambda) \qquad \text{for all } w \in \tilde{W}_C. \tag{6.4}$$

(In other words $b_R(\tau, C; x, \lambda)$ only depends on the subset R_C determined by C.) It is trivial that (6.4) holds for all $w \in W_C$, but to prove it for $w \in \tilde{W}_C$, we need to use Lemma 1.5 (applied to $R^\vee$), just as in the previous proof. Indeed, by Lemma 1.5 the expression (6.2) vanishes unless $s^2 = 1$. Therefore summing (6.2) over $\{s \in \hat{A}_{sc} \mid s^2 = 1\}$ yields the same result as summing over $\{s \in \hat{A}_{sc} \mid s^2 \in Z^\vee\}$. It follows that (drop the subscript 0 from D)

$$\sum_{w \in W_D} \varepsilon(w)\psi_{wD^\vee}(\lambda, x)$$

is equal to

$$|Z^\vee|^{-1} \sum_{w \in \tilde{W}_D} \varepsilon(w) \psi_{wD^\vee}(\lambda, x),$$

and it is clear that this expression is multiplied by $\varepsilon(w)$ if D is replaced by wD for $w \in \tilde{W}_D$. This proves (6.4). (Take $D = w_0 C$ for any $w_0 \in W(\tau, C, \lambda)$.)

The next two properties of $b_R(\tau, C; x, \lambda)$ are obvious:

(6.5) $$b_R(w\tau, wC; x, \lambda) = b_R(\tau, C; x, \lambda) \qquad \text{for all } w \in W;$$

(6.6) $$b_R(\tau, C; wx, w\lambda) = b_R(\tau, C; x, \lambda) \qquad \text{for all } w \in W.$$

Moreover, it follows from Proposition A.5 that

$$b_R(\tau, C; x, \lambda) = 0 \qquad \text{unless } \lambda(\mathrm{x}) \leqslant 0,$$

and since $b_R(\tau, C; x, \lambda)$ depends only on the chamber C_x containing x, we find that

(6.7) $$b_R(\tau, C; x, \lambda) = 0 \qquad \text{unless } \lambda \leqslant 0 \text{ on } C_x.$$

There is one more elementary property of the constants:

(6.8) $$b_R(\tau, C; x, \lambda) = 1 \qquad \text{if } R \text{ is empty}.$$

The last property we need requires a bit more work. Suppose that $\alpha \in R$, and put $Y = \ker(\alpha) \subset X$. Define R_α, $R_\alpha^\vee$ as in the discussion at the beginning of §2. Recall that R_α generates Y^* and that $-1_Y \in W(R_\alpha)$. Let $s = s_\alpha \in W$ be the reflection in the root α. Assume further that C is a chamber in X such that α belongs to the closure of $C^\vee$. Let x, x' be $R^\vee$-regular elements in X that lie in adjacent chambers separated by the wall Y. We are going to derive a formula for

$$b_R(\tau, C; x, \lambda) + b_R(\tau, C; x', \lambda)$$

in terms of the constants b_{R_α} associated to the root system R_α.

To get a clean formula, we need to use the constants for the root system R_α to define constants $b_{R_\alpha}^R(\tau, C; y, \lambda)$ for $R_\alpha^\vee$-regular $y \in Y$ and $\tau \in X^*$, $\lambda \in W \cdot \tau$, C as before (subject to the requirement that α belongs to the closure of $C^\vee$). Write W_α for the Weyl group of R_α. Then we define

$$b_{R_\alpha}^R(\tau, C; y, \lambda) = 0 \qquad \text{unless } \lambda \in W_\alpha W_C \cdot \tau.$$

If λ does belong to $W_\alpha W_C \cdot \tau$, choose $\tau' \in W_C \cdot \tau$ such that $\lambda \in W_\alpha \cdot \tau'$, and put

$$b_{R_\alpha}^R(\tau, C; y, \lambda) = b_{R_\alpha}(\tilde{\tau}', C_Y; y, \tilde{\lambda}).$$

Here $\tilde{\tau}', \tilde{\lambda} \in Y^*$ denote the restrictions of $\tau', \lambda \in X^*$ to Y and C_Y is the chamber in Y determined by C. (Thus $C_Y = Y \cap \tilde{C}$, where $\tilde{C}$ is the unique chamber in X relative to R_α that contains C.) Since τ' is well determined up to an element of $W_\alpha \cap W_C = W_{C_Y}$, we see from (6.4) that $b_{R_\alpha}(\tilde{\tau}', C_Y; y, \tilde{\lambda})$ is independent of the choice of τ'. (The equality $W_\alpha \cap W_C = W_{C_Y}$ is a consequence of our assumption that α belongs to the closure of $C^\vee$.) It is easy to see that for any $\lambda \in W \cdot \tau$, we have the formula

$$(6.9) \qquad b^R_{R_\alpha}(\tau, C; y, \lambda) = (-1)^{q(R_\alpha)} \sum_{w \in W_\alpha \cap W(\tau, C, \lambda)} \varepsilon(y, wC_Y) \psi_{wC_Y^\vee}(\tilde{\lambda}, y).$$

Now we are ready to formulate the last property of our constants: for α, s, C, x, x' as above

$$(6.10) \qquad b_R(\tau, C; x, \lambda) + b_R(\tau, C; x', \lambda) = b^R_{R_\alpha}(\tau, C; y, \lambda) + b^R_{R_\alpha}(\tau, C; y, s\lambda),$$

where y is the unique point of Y lying on the line segment joining x and x'. Note that by (6.6) the left-hand side of (6.10) can also be written as

$$b_R(\tau, C; x, \lambda) + b_R(\tau, C; x, s\lambda).$$

Let us now prove (6.10). Assume without loss of generality that $\alpha(x) > 0$ and $\alpha(x') < 0$. Then by Corollary A.3 the left-hand side of (6.10) is equal to

$$(-1)^{q(R)} \sum_w \varepsilon(x, wC) \cdot \psi_{(wC)_Y^\vee}(\tilde{\lambda}, y) \cdot \eta(w),$$

where the sum is taken over the set of all $w \in W(\tau, C, \lambda)$ such that the closure of $wC^\vee$ contains either α or $-\alpha$, and where $\eta(w) = \pm 1$ is defined by

$$\eta(w) = \begin{cases} -1 & \text{if } w\bar{C}^\vee \text{ contains } \alpha, \\ 1 & \text{if } w\bar{C}^\vee \text{ contains } -\alpha. \end{cases}$$

Note that $w\bar{C}^\vee$ contains α if and only if $w \in W_\alpha$, and $w\bar{C}^\vee$ contains $-\alpha$ if and only if $sw \in W_\alpha$ (since $s\alpha = -\alpha$). Looking back at the proof of Lemma 2.2, we see that if $w \in W_\alpha$, then

$$-(-1)^{q(R)} \varepsilon(x, wC) = (-1)^{q(R_\alpha)} \varepsilon(y, wC_Y).$$

Therefore the left-hand side of (6.10) is equal to the difference of

$$(-1)^{q(R_\alpha)} \sum_{w \in W_\alpha \cap W(\tau, C, \lambda)} \varepsilon(y, wC_Y) \psi_{wC_Y^\vee}(\tilde{\lambda}, y)$$

and

$$(-1)^{q(R_\alpha)} \sum_{w \in sW_\alpha \cap W(\tau, C, \lambda)} \varepsilon(y, wC_Y)\psi_{(wC)_Y^\vee}(\tilde{\lambda}, y).$$

Replacing w by sw in the second sum (and using that $(swC)_Y = (wC)_Y$ and $s\tilde{\lambda} = \tilde{\lambda}$), we see that the left-hand side of (6.10) is equal to the right-hand side of (6.10), as we wished to show.

The constants b_R are determined uniquely by properties (6.4)–(6.8) and (6.10) (and the property that $b_R(\tau, C; x, \lambda)$ depends only on the chamber in which x lies). To see this, fix τ,C,λ and regard $b_R(\tau, C; x, \lambda)$ as a function of x. If R is empty, b_R is given by (6.8). If it is nonempty, the value of $b_R(\tau, C; x, \lambda)$ is given by (6.7) for x in at least one chamber in X. Therefore it is enough to know

$$b_R(\tau, C; x, \lambda) + b_R(\tau, C; x', \lambda)$$

whenever x, x' lie in adjacent chambers. But by (6.5) (which we use to put C in a good position relative to the wall separating x, x') and (6.10), the sum above can be written in terms of the constants for a root system of lower rank, which we may assume have already been determined.

It is known (see [K, 13.4]) that the discrete series constants satisfy these same properties; therefore they are equal to the constants $b(\tau, C; x, \lambda)$. Before making this statement more precise, we need to change the indexing of our constants in order to facilitate comparison with [K]. We now fix a chamber C such that $R_C = R_c$ and define constants $c(w, \lambda, \Delta^+)$ as follows. For a regular element $\lambda \in X^*$, $w \in W$, and a system Δ^+ of positive roots for R, we put

$$c(w, \lambda, \Delta^+) := b(\lambda, C; x, w\lambda),$$

where $x \in X$ is any $R^\vee$-regular element in the (positive) Weyl chamber in X determined by Δ^+. It follows from (6.4) that the right side in this definition is independent of the choice of chamber C such that $R_C = R_c$. The constants $c(w, \lambda, \Delta^+)$ are those in [K] (see properties (13.32)–(13.34) in [K]). In other words, our τ corresponds to Knapp's λ, and our λ corresponds to Knapp's $w\lambda$.

7. Lefschetz formula on reductive Borel-Serre compactifications

7.1. The group G. Let G be a connected reductive group over $\mathbb{Q}$, and let A_G denote the maximal $\mathbb{Q}$-split torus in the center of G. Choose a maximal $\mathbb{Q}$-split torus A_0 in G, and let M_0 denote its centralizer, a Levi subgroup of G. Fix a parabolic subgroup P_0 of G over $\mathbb{Q}$ having M_0 as Levi component; then P_0 is a minimal parabolic subgroup of G over $\mathbb{Q}$. See §5 for notation and terminology concerning parabolic and Levi subgroups. In particular, for any standard parabolic subgroup P, we write M for the unique Levi component of P containing M_0, and N for the unipotent radical of P; thus $P = MN$.

7.2. The locally symmetric spaces S_K. Let K be a suitably small compact open subgroup of $G(\mathbb{A}_f)$. Choose a maximal compact subgroup K_G of $G(\mathbb{R})$ in good position relative to M_0 in the sense that the Cartan involution on G associated to K_G preserves M_0. For each standard parabolic subgroup $P = MN$, we denote by K_M the intersection of K_G with $M(\mathbb{R})$, a maximal compact subgroup in $M(\mathbb{R})$. We denote by $A_G(\mathbb{R})^0$ the identity component of the topological group $A_G(\mathbb{R})$, and we denote by X_G the homogeneous space

$$G(\mathbb{R})/(K_G \cdot A_G(\mathbb{R})^0)$$

for $G(\mathbb{R})$. We then denote by S_K the space

$$G(\mathbb{Q})\backslash[(G(\mathbb{A}_f)/K) \times X_G].$$

7.3. The local system $\mathbf{E}_K$ on S_K. Let E be an irreducible representation of the algebraic group G on a finite-dimensional complex vector space. Then E gives rise to a local system $\mathbf{E}_K$ on S_K. By definition $\mathbf{E}_K$ is the sheaf of flat sections of the flat vector bundle

$$G(\mathbb{Q})\backslash[(G(\mathbb{A}_f)/K) \times X_G \times E]$$

over S_K.

7.4. The Hecke correspondence (c_1, c_2) on S_K. Now fix an element g in $G(\mathbb{A}_f)$, and let K' be any compact open subgroup of $G(\mathbb{A}_f)$ that is contained in $K \cap g^{-1}Kg$. We use g, K' to form a Hecke correspondence on S_K as follows. The inclusion $K' \subset K$ induces a surjection

$$c_1 \colon S_{K'} \to S_K.$$

The inclusion $K' \subset g^{-1}Kg$ induces a surjection

$$S_{K'} \to S_{g^{-1}Kg},$$

which we compose with the canonical isomorphism (use the element g)

$$S_{g^{-1}Kg} \simeq S_K$$

to get a second surjection

$$c_2 \colon S_{K'} \to S_K.$$

There are canonical isomorphisms

$$c_1^*\mathbf{E}_K \simeq \mathbf{E}_{K'} \simeq c_2^*\mathbf{E}_K. \tag{7.4.1}$$

7.5. The reductive Borel-Serre compactification $\bar{S}_K$ of S_K. For any standard parabolic subgroup $P = MN$, we denote by S_K^P the space

$$S_K^P := M(\mathbb{Q})\backslash[(N(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K) \times X_M],$$

where X_M denotes the analog for M of X_G, namely, $M(\mathbb{R})/(K_M \cdot A_M(\mathbb{R})^0)$. Now we can make precise what it means for K to be suitably small: we require that for each standard P the group $M(\mathbb{Q})$ acts freely on

$$(N(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K) \times X_M.$$

The reductive Borel-Serre compactification (see [GHM]) $\bar{S}_K$ of S_K is a stratified space whose strata are indexed by the standard parabolic subgroups P of G, the stratum indexed by P being the manifold S_K^P described above.

7.6. The weighted cohomology complex $\bar{\mathbf{E}}_K$ on $\bar{S}_K$. Let p be a weight profile (see §1.1 of [GHM]). Associated to the representation E and the weight profile p is a constructible complex of sheaves $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$ of complex vector spaces on $\bar{S}_K$ (see §1.3 of [GHM]). In this paper we will denote this complex of sheaves by $\bar{\mathbf{E}}_K$; as the notation suggests, the restriction of $\bar{\mathbf{E}}_K$ to S_K may be identified with $\mathbf{E}_K$.

7.7. The Hecke correspondence $(\bar{c}_1, \bar{c}_2)$ on $\bar{S}_K$. The maps

$$c_1, c_2 \colon S_{K'} \to S_K$$

have unique continuous extensions

$$\bar{c}_1, \bar{c}_2 \colon \bar{S}_{K'} \to \bar{S}_K.$$

These maps carry $S_{K'}^P$ onto S_K^P; in fact, representing points of $S_{K'}^P$ by pairs (x, x_∞), where $x \in G(\mathbb{A}_f)$ and $x_\infty \in X_M$, we have that the image of the pair (x, x_∞) under $\bar{c}_1$ (respectively, $\bar{c}_2$) is the point of S_K^P represented by (x, x_∞) (respectively, (xg^{-1}, x_∞)).

It follows from the definition of weighted cohomology complexes that there are canonical isomorphisms

$$\bar{c}_1^*\bar{\mathbf{E}}_K \simeq \bar{\mathbf{E}}_{K'} \simeq \bar{c}_2^*\bar{\mathbf{E}}_K. \tag{7.7.1}$$

The Verdier dual of the weighted cohomology complex $\bar{\mathbf{E}}_K$ is (a shift of) the weighted cohomology complex obtained from the contragredient of the representation E and the weight profile $\bar{p}$ dual to p (see §1.3 of [GHM]). Thus, applying Verdier duality to (7.7.1), we find that there are canonical isomorphisms

$$\bar{c}_1^!\bar{\mathbf{E}}_K \simeq \bar{\mathbf{E}}_{K'} \simeq \bar{c}_2^!\bar{\mathbf{E}}_K. \tag{7.7.2}$$

It follows that there is a canonical isomorphism

$$\bar{c}_2^*\bar{\mathbf{E}}_K \to \bar{c}_1^!\bar{\mathbf{E}}_K, \tag{7.7.3}$$

obtained as the composition of the isomorphism (7.7.1) from $\bar{c}_2^*\bar{\mathbf{E}}_K$ to $\bar{\mathbf{E}}_{K'}$ and the isomorphism (7.7.2) from $\bar{\mathbf{E}}_{K'}$ to $\bar{c}_1^!\bar{\mathbf{E}}_K$. Thus there is a canonical extension, namely, the morphism (7.7.3), of the Hecke correspondence $(\bar{c}_1, \bar{c}_2)$ to the weighted cohomology complex $\bar{\mathbf{E}}_K$.

7.8. The goal. The canonical morphism (7.7.3) induces self-maps on hypercohomology groups

$$H^i(\bar{S}_K, \bar{\mathbf{E}}_K) \to H^i(\bar{S}_K, \bar{\mathbf{E}}_K). \tag{7.8.1}$$

These maps are obtained as the composition of the canonical pullback map

$$H^i(\bar{S}_K, \bar{\mathbf{E}}_K) \to H^i(\bar{S}_{K'}, \bar{c}_2^*\bar{\mathbf{E}}_K),$$

the map

$$H^i(\bar{S}_{K'}, \bar{c}_2^*\bar{\mathbf{E}}_K) \to H^i(\bar{S}_{K'}, \bar{c}_1^!\bar{\mathbf{E}}_K)$$

induced by (7.7.3), and the canonical proper pushforward map

$$H^i(\bar{S}_{K'}, \bar{c}_1^!\bar{\mathbf{E}}_K) \to H^i(\bar{S}_K, \bar{\mathbf{E}}_K).$$

The Lefschetz fixed-point formula is a formula for the alternating sum of the traces of the self-maps (7.8.1). An explicit version of the Lefschetz formula (for the case at hand) is given in the theorem on page 474 of [GM1]; our goal here is to rewrite that formula in terms of stable virtual characters on the group $G(\mathbb{R})$ using the results in §5 of this paper.

7.9. Fixed points. First we need to determine the fixed points of the correspondence. Of course, a fixed point is an element x of $\bar{S}_{K'}$ such that $\bar{c}_1(x) = \bar{c}_2(x)$. Let us fix a standard parabolic subgroup $P = MN$ and determine the fixed points of the correspondence that lie in the subset $S_{K'}^P$ of $\bar{S}_{K'}$. The group $P(\mathbb{A}_f)$ acts on $G(\mathbb{A}_f)/K'$ with finitely many orbits. Choose a set of representatives $x_0 \in G(\mathbb{A}_f)$ for these orbits and put

$$K'_P(x_0) = P(\mathbb{A}_f) \cap x_0 K' x_0^{-1}$$

$$K'_M(x_0) = \text{image of } K'_P(x_0) \text{ in } M(\mathbb{A}_f).$$

Then $S^P_{K'}$ is the disjoint union of the subsets

$$S^P_{K'}(x_0) := M(\mathbb{Q})\backslash[(M(\mathbb{A}_f)/K'_M(x_0)) \times X_M],$$

the disjoint union being indexed by the set of representatives x_0 chosen above.

A pair $(y, y_\infty) \in M(\mathbb{A}_f) \times X_M$ represents a fixed point in $S^P_{K'}(x_0)$ of our Hecke correspondence if and only if there exists $\gamma \in M(\mathbb{Q})$ such that

(1) $\gamma y_\infty = y_\infty$, and

(2) there exists $n \in N(\mathbb{A}_f)$ such that $y^{-1}n\gamma y \in x_0 K g x_0^{-1}$.

The conjugacy class of γ in $M(\mathbb{Q})$ depends only on the fixed point we started with. Now fix an element $\gamma \in M(\mathbb{Q})$, and denote by $\mathrm{Fix}(P, x_0, \gamma)$ the subset of $S^P_{K'}(x_0)$ consisting of all fixed points of our correspondence for which the associated conjugacy class in $M(\mathbb{Q})$ is equal to that of γ. The discussion above shows that $\mathrm{Fix}(P, x_0, \gamma)$ is equal to

$$M_\gamma(\mathbb{Q})\backslash(Y^\infty \times Y_\infty),$$

where M_γ denotes the centralizer of γ in M, Y^∞ denotes the subset of $M(\mathbb{A}_f)/K'_M(x_0)$ consisting of elements in that set represented by elements $y \in M(\mathbb{A}_f)$ such that $y^{-1}\gamma y$ belongs to the image in $M(\mathbb{A}_f)$ of $P(\mathbb{A}_f) \cap x_0 K g x_0^{-1}$, and Y_∞ denotes the set of fixed points of γ in X_M.

The group $M_\gamma(\mathbb{Q})$ acts freely on $Y^\infty \times Y_\infty$. We write I for the identity component of M_γ. The group $I(\mathbb{A}_f)$ acts on Y^∞ with finitely many orbits. The space Y_∞ is empty unless γ is conjugate in $M(\mathbb{R})$ to an element of $K_M \cdot A_M(\mathbb{R})^0$, in which case $I(\mathbb{R})$ acts transitively on Y_∞ and in fact

$$Y_\infty = I(\mathbb{R})/(K_I \cdot A_M(\mathbb{R})^0)$$

for some maximal compact subgroup K_I of $I(\mathbb{R})$. In line with our usual notational conventions, we write X_I for the homogeneous space

$$I(\mathbb{R})/(K_I \cdot A_I(\mathbb{R})^0);$$

note that Y_∞ maps onto X_I and is in fact a principal fiber bundle over that space for the vector group

$$A_I(\mathbb{R})^0/A_M(\mathbb{R})^0.$$

7.10. *Euler characteristic of S_K.* We need to recall Harder's formula (see [H]) for the Euler characteristic of the space S_K. (We should also note that this Euler characteristic coincides with the Euler characteristic with compact support of S_K.) Harder's formula involves several ingredients, which we now explain. Let us choose a Haar measure dg_f on $G(\mathbb{A}_f)$. Then the Euler characteristic of S_K has the form

$$\chi(G) \cdot \mathrm{vol}(K)^{-1}.$$

Of course, $\mathrm{vol}(K)$ denotes the measure of K with respect to dg_f. The quantity $\chi(G)$ depends on G and the Haar measure dg_f, but not on K. Moreover, $\chi(G)$ is 0 unless the group G has a maximal torus T over $\mathbb{R}$ such that T/A_G is anisotropic over $\mathbb{R}$. Assume now that this condition is satisfied. Let $\mathscr{D}(G)$ denote the finite set

$$\mathscr{D}(G) := \ker[H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)].$$

(As usual we write $H^1(\mathbb{R}, G)$ as an abbreviation for $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), G(\mathbb{C}))$.) Since G/A_G has an anisotropic maximal torus over $\mathbb{R}$, there is an inner form $\bar{G}$ of G over $\mathbb{R}$ such that $\bar{G}/A_G$ is anisotropic over $\mathbb{R}$. We pick a Haar measure dg_∞ on $G(\mathbb{R})$ and transport it to the inner form $\bar{G}(\mathbb{R})$ in the usual way by identifying the space of invariant top-degree differential forms on G with the analogous space for $\bar{G}$. (This identification is defined over $\mathbb{R}$ since $\bar{G}$ is an *inner* form of G.) We define $q(G)$ to be half the real dimension of the symmetric space associated to the real points of the adjoint group of G. Then $\chi(G)$ is equal to

$$(-1)^{q(G)} \mathrm{vol}(G(\mathbb{Q})A_G(\mathbb{R})^0 \backslash G(\mathbb{A})) \ \mathrm{vol}(A_G(\mathbb{R})^0 \backslash \bar{G}(\mathbb{R}))^{-1} |\mathscr{D}(G)|.$$

Of course, we use dg_f and dg_∞ to get a Haar measure on $G(\mathbb{A})$; note that $\chi(G)$ is independent of the choice of Haar measures on $G(\mathbb{R})$ and $A_G(\mathbb{R})^0$.

7.11. Euler characteristic with compact support of $\mathrm{Fix}(P, x_0, \gamma)$. Assume that Y_∞ is nonempty. The Euler characteristic with compact support of the space $\mathrm{Fix}(P, x_0, \gamma)$ (which is one of the ingredients in the Lefschetz formula) is equal to

$$|M_\gamma(\mathbb{Q})/I(\mathbb{Q})|^{-1}$$

times the Euler characteristic with compact support of

$$I(\mathbb{Q}) \backslash (Y^\infty \times Y_\infty),$$

and this latter Euler characteristic with compact support is equal to $(-1)^{\dim(A_I/A_M)}$ times that of

$$I(\mathbb{Q}) \backslash (Y^\infty \times X_I),$$

since the natural surjection

$$I(\mathbb{Q}) \backslash (Y^\infty \times Y_\infty) \to I(\mathbb{Q}) \backslash (Y^\infty \times X_I)$$

is a principal fiber bundle under the vector group

$$A_I(\mathbb{R})^0/A_M(\mathbb{R})^0.$$

It follows from Harder's theorem (see (7.10)) that the Euler characteristic with compact support of $\mathrm{Fix}(P, x_0, \gamma)$ is equal to

$$(-1)^{\dim(A_I/A_M)}|M_\gamma(\mathbb{Q})/I(\mathbb{Q})|^{-1}\cdot\chi(I)\cdot\sum_y \mathrm{vol}(I(\mathbb{A}_f)\cap yK'_M(x_0)y^{-1})^{-1},$$

where the index set for the sum is the subset of

$$I(\mathbb{A}_f)\backslash M(\mathbb{A}_f)/K'_M(x_0)$$

consisting of elements that can be represented by an element $y\in M(\mathbb{A}_f)$ such that $y^{-1}\gamma y$ belongs to the image in $M(\mathbb{A}_f)$ of $P(\mathbb{A}_f)\cap x_0Kgx_0^{-1}$. Of course, we have chosen a Haar measure di_f on $I(\mathbb{A}_f)$. Let us fix a Haar measure dm on $M(\mathbb{A}_f)$ as well. Define a locally constant compactly supported function f_{P,x_0} on $M(\mathbb{A}_f)$ as follows: f_{P,x_0} is $\mathrm{vol}(K'_M(x_0))^{-1}$ times the characteristic function of the image in $M(\mathbb{A}_f)$ of $P(\mathbb{A}_f)\cap x_0Kgx_0^{-1}$. For any locally constant compactly supported function f on $M(\mathbb{A}_f)$, write $O_\gamma(f)$ for the orbital integral

$$\int_{I(\mathbb{A}_f)\backslash M(\mathbb{A}_f)} f(m^{-1}\gamma m)\,dm/di_f.$$

Then

$$O_\gamma(f_{P,x_0})=\sum_y \mathrm{vol}(I(\mathbb{A}_f)\cap yK'_M(x_0)y^{-1})^{-1}$$

with the same index set as above. Therefore the Euler characteristic with compact support of $\mathrm{Fix}(P, x_0, \gamma)$ is equal to

$$(-1)^{\dim(A_I/A_M)}|M_\gamma(\mathbb{Q})/I(\mathbb{Q})|^{-1}\cdot\chi(I)\cdot O_\gamma(f_{P,x_0}). \tag{7.11.1}$$

7.12. Lefschetz formula (qualitative version). We now need to recall the general form taken by the Lefschetz formula in [GM1]. The formula is a sum of contributions, one for each connected component C of the fixed-point set of the Hecke correspondence. We further decompose each such connected component into locally closed pieces

$$C_P := C\cap S^P_{K'}.$$

There are two natural ways to break up the contribution of C to the Lefschetz formula as a sum of contributions from the pieces C_P. In [GM1] one of these two ways was chosen; it leads to the version of the Lefschetz formula given in that paper. However, here we use the other version.

This alternative version differs in two respects from the one chosen in [GM1]. The first is that it involves the Euler characteristic with compact support of $\mathrm{Fix}(P, x_0, \gamma)$ (rather than its Euler characteristic). The second is that neutral directions are treated as being contracting (rather than expanding); this change affects the definition of the set $I(\gamma)$ appearing in (7.14), as we explain in more detail when we make the definition.

The subset $\mathrm{Fix}(P, x_0, \gamma)$ of the fixed-point set is a disjoint union of certain sets of the form C_P, and from [GM1] we see that the total contribution of $\mathrm{Fix}(P, x_0, \gamma)$ to the Lefschetz formula is given by the product of three factors:

(1) the Euler characteristic with compact support of $\mathrm{Fix}(P, x_0, \gamma)$;

(2) the ramification index

$$r(x_0) := [N(\mathbb{A}_f) \cap x_0 K x_0^{-1} \colon N(\mathbb{A}_f) \cap x_0 K' x_0^{-1}]$$

of the map $\bar{c}_1$ at any point in $S^P_{K'}(x_0)$;

(3) a factor $L_P(\gamma)$ that depends only on the $G(\mathbb{R})$-conjugacy class of the pair (P, γ) (and, of course, the representation E and the weight profile p as well).

We will review the precise form of the factor $L_P(\gamma)$ later. All that matters for the moment is the property stated in (3). The discussion above shows that the Lefschetz formula (for the alternating sum of the traces of the self-maps (7.8.1)) is given by the following sum

$$\sum_P \sum_\gamma (-1)^{\dim(A_I/A_M)} \cdot |M_\gamma(\mathbb{Q})/I(\mathbb{Q})|^{-1} \cdot \chi(I) \cdot L_P(\gamma) \cdot O_\gamma(f_P), \tag{7.12.1}$$

where f_P is the locally constant compactly supported function on $M(\mathbb{A}_f)$ defined by

$$f_P := \sum_{x_0} r(x_0) f_{P,x_0}. \tag{7.12.2}$$

In the sum defining f_P, the index x_0 runs over a set of representatives for the orbits of $P(\mathbb{A}_f)$ on $G(\mathbb{A}_f)/K'$, as before. In the first sum in (7.12.1), P runs through the standard parabolic subgroups of G, and in the second sum, γ runs through the set of $M(\mathbb{Q})$-conjugacy classes of elements $\gamma \in M(\mathbb{Q})$ such that the fixed-point set of γ in X_M is nonempty.

7.13. Some familiar harmonic analysis. Let $P = MN$ be a standard parabolic subgroup of G. In (7.11) we fixed a Haar measure dm on $M(\mathbb{A}_f)$. Now we fix a Haar measure dg on $G(\mathbb{A}_f)$ as well. Pick a compact open subgroup K_0 of $G(\mathbb{A}_f)$

such that

$$G(\mathbb{A}_f) = P(\mathbb{A}_f)K_0.$$

Choose Haar measures dn on $N(\mathbb{A}_f)$, and choose dk on K_0 so that the usual integration formula holds:

$$\int_{G(\mathbb{A}_f)} f(g)\,dg = \int_{M(\mathbb{A}_f)} \int_{N(\mathbb{A}_f)} \int_{K_0} f(mnk)\,dk\,dn\,dm \tag{7.13.1}$$

for any f in $C_c^\infty(G(\mathbb{A}_f))$, the space of all locally constant compactly supported functions on $G(\mathbb{A}_f)$. Let $\delta_{P(\mathbb{A}_f)}$ denote the modulus function on $P(\mathbb{A}_f)$; thus, for $x \in P(\mathbb{A}_f)$ we have

$$\delta_{P(\mathbb{A}_f)}(x) := |\det(\mathrm{Ad}(x); \mathrm{Lie}(N) \otimes \mathbb{A}_f)|_{\mathbb{A}_f},$$

where $|\cdot|_{\mathbb{A}_f}$ is the normalized absolute value on $\mathbb{A}_f^\times$. Given $f \in C_c^\infty(G(\mathbb{A}_f))$ we define a function $f_M \in C_c^\infty(M(\mathbb{A}_f))$ in the usual way, by putting

$$f_M(m) := \delta_{P(\mathbb{A}_f)}^{-1/2}(m) \int_{N(\mathbb{A}_f)} \int_{K_0} f(k^{-1}nmk)\,dk\,dn. \tag{7.13.2}$$

The function f_M depends on P and even on K_0, but its orbital integrals do not. It is worth noting that its orbital integrals are one of the ingredients in Arthur's trace formula [A1].

We are interested in a particular function $f^\infty \in C_c^\infty(G(\mathbb{A}_f))$, namely, the Hecke operator associated to the Hecke correspondence in (7.4). Explicitly, f^∞ is by definition $\mathrm{vol}_{dg}(K')^{-1}$ times the characteristic function of the coset Kg (with K, g, K' as in (7.4)). Applying the discussion above to f^∞, we get $f_M^\infty \in C_c^\infty(M(\mathbb{A}_f))$ (defined by (7.13.2), with f replaced by f^∞). In (7.12.2) we defined a function $f_P \in C_c^\infty(M(\mathbb{A}_f))$.

Lemma 7.13.A. *The functions f_M^∞ and $\delta_{P(\mathbb{A}_f)}^{-1/2} \cdot f_P$ have the same orbital integrals.*

Proof. It is equivalent to prove that $\delta_{P(\mathbb{A}_f)}^{1/2} \cdot f_M^\infty$ and f_P have the same orbital integrals. Note that although f_P depends on the choice of representatives x_0 for the double cosets

$$P(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K',$$

its orbital integrals do not. Suppose that we replace K' by a compact open subgroup K'' of $G(\mathbb{A}_f)$ contained in K'. Then f^∞ is multiplied by the index $[K' : K'']$, as is f_M^∞. An easy calculation shows that f_P is also multiplied by $[K' : K'']$, as long as we take representatives for

$$P(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K''$$

of the form $x_0 k'$, where x_0 is one of our previous representatives and $k' \in K'$. Therefore by shrinking K' if necessary, it is enough to prove the lemma in the case that K' is contained in K_0. Then we have

$$\delta_{P(\mathbb{A}_f)}^{1/2}(m) f_M^{\infty}(m) = \sum_{x \in K_0/K'} \mathrm{vol}_{dk}(K') \int_{N(\mathbb{A}_f)} f^{\infty}(x^{-1}nmx)\, dn.$$

Thus $\delta_{P(\mathbb{A}_f)}^{1/2} \cdot f_M^{\infty}$ has the same orbital integrals as the function of $m \in M(\mathbb{A}_f)$ given by

$$\sum_{x \in (P(\mathbb{A}_f) \cap K_0)\backslash K_0/K'} a(x) \int_{N(\mathbb{A}_f)} f^{\infty}(x^{-1}nmx)\, dn,$$

where

$$a(x) = \mathrm{vol}_{dk}(K') \cdot [P(\mathbb{A}_f) \cap K_0 \colon P(\mathbb{A}_f) \cap xK'x^{-1}]$$

$$= \mathrm{vol}_{dg}(K') \cdot \mathrm{vol}_{dm\,dn}(P(\mathbb{A}_f) \cap xK'x^{-1})^{-1}.$$

Here we used that

$$\mathrm{vol}_{dm\,dn}(P(\mathbb{A}_f) \cap K_0) = \mathrm{vol}_{dg}(K_0) \cdot \mathrm{vol}_{dk}(K_0)^{-1}$$

$$= \mathrm{vol}_{dg}(K') \cdot \mathrm{vol}_{dk}(K')^{-1},$$

which follows from (7.13.1), applied to the characteristic function of K_0. From the equality

$$G(\mathbb{A}_f) = P(\mathbb{A}_f)K_0,$$

it follows that

$$(P(\mathbb{A}_f) \cap K_0)\backslash K_0/K' \simeq P(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K'.$$

Thus the elements x used here can serve as the elements x_0 used to define f_P. Moreover, it is easy to see that

$$\int_{N(\mathbb{A}_f)} f^{\infty}(x^{-1}nmx)\, dn$$

is 0 unless m lies in the image in $M(\mathbb{A}_f)$ of

$$P(\mathbb{A}_f) \cap xKgx^{-1},$$

in which case it equals

$$\mathrm{vol}_{dn}(N(\mathbb{A}_f) \cap xKx^{-1}) \cdot \mathrm{vol}_{dg}(K')^{-1};$$

this shows that (with $x_0 = x$)

$$a(x) \int_{N(\mathbb{A}_f)} f^{\infty}(x^{-1}nmx)\, dn = r(x_0) f_{P,x_0}(m).$$

The proof of the lemma is now complete.

7.14. Manipulation of the Lefschetz formula. From Lemma 7.13.A, we see that the orbital integral $O_\gamma(f_P)$ appearing in (7.12.1) can be rewritten as

$$O_\gamma(f_P) = \delta_{P(\mathbb{A}_f)}^{1/2}(\gamma) \cdot O_\gamma(f_M^{\infty}),$$

with f_M^{∞} as in (7.13). Moreover, since $\gamma \in M(\mathbb{Q})$, the product formula shows that

$$\delta_{P(\mathbb{A}_f)}(\gamma) = \delta_{P(\mathbb{R})}^{-1}(\gamma),$$

where for $x \in P(\mathbb{R})$ we put

$$\delta_{P(\mathbb{R})}(x) = |\det(\mathrm{Ad}(x); \mathrm{Lie}(N) \otimes \mathbb{R})|.$$

Therefore the Lefschetz formula (7.12.1) can be rewritten as

$$(7.14.1) \quad \sum_P \sum_\gamma |M_\gamma(\mathbb{Q})/I(\mathbb{Q})|^{-1} \cdot \chi(I) \cdot O_\gamma(f_M^{\infty}) \cdot (-1)^{\dim(A_I/A_M)} \cdot \delta_{P(\mathbb{R})}^{-1/2}(\gamma) \cdot L_P(\gamma),$$

with the two index sets for the sum the same as in (7.12.1).

The factors $|M_\gamma(\mathbb{Q})/I(\mathbb{Q})|^{-1}$ and $\chi(I)$ depend only on M_γ. The factor $O_\gamma(f_M^{\infty})$ depends only on the $G(\mathbb{Q})$-conjugacy class of the pair (M, γ). Therefore we could rewrite (7.14.1) by grouping together terms according to the $G(\mathbb{Q})$-conjugacy class of (M, γ). However, there are still more terms that can be grouped together, and it is this phenomenon that we must study next.

The elements γ appearing in (7.14.1) are semisimple, since they are required to fix some point in X_M. We now fix a semisimple element $\gamma \in G(\mathbb{Q})$ and consider the set $\mathscr{M}_\gamma$ of Levi subgroups of G such that $\gamma \in M(\mathbb{Q})$. Recall that M coincides with the centralizer $\mathrm{Cent}_G(A_M)$ of A_M in G. (A_M denotes the maximal $\mathbb{Q}$-split torus in the center of M, in line with the notational conventions established in

(7.1).) Thus a Levi subgroup M of G contains γ if and only if A_M is contained in G_γ^0, the identity component of the centralizer of γ in G. We are now going to define a map $M \mapsto M^*$ from $\mathcal{M}_\gamma$ to itself. Let $M \in \mathcal{M}_\gamma$. Put $I := M_\gamma^0$ and note that A_I contains A_M. Then put

$$M^* := \mathrm{Cent}_G(A_I) = \mathrm{Cent}_M(A_I).$$

Clearly, M^* is a member of $\mathcal{M}_\gamma$, and M^* is contained in M.

LEMMA 7.14.A. *The following statements hold.*

(1) $\gamma \in I \subset M^*$.

(2) $I = (M^*)_\gamma^0$.

(3) $A_{M^*} = A_I$.

(4) $M^{**} = M^*$.

(5) *Suppose that* $M^* = M$ *and that* M_1 *is a Levi subgroup of* G *containing* M. *Then* $M_1^* = M$ *if and only if* $(M_1)_\gamma^0 = M_\gamma^0$.

Proof. The first statement is obvious. The second statement follows from the first and the fact that M^* is contained in M. As for the third statement, the inclusion $A_{M^*} \supset A_I$ is trivial, and the opposite inclusion follows from the obvious inclusion $A_{M^*} \subset A_{(M^*)_\gamma^0}$, since $(M^*)_\gamma^0 = I$ by the second statement. The fourth statement follows immediately from the second. Finally, we verify the fifth statement. If $M_1^* = M$, apply the second statement to M_1 to see that $(M_1)_\gamma^0 = M_\gamma^0$. Conversely, suppose that $(M_1)_\gamma^0 = M_\gamma^0$. Then, applying the third statement to both M_1 and M, we see that $A_{M_1^*} = A_M$, which in turn implies that $M_1^* = M$.

The first and second statements of the lemma together allow us to apply the usual descent theory for orbital integrals. Put (as usual)

$$D_{M^*}^M(\gamma) := \det(1 - \mathrm{Ad}(\gamma); \mathrm{Lie}(M)/\mathrm{Lie}(M^*)) \in \mathbb{Q}.$$

Since $M_\gamma^0 = (M^*)_\gamma^0$, it follows that

$$D_{M^*}^M(\gamma) \neq 0,$$

and from descent theory we have the equality

$$O_\gamma(f_{M^*}^\infty) = |D_{M^*}^M(\gamma)|_{\mathbb{A}_f}^{1/2} \cdot O_\gamma(f_M^\infty). \tag{7.14.2}$$

Furthermore, the product formula implies that

$$|D_{M^*}^M(\gamma)|_{\mathbb{A}_f}^{1/2} = |D_{M^*}^M(\gamma)|_{\mathbb{R}}^{-1/2}.$$

The Lefschetz formula (7.14.1) can now be rewritten as

$$(7.14.3)\qquad \sum_{(P,M,\gamma)} |M_\gamma(\mathbb{Q})/I(\mathbb{Q})|^{-1} \cdot \chi(I) \cdot O_\gamma(f_{M^*}^\infty) \cdot (-1)^{\dim(A_I/A_M)} \cdot |D_{M^*}^M(\gamma)|_{\mathbb{R}}^{1/2} \cdot \delta_{P(\mathbb{R})}^{-1/2}(\gamma) \cdot L_P(\gamma),$$

where the sum runs through a set of representatives for the $G(\mathbb{Q})$-conjugacy classes of triples (P, M, γ) consisting of a parabolic subgroup P of G, a Levi factor M of P, and an element $\gamma \in M(\mathbb{Q})$, satisfying the condition that the fixed-point set X_M^γ of γ on X_M be nonempty. As usual, we write I for M_γ^0. We may impose the additional condition that the real group $(I/A_I)_{\mathbb{R}}$ contain some anisotropic maximal $\mathbb{R}$-torus, since otherwise $\chi(I) = 0$ (see (7.10)).

For any triple (P, M, γ) satisfying these conditions, the real group $(M^*/A_{M^*})_{\mathbb{R}}$ contains some anisotropic maximal $\mathbb{R}$-torus, and, moreover, γ is elliptic in $M^*(\mathbb{R})$ (in other words, is contained in some maximal $\mathbb{R}$-torus in M^* that is anisotropic modulo A_{M^*}). Indeed, by assumption there exists a maximal torus T in I over $\mathbb{R}$ such that T/A_I is anisotropic over $\mathbb{R}$. But T is also a maximal torus in M^* (use the second statement in Lemma 7.14.A), and $A_I = A_{M^*}$ (use the third statement in Lemma 7.14.A), so that T/A_{M^*} is an anisotropic maximal torus in $(M^*/A_{M^*})_{\mathbb{R}}$ containing (the image of) γ.

Therefore the Lefschetz formula (7.14.3) can be rewritten as

$$(7.14.4)\qquad \sum_{(M,\gamma)} \chi(I) \cdot O_\gamma(f_M^\infty) \cdot \sum_{(P_1,M_1)} |(M_1)_\gamma(\mathbb{Q})/I(\mathbb{Q})|^{-1} \cdot |D_M^{M_1}(\gamma)|_{\mathbb{R}}^{1/2} \cdot (-1)^{\dim(A_M/A_{M_1})} \cdot \delta_{P_1(\mathbb{R})}^{-1/2}(\gamma) \cdot L_{P_1}(\gamma).$$

In (7.14.4) I again denotes M_γ^0, and the index sets for the sums are as follows. The first sum runs over a set of representatives for the $G(\mathbb{Q})$-conjugacy classes of pairs (M, γ), where M is a Levi subgroup of G, and $\gamma \in M(\mathbb{Q})$, satisfying the conditions that $(M/A_M)_{\mathbb{R}}$ contain some anisotropic maximal $\mathbb{R}$-torus and that γ be elliptic in $M(\mathbb{R})$. Write $N_G(M)$ for the normalizer of M in G. Then the second sum in (7.14.4) runs over a set of representatives for the $N_G(M)(\mathbb{Q}) \cap G_\gamma(\mathbb{Q})$-conjugacy classes of pairs (P_1, M_1) consisting of a parabolic subgroup P_1 in G and a Levi factor M_1 of P_1, satisfying these three conditions:

(1) $M_1 \supset M$ (which implies that $\gamma \in M_1(\mathbb{Q})$);
(2) $M_1^* = M$;
(3) $X_{M_1}^\gamma$ is nonempty.

Consider two pairs (M, γ) and (P_1, M_1) appearing in (7.14.4), and let $I = M_\gamma^0$, as before. Our assumptions on (M, γ) imply that $(I/A_I)_{\mathbb{R}}$ contains some anisotropic maximal $\mathbb{R}$-torus and that $A_I = A_M$. (Note in passing that this implies that $M^* = M$.) Let $n(P_1, M_1)$ denote the number of elements in the conjugacy class of

(P_1, M_1) under $N_G(M)(\mathbb{Q}) \cap G_\gamma(\mathbb{Q})$. We are now going to check that

$$|(M_1)_\gamma(\mathbb{Q})/I(\mathbb{Q})| \cdot n(P_1, M_1) = |(N_G(M)(\mathbb{Q}) \cap G_\gamma(\mathbb{Q}))/I(\mathbb{Q})|. \tag{7.14.5}$$

Indeed, it follows from the equality $A_M = A_I$ that

$$\begin{aligned} N_G(M)(\mathbb{Q}) &= N_G(A_M)(\mathbb{Q}) \\ &= N_G(A_I)(\mathbb{Q}). \end{aligned}$$

The stabilizer of (P_1, M_1) in $G(\mathbb{Q})$ is $M_1(\mathbb{Q})$, and hence its stabilizer in $N_G(M)(\mathbb{Q}) \cap G_\gamma(\mathbb{Q})$ is $N_{M_1}(M)(\mathbb{Q}) \cap (M_1)_\gamma(\mathbb{Q})$. Since any element of $(M_1)_\gamma(\mathbb{Q})$ normalizes the split component A_I of $(M_1)_\gamma^0 = I$, we see that

$$N_{M_1}(M)(\mathbb{Q}) \supset (M_1)_\gamma(\mathbb{Q}),$$

and therefore the stabilizer of (P_1, M_1) in $N_G(M)(\mathbb{Q}) \cap G_\gamma(\mathbb{Q})$ is simply $(M_1)_\gamma(\mathbb{Q})$, which proves (7.14.5). Therefore the Lefschetz formula (7.14.4) can be rewritten as

(7.14.6)
$$\sum_{(M,\gamma)} |(N_G(M)(\mathbb{Q}) \cap G_\gamma(\mathbb{Q}))/I(\mathbb{Q})|^{-1} \cdot \chi(I) \cdot O_\gamma(f_M^\infty) \cdot (-1)^{\dim A_M/A_G} \cdot L_M(\gamma),$$

where the index set for the sum is the same as that for the first sum in (7.14.4), and where

$$L_M(\gamma) := \sum_{(P_1,M_1)} (-1)^{\dim A_{M_1}/A_G} |D_M^{M_1}(\gamma)|_{\mathbb{R}}^{1/2} \cdot \delta_{P_1(\mathbb{R})}^{-1/2}(\gamma) \cdot L_{P_1}(\gamma), \tag{7.14.7}$$

with the index set equal to the set of *all* pairs (P_1, M_1) satisfying (1), (2), and (3). (In other words, we no longer divide by the action of $N_G(M)(\mathbb{Q}) \cap G_\gamma(\mathbb{Q})$ on the set of pairs.)

Note that condition (2) is equivalent to the condition that the connected centralizers of γ in M and M_1 coincide (recall that $M^* = M$ and apply Lemma 7.14.A(5)), and this in turn is equivalent to the condition that $D_M^{M_1}(\gamma)$ be nonzero. Therefore condition (2) can be dropped without changing (7.14.7).

Our assumptions on (M, γ) imply that there exists a maximal torus T of G over $\mathbb{R}$ containing γ such that T/A_M is anisotropic over $\mathbb{R}$. The element γ can be written as

$$\gamma = \exp(x) \cdot \gamma_1$$

for unique elements $x \in \mathfrak{A}_M$ and $\gamma_1 \in T(\mathbb{R})_1$, where $T(\mathbb{R})_1$ denotes the maximal compact subgroup of $T(\mathbb{R})$ (see §5 for the definition of $\mathfrak{A}_M$). It is easy to see that M_1 satisfies condition (3) if and only if x belongs to the subspace $\mathfrak{A}_{M_1}$ of $\mathfrak{A}_M$. We conclude that

$$L_M(\gamma) = \sum_Q (-1)^{\dim(A_L/A_G)} \cdot |D_M^L(\gamma)|_{\mathbb{R}}^{1/2} \cdot \delta_{Q(\mathbb{R})}^{-1/2}(\gamma) \cdot L_Q(\gamma), \tag{7.14.8}$$

where the sum ranges over all parabolic subgroups $Q = LU$ containing M such that x lies in $\mathfrak{A}_L$. (Here, as usual, L denotes the unique Levi component of Q containing M, and U denotes the unipotent radical of Q.)

At this point we need to be more precise about the complex numbers $L_Q(\gamma)$, and in order to do so, we must first discuss weight profiles. As in §5, let ν be an element in $(\mathfrak{A}_{P_0})^*$ whose restriction to $\mathfrak{A}_G$ coincides with the character by which A_G acts on E. Then ν determines a weight profile (see §1.1 of [GHM]) in the following way. As in §5, ν determines elements

$$\nu_P \in \mathfrak{A}_P^*$$

for every $P \in \mathscr{F}(M_0)$, and in particular for every standard parabolic subgroup P. The weight profile we associate to ν is the one such that the restriction to the stratum S_K^P (for standard $P = MN$) of the ith cohomology sheaf of the weighted cohomology complex $\overline{\mathbf{E}}_K$ is the local system associated to the finite-dimensional representation

$$H^i(\mathrm{Lie}(N), E)_{\geqslant \nu_P}$$

of M defined in §5. See Proposition 17.2 of [GHM] in order to see how to describe this weight profile in the language of that paper; note that in [GHM] the subspace

$$H^i(\mathrm{Lie}(N), E)_{\geqslant \nu_P}$$

is denoted by

$$H^i(\mathfrak{n}_P, E)_+.$$

Now we are in a position to give a precise formula for $L_Q(\gamma)$. Here, as above, $Q = LU$ is a parabolic subgroup containing M such that x lies in the subspace $\mathfrak{A}_L$ of $\mathfrak{A}_M$. This formula is essentially the one given in [GM1, §11], although that paper only discusses two particular weight profiles and uses the other natural way of breaking up the Lefschetz formula as a sum over strata of fixed-point components. Let $\alpha_1, \dots, \alpha_n \in \mathfrak{A}_L^*$ be the simple roots of A_L in $\mathrm{Lie}(U)$. Let

$I = \{1, \dots, n\}$. (We temporarily reuse the notation I in this way for the sake of compatibility with [GM1].) Put

$$I(\gamma) := \{i \in I \mid \langle \alpha_i, x \rangle < 0\}$$

(the set of "expanding directions"). In [GM1] the set $I(\gamma)$ is defined instead by the condition

$$\langle \alpha_i, x \rangle \leqslant 0;$$

this, together with the use of Euler characteristics rather than Euler characteristics with compact support, is the other natural way of breaking up the Lefschetz formula.

Choose a Borel subgroup B of G over $\mathbb{C}$ containing T and contained in Q, and let W' be the corresponding Kostant representatives for the cosets $W_L \backslash W$. Let $\lambda_B, \rho_B \in X^*(T)_{\mathbb{R}}$ be as usual (see §5). Let $t_1, \dots, t_n \in \mathfrak{A}_L/\mathfrak{A}_G$ be the basis of $\mathfrak{A}_L/\mathfrak{A}_G$ dual to the basis $\alpha_1, \dots, \alpha_n$ of $(\mathfrak{A}_L/\mathfrak{A}_G)^*$. For $w \in W'$ put

$$I(w) = \{i \in I \mid \langle p_L(w(-\lambda_B - \rho_B) + \rho_B) + \nu_Q, t_i \rangle > 0\}.$$

In [GM1] the set $I(w)$ was defined using $\leqslant$ rather than $>$, but this was a misprint and should have been $\geqslant$. If we take $\nu = \nu_m$, our set $I(w)$ still differs ($>$ rather than $\geqslant$) from the (corrected) one in [GM1]: there are two middle-weighted cohomology complexes, upper and lower, and the formula in [GM1] arises from one of them while the formula here arises from the other.

The complex number $L_Q(\gamma)$ is defined by

$$(-1)^{|I(\gamma)|} \sum_{w} \varepsilon(w) \cdot \operatorname{tr}(\gamma^{-1}; V^L_{w(\lambda_B+\rho_B)-\rho_B}),$$

where the index set for the sum is the set of $w \in W'$ such that $I(w) = I(\gamma)$. Perhaps the following remarks will help the reader understand the number $L_Q(\gamma)$. The local contribution to the Lefschetz fixed-point formula from a connected component of the fixed-point set is given (see [GM2]) by the (weighted) cohomology (of a regular neighborhood of the fixed-point component) with supports that are compact in the expanding directions and that are closed in the contracting directions. The piece of weighted cohomology that is indexed by $w \in W'$ (via Kostant's theorem) contributes to stalk cohomology with compact supports in the directions $I(w)$ and with degree shift by $|I(w)|$. The Hecke correspondence is expanding away from the stratum S_K^Q in the directions $I(\gamma)$. Thus the contributions to the Lefschetz formula occur when $I(w) = I(\gamma)$.

As in §5 let C_Q be the (open) chamber in $\mathfrak{A}_L$ corresponding to $Q \in \mathscr{P}(L)$; of course, the image of the cone C_Q in $\mathfrak{A}_L/\mathfrak{A}_G$ is generated by $t_1, \dots, t_n$. Again, as in §5, let φ_Q denote the function

$$\varphi_{C_Q}(\cdot,\cdot)$$

on $\mathfrak{A}_L \times \mathfrak{A}_L^*$ determined by the *open* cone C_Q (see the last part of Appendix A). By Lemma A.7 and the analog of (A.2) mentioned just before Lemma A.7, the value of $\varphi_Q(-x, p_L(w(\lambda_B+\rho_B)-\rho_B)-\nu_Q)$ is given by

$$(-1)^{\dim(A_G)} \cdot (-1)^{\dim(A_L/A_G)-|I(\gamma)|}$$

if $I(w) = I(\gamma)$ and is 0 otherwise. Therefore

$$L_Q(\gamma) = \sum_{w \in W'} \varepsilon(w) \cdot (-1)^{\dim(A_L)} \cdot \varphi_Q(-x, p_L(w(\lambda_B+\rho_B)-\rho_B)-\nu_Q) \cdot \operatorname{tr}(\gamma^{-1}; V^L_{w(\lambda_B+\rho_B)-\rho_B}).$$

This expression for $L_Q(\gamma)$ coincides exactly with the one used to define $L_Q^\nu(\gamma)$ in §5. Moreover, comparing the definition of $L_M^\nu(\gamma)$ given in §5 with (7.14.8), we now see that the numbers $L_M^\nu(\gamma)$ from §5 and $L_M(\gamma)$ from this section are equal. Applying Theorem 5.1 yields the following result. (We now let I once again denote M_γ^0.)

THEOREM 7.14.B. *The Lefschetz formula for the alternating sum of the traces of the self-maps (7.8.1) is given by*

$$\sum_{(M,\gamma)} |(N_G(M)(\mathbb{Q}) \cap G_\gamma(\mathbb{Q}))/I(\mathbb{Q})|^{-1} \cdot \chi(I) \cdot O_\gamma(f_M^\infty) \cdot (-1)^{\dim(A_M/A_G)} \cdot \Phi_M(\gamma, \Theta_\nu)$$

with Θ_ν and $\Phi_M(\gamma,\Theta_\nu)$ as in §5, the index set for the sum being the same as that for the first sum in (7.14.4).

7.15. $\mathbb{Q}$-equivalence. There are three special cases in which the statement of Theorem 7.14.B can be simplified, but before we can do this we need to make a couple of definitions. Let Θ, Θ' be stable virtual characters on $G(\mathbb{R})$. We say that Θ and Θ' are *$\mathbb{Q}$-equivalent* if they agree on $T_{\mathrm{reg}}(\mathbb{R})$ for every maximal $\mathbb{R}$-torus T in G whose $\mathbb{R}$-split component is both defined and split over $\mathbb{Q}$. Note that Θ and Θ' are $\mathbb{Q}$-equivalent if and only if the functions $\Phi_M(\cdot,\Theta)$ and $\Phi_M(\cdot,\Theta')$ coincide for every Levi subgroup M of G over $\mathbb{Q}$ such that $(M/A_M)_{\mathbb{R}}$ contains an anisotropic maximal $\mathbb{R}$-torus. Clearly the expression for the Lefschetz formula given in Theorem 7.14.B remains valid when Θ_ν is replaced by any stable virtual character Θ' that is $\mathbb{Q}$-equivalent to Θ_ν.

7.16. The orientation character χ_G. We will also need the following sign character

$$\chi_G \colon G(\mathbb{R}) \to \{\pm 1\}.$$

Recall from 7.2 the real manifold

$$X_G = G(\mathbb{R})/(K_G \cdot A_G(\mathbb{R})^0),$$

on which $G(\mathbb{R})$ acts by left translations. Of course, X_G is diffeomorphic to a Euclidean space and hence is orientable. For $g \in G(\mathbb{R})$ we define $\chi_G(g)$ to be -1 if g reverses the orientation of X_G and $+1$ if g preserves the orientation of X_G.

Let $P = MN$ be a parabolic subgroup of G whose Levi component M contains M_0. Suppose that M contains a maximal $\mathbb{R}$-torus T such that T/A_M is anisotropic over $\mathbb{R}$. We claim that

$$\chi_G(\gamma) = \operatorname{sgn}(\det(\gamma; \operatorname{Lie}(N))) \tag{7.16.1}$$

for all $\gamma \in T(\mathbb{R})$.

Indeed, since $P(\mathbb{R})$ acts transitively on X_G, we see that as an $M(\mathbb{R})$-space, X_G is given by

$$X_G = \operatorname{Lie}(N) \times X_M \times (A_M(\mathbb{R})^0/A_G(\mathbb{R})^0). \tag{7.16.2}$$

It follows that

$$\chi_G(m) = \chi_M(m) \cdot \operatorname{sgn}(\det(m; \operatorname{Lie}(N))) \tag{7.16.3}$$

for all $m \in M(\mathbb{R})$. Since $T(\mathbb{R})/A_M(\mathbb{R})$ is connected, and since $A_M(\mathbb{R})$ acts trivially on X_M, we see that χ_M is trivial on $T(\mathbb{R})$, so that (7.16.1) follows from (7.16.3).

7.17. Very positive v. Now suppose that v is sufficiently positive. Then the weighted cohomology complex $\bar{\mathbf{E}}_K$ is equal to $Rj_!\mathbf{E}_K$, where j denotes the inclusion of S_K in $\bar{S}_K$. Moreover it is obvious from the definition of Θ_v that Θ_v is equal to (the character of) the representation E^* contragredient to E. (For sufficiently positive v, the virtual modules E_P^v used to define Θ_v are trivial except when $P = G$.) Therefore in this case the expression for the Lefschetz formula given by Theorem 7.14.B involves the character of the representation E^*, as was also observed by J. Franke [F] and G. Harder.

7.18. Very negative v. Now suppose that v is sufficiently negative. Then the weighted cohomology complex $\bar{\mathbf{E}}_K$ is equal to the full direct image $Rj_*\mathbf{E}_K$. Moreover, we claim that Θ_v is $\mathbb{Q}$-equivalent to (the character of) the virtual representation $(-1)^{\dim(X_G)}\chi_G \otimes E^*$ of $G(\mathbb{R})$, where X_G is the space defined in 7.2 and χ_G is the orientation character defined in 7.16.

Let T be a maximal $\mathbb{R}$-torus in G whose $\mathbb{R}$-split component A is defined and split over $\mathbb{Q}$. Let M be the centralizer of A in G, a Levi subgroup of G over $\mathbb{Q}$. Replacing T by a conjugate under $G(\mathbb{Q})$, we may assume that M contains M_0. We must show that for all $\gamma \in T(\mathbb{R})$

$$\Phi_M(\gamma, \Theta_v) = (-1)^{\dim(X_G)} \cdot \chi_G(\gamma) \cdot \Phi_M(\gamma, E^*). \tag{7.18.1}$$

As usual we write

$$\gamma = \exp(x) \cdot \gamma_1$$

for unique elements $x \in \mathfrak{A}_M$ and $\gamma_1 \in T(\mathbb{R})_1$. By continuity it is enough to prove (7.18.1) when γ is regular in $T(\mathbb{R})$ and x is regular in $\mathfrak{A}_M$.

We use Theorem 5.1 to evaluate $\Phi_M(\gamma, \Theta_\nu)$. Since we are assuming that ν is sufficiently negative, the factor

$$\varphi_Q(-x, p_L(w(\lambda_B + \rho_B) - \rho_B) - \nu_Q)$$

entering into the definition of $L_Q^\nu(\gamma)$ is 0 unless $L = M$ and x lies in the chamber C_Q in $\mathfrak{A}_M$ determined by Q, in which case the factor is $(-1)^{\dim(A_M)}$ (use Lemma A.7 and the analog for φ_Q of (A.2) to evaluate $\varphi_Q(-x, \mu)$ for very positive μ). Therefore Theorem 5.1 tells us that $\Phi_M(\gamma, \Theta_\nu)$ is equal to

$$(-1)^{\dim(A_M/A_G)} \cdot \delta_Q^{-1/2}(\gamma)$$

times

$$\sum_{w \in W'} \varepsilon(w) \cdot \operatorname{tr}(\gamma^{-1}; V^M_{w(\lambda_B+\rho_B)-\rho_B}), \tag{7.18.2}$$

where $Q = MN$ is the unique parabolic subgroup with Levi component M such that x lies in the chamber C_Q. The number (7.18.2) is equal to

$$\operatorname{tr}(\gamma^{-1}; E) \cdot \operatorname{tr}(\gamma; \textstyle\bigwedge^\bullet(\operatorname{Lie}(N))),$$

where $\bigwedge^\bullet(\operatorname{Lie}(N))$ denotes the virtual M-module

$$\sum_{i \geq 0} (-1)^i \bigwedge^i(\operatorname{Lie}(N)).$$

Therefore, in order to prove the equality (7.18.1), it is enough to prove that

(7.18.3)
$$(-1)^{\dim(A_M/A_G)} \cdot \delta_Q^{-1/2}(\gamma) \cdot \prod_\alpha (1 - \alpha(\gamma)) = (-1)^{\dim(X_G)} \cdot \chi_G(\gamma) \cdot |D_M^G(\gamma)|^{1/2},$$

where the product is taken over roots α of T in $\operatorname{Lie}(N)$. The two sides of (7.18.3) have the same absolute value, and therefore it is enough to prove that the number

$$(-1)^{\dim(X_G)} \cdot (-1)^{\dim(A_M/A_G)} \cdot \chi_G(\gamma) \cdot \prod_\alpha (1 - \alpha(\gamma)) \tag{7.18.4}$$

is positive. Using (7.16.1) we see that the number (7.18.4) has the same sign as

$$(-1)^{\dim(X_G)} \cdot (-1)^{\dim(A_M/A_G)} \cdot \prod_{\alpha} (\alpha^{-1}(\gamma) - 1).$$

Complex conjugation preserves the set of roots α of T in $\mathrm{Lie}(N)$. If α is not real, then the product of $\alpha^{-1}(\gamma) - 1$ and $\bar{\alpha}^{-1}(\gamma) - 1$ is positive. If α is real, then $\alpha^{-1}(\gamma) - 1$ is negative (use that x lies in the chamber C_Q). Therefore the sign of the number (7.18.4) is equal to

$$(-1)^{\dim(X_G)} \cdot (-1)^{\dim(A_M/A_G)} \cdot (-1)^{\dim(N)}.$$

It follows from (7.16.2) that

$$\dim(X_G) = \dim(A_M/A_G) + \dim(N) + \dim(X_M).$$

Moreover, $\dim(X_M)$ is even, since $(M/A_M)_{\mathbb{R}}$ contains an anisotropic maximal $\mathbb{R}$-torus. Therefore the sign of the number (7.18.4) is indeed $+1$, as we wished to show.

7.19. Middle v. Now suppose that $v = v_m$. Then $\bar{\mathbf{E}}_K$ is the upper middle-weighted cohomology complex. Assume further that $(G/A_G)_{\mathbb{R}}$ has an anisotropic maximal $\mathbb{R}$-torus. Then by Theorem 5.2 the virtual character Θ_v is $\mathbb{Q}$-equivalent to Θ^{E^*}. Thus the expression for the Lefschetz formula given in Theorem 7.14.B essentially agrees with Arthur's formula [A1, Theorem 6.1] for the alternating sum of the traces of a Hecke operator on the L^2-cohomology groups of S_K with coefficients in $\mathbf{E}_K$. Actually, Theorem 7.14.B appears at first to disagree with Arthur's formula, which contains the factor $\Phi_M(\gamma, \Theta^E)$ rather than $\Phi_M(\gamma, \Theta^{E^*})$. But by replacing γ by γ^{-1} in one of the two formulas, we come closer to agreement, since Arthur uses the usual *left* action of the Hecke algebra on the L^2-cohomology of S_K, while in this paper we have (implicitly) used the *right* action of the Hecke algebra on the upper middle-weighted cohomology of $\bar{S}_K$. In other words, it is the *right* action of f^∞ on $H^i(\bar{S}_K, \bar{\mathbf{E}}_K)$ that coincides with the self-map defined by our Hecke correspondence $(\bar{c}_1, \bar{c}_2)$; of course, the right action of f^∞ coincides with the left action of the Hecke operator f_r^∞ defined by

$$f_r^\infty(x) = f^\infty(x^{-1}) \qquad (x \in G(\mathbb{A}_f)).$$

We should also note that the index set for the sum in our formula is somewhat different from the index set for the double sum in Arthur's formula, and that our formula has the factor

$$|(N_G(M)(\mathbb{Q}) \cap G_\gamma(\mathbb{Q}))/I(\mathbb{Q})|^{-1}$$

while Arthur's has the factor

$$|W_0^M||W_0^G|^{-1}|\iota^M(\gamma)|^{-1};$$

however, it is easy to see that these are just two different ways of writing the same thing.

However, our formula disagrees with Arthur's in that we sum only over Levi subgroups M such that M/A_M contains an *anisotropic* maximal torus over $\mathbb{R}$. In [A1] all Levi subgroups M are allowed, although the term indexed by M vanishes (due to the factor Φ_M) unless M/A_M contains an *elliptic* maximal torus over $\mathbb{R}$. Thus Arthur's formula may have more nonzero terms than ours (the split component of the center of M over $\mathbb{R}$ may be strictly bigger than A_M); however, these extra terms in Arthur's formula should not actually be there. (The error occurs in equation (4.1) of [A1], which is only valid if the split components of the center of M over $\mathbb{Q}$ and $\mathbb{R}$ are the same.)

Appendices

A. Functions associated to convex polyhedral cones. Let X be a finite-dimensional real vector space, and let C be a closed convex polyhedral cone in X. Recall that this means that there exists a finite subset S of X such that C is equal to the set of nonnegative linear combinations of elements of S; equivalently, it means that there exists a finite subset T of the dual vector space X^* such that C is equal to the intersection of the sets

$$\{x \in X \mid \lambda(x) \geqslant 0\},$$

where λ ranges through T. We denote by C^* the closed convex polyhedral cone in X^* dual to C. Recall that C^* is defined by

$$C^* := \{\lambda \in X^* \mid \lambda(x) \geqslant 0 \text{ for all } x \in C\},$$

and that the map

$$F \mapsto F^\perp := C^* \cap \{\lambda \in X^* \mid \lambda(x) = 0 \text{ for all } x \in F\}$$

sets up a bijection between the set $\mathscr{F}$ of (closed) faces of C to the set $\mathscr{F}^*$ of (closed) faces of C^*. Moreover, $C^{**} = C$ and $F^{\perp\perp} = F$. The map $F \mapsto F^\perp$ is order reversing (we order faces by inclusion), and

$$\dim(F) + \dim(F^\perp) = \dim(X),$$

where $\dim(F)$ denotes the dimension of the linear span of F.

Define an integer-valued function ψ_C on $X \times X^*$ by

$$\psi_C = \sum_{F \in \mathscr{F}} (-1)^{\dim(F)} \xi_{(F^\perp)^* \times F^*},$$

where we have denoted by $\xi_{(F^\perp)^* \times F^*}$ the characteristic function of the subset $(F^\perp)^* \times F^*$ of $X \times X^*$. (Since $F^\perp$ and F are themselves closed convex polyhedral cones, it makes sense to consider their dual cones $(F^\perp)^*$ and F^*.) Note that $(F^\perp)^*$ is equal to $C + \mathrm{span}(F)$, where $\mathrm{span}(F)$ denotes the linear span of F. Clearly

$$\psi_{C^*}(\lambda, x) = (-1)^{\dim(X)} \psi_C(x, \lambda). \tag{A.1}$$

Let X_1 denote the linear span of C and let X_2 denote the largest linear subspace of X contained in C. In X^* we have the perpendicular subspaces

$$X_i^\perp = \{\lambda \in X^* \mid \lambda(x) = 0 \text{ for all } x \in X_i\} \qquad (i = 1, 2).$$

The cone C gives rise to a cone $\tilde{C}$ in X_1/X_2, and it is easy to see that $\psi_C(x, \lambda)$ is 0 unless $x \in X_1$ and $\lambda \in X_2^\perp$, in which case

$$\psi_C(x, \lambda) = (-1)^{\dim(X_2)} \psi_{\tilde{C}}(\tilde{x}, \tilde{\lambda}), \tag{A.2}$$

where $\tilde{x}, \tilde{\lambda}$ denote the images of x, λ under the natural surjections

$$X_1 \to X_1/X_2$$

$$X_2^\perp \to X_2^\perp / X_1^\perp = (X_1/X_2)^*,$$

respectively.

Suppose that C is a simplicial cone in X. Then the pair (X, C) is isomorphic to the pair $(\mathbb{R}^n, (\mathbb{R}_{\geqslant 0})^n)$, where $n = \dim X$ and $\mathbb{R}_{\geqslant 0}$ denotes the set of nonnegative real numbers. We will now calculate ψ_C for the pair $(\mathbb{R}^n, (\mathbb{R}_{\geqslant 0})^n)$. Of course, we identify X^* with $\mathbb{R}^n$ as well. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. Write I for $\{1, \dots, n\}$ and then put

$$I_x = \{i \in I \mid x_i \geqslant 0\}$$

$$I_\lambda = \{i \in I \mid \lambda_i \geqslant 0\}.$$

Lemma A.1. *The number $\psi_C(x, \lambda)$ is 0 unless the subsets I_x and I_λ of I are complementary, in which case*

$$\psi_C(x, \lambda) = (-1)^{|I_\lambda|},$$

where $|I_\lambda|$ denotes the cardinality of I_λ.

Proof. Indeed, the faces of C are indexed by the subsets $J \subset I$, the face corresponding to J being

$$\{x \in \mathbb{R}^n \mid x_i \geqslant 0 \text{ for all } i \in J \text{ and } x_i = 0 \text{ for all } i \in I \backslash J\}.$$

Thus

$$\psi_C(x, \lambda) = \sum_J (-1)^{|J|},$$

where J ranges through all subsets of I such that

$$(I \backslash I_x) \subset J \subset I_\lambda,$$

and the lemma follows from the elementary fact that for any finite set S, the sum

$$\sum_{T \subset S} (-1)^{|T|}$$

is 0 unless S is empty, in which case it is 1.

In the next lemma and corollary it will be convenient to assume that

$$\dim(C) = \dim(C^*) = \dim(X).$$

We refer to the faces of C having codimension 1 in C as *facets* of C, and we refer to the linear spans of the facets of C as the *walls* of C. The walls of C form a finite set of hyperplanes in X. We say that a point in X is C-regular if it lies on no wall of C. We refer to the connected components of the set of C-regular points in X as *chambers* in X. Of course, the interior of C is one of these chambers. Similarly the walls of C^* divide X^* into chambers. The value of $\psi_C(x, \lambda)$ on C-regular $x \in X$ and C^*-regular $\lambda \in X^*$ depends only on the chambers in which x, λ lie.

LEMMA A.2. *Let x, x' be C-regular elements of X, and suppose that there is exactly one wall Y of C separating x and x' (in other words x, x' lie in adjacent chambers). Put $C_Y := C \cap Y$, a facet of C which can also be regarded as a closed convex polyhedral cone in Y, so that the function ψ_{C_Y} on $Y \times Y^*$ is defined. Assume that x, C lie on the same side of Y. Then*

$$\psi_C(x, \lambda) - \psi_C(x', \lambda) = \psi_{C_Y}(y, \lambda_Y),$$

where $\lambda_Y \in Y^*$ is the restriction of λ to Y and $y \in Y$ is the unique point of Y lying on the line segment joining x and x'.

Proof. Let $\alpha \in C^*$ be a nonzero element of the 1-dimensional face $(C_Y)^\perp$ of C^*. Thus Y is the kernel of the linear form α, and α is positive on x, negative on

x'. Since x, x' lie in adjacent chambers, $\beta(x)$ and $\beta(x')$ have the same sign for any nonzero element β of any 1-dimensional face of C^* other than $(C_Y)^\perp$, and this sign is the same as that of $\beta(y)$. Therefore

$$\xi_{(F^\perp)^* \times F^*}(x, \lambda) - \xi_{(F^\perp)^* \times F^*}(x', \lambda)$$

is 0 unless F is contained in Y (equivalently, unless $F^\perp$ contains α), in which case it is equal to

$$\xi_{G \times H}(y, \lambda_Y),$$

where $G = (F^\perp)^* \cap Y$ and H is the image of F^* under the canonical surjection from X^* to Y^*. But the faces of C_Y are precisely the faces F of C contained in Y, and, writing F_Y to indicate that we are regarding such a face F as a face of C_Y, we have $F_Y^* = H$ and

$$\begin{aligned}(F_Y^\perp)^* &= C_Y + \operatorname{span}(F) \\ &= (C + \operatorname{span}(F)) \cap Y \\ &= (F^\perp)^* \cap Y \\ &= G.\end{aligned}$$

This proves the lemma.

Applying the lemma to C^* and using (A.1), we obtain the following result.

COROLLARY A.3. *Let λ, λ' be C^*-regular elements of X, and suppose that there is exactly one wall Z of C^* separating λ and λ'. Let $\omega \in C$ be a nonzero element of the 1-dimensional face $(C^* \cap Z)^\perp$ of C corresponding to the facet $C^* \cap Z$ of C^*. Put $\tilde{X} := X/\mathbb{R}\omega$, and let $\tilde{C}$ denote the image of C under the canonical surjection $X \to \tilde{X}$; note that Z is the dual vector space to $\tilde{X}$ and that $\tilde{C}$ is the closed convex polyhedral cone $(C^* \cap Z)^*$ in $\tilde{X}$ dual to the cone $C^* \cap Z$ in Z. In particular, the function $\psi_{\tilde{C}}$ on $\tilde{X} \times Z$ is defined. Assume that λ, C^* lie on the same side of Z (equivalently, assume that $\lambda(\omega) > 0$ and $\lambda'(\omega) < 0$). Then*

$$\psi_C(x, \lambda) - \psi_C(x, \lambda') = -\psi_{\tilde{C}}(\tilde{x}, \tilde{\lambda}),$$

where $\tilde{x}$ denotes the image of x under the canonical surjection $X \to \tilde{X}$ and $\tilde{\lambda}$ denotes the unique point of Z lying on the line segment joining λ and λ'.

Now we come to the main point of the appendix, which is to show that if C decomposes as a union of cones, then ψ_C decomposes accordingly. For this we need a little preparation. For any closed convex polyhedral cone C in X, we write $\mathring{C}$ for its relative interior (the interior of C in span(C)). We say that a function $f: X \to \mathbb{Z}$ is *conic* if it can be written as a finite $\mathbb{Z}$-linear combination of characteristic functions ξ_C of closed convex polyhedral cones C in X. Let λ be a nonzero element of X^*, let C be a closed convex polyhedral cone in X, and put

$$C_+ = \{x \in C \mid \lambda(x) \geqslant 0\}$$

$$C_- = \{x \in C \mid \lambda(x) \leqslant 0\}$$

$$C_0 = \{x \in C \mid \lambda(x) = 0\}.$$

Then C_+, C_-, C_0 are closed convex polyhedral cones in X, and we have the relation

$$\xi_C + \xi_{C_0} - \xi_{C_+} - \xi_{C_-} = 0. \tag{A.3}$$

It is not difficult to see that the abelian group $C(X)$ of conic functions on X is presented by the generators ξ_C and the relations (A.3).

The characteristic function $\xi_{\mathring{C}}$ of the relative interior $\mathring{C}$ of C is a conic function on X. Indeed, it can be expressed, in terms of our generators, as

$$\xi_{\mathring{C}} = (-1)^{\dim(C)} \sum_F (-1)^{\dim(F)} \xi_F, \tag{A.4}$$

where the sum is taken over the set of faces F of C. To prove this note that the value of the right-hand side of (A.4) at a point $x \in X$ is 0 unless $x \in C$, in which case it is

$$(-1)^{\dim(C)} \sum_F (-1)^{\dim(F)},$$

where F now ranges through all faces of C containing $F(x)$, the smallest face of C containing x. These faces correspond bijectively to the faces of a new cone, namely, $C + \text{span}(F(x))$. Therefore (A.4) follows from the following well-known fact: for any closed convex polyhedral cone C

$$\sum_F (-1)^{\dim(F)} = \begin{cases} (-1)^{\dim(C)} & \text{if } C \text{ is a linear subspace of } X, \\ 0 & \text{otherwise,} \end{cases} \tag{A.5}$$

where the sum is taken over all faces F of C. (To prove this fact, reduce to the case in which $C \neq \{0\}$ and C contains no nonzero linear subspace, and note that in this case $C \backslash \{0\}$ is contractible and hence has Euler characteristic 1; on the other hand, this Euler characteristic is equal to

$$-\sum_{F}(-1)^{\dim(F)},$$

where F ranges through all faces of C other than $\{0\}$.)

We now define a homomorphism $f \mapsto f^*$ from $C(X)$ to itself by putting

$$(\xi_C)^* := (-1)^{\dim(C)}\xi_{\mathring{C}}.$$

Of course, we must check that if this definition is applied to the left-hand side of (A.3), we get 0. We may as well assume that λ takes both strictly positive and strictly negative values on C. (Otherwise the relation (A.3) is itself trivial.) Then

$$\dim(C_+) = \dim(C_-) = \dim(C)$$

$$\dim(C_0) = \dim(C) - 1$$

and $\mathring{C}$ is the disjoint union of $\mathring{C}_+$, $\mathring{C}_-$, and $\mathring{C}_0$, which gives what we want.

Next we define a homomorphism $f \mapsto \hat{f}$ from $C(X)$ to $C(X^*)$ by putting

$$(\xi_C)^\wedge := (-1)^{\dim(X)-\dim(C^*)}\xi_{\mathring{C}^*}.$$

Here, $\mathring{C}^*$ denotes the relative interior of the dual cone C^*. Again, we must check that when we apply this definition to the left-hand side of (A.3), we get 0. We will see in a moment that the operation $*$ is an isomorphism; therefore it is enough to consider instead the operation obtained by following $\wedge$ by $*$ and multiplying by $(-1)^{\dim(X)}$; this operation sends ξ_C to ξ_{C^*}. We must show that

$$\xi_{C^*} + \xi_{C_0^*} - \xi_{C_+^*} - \xi_{C_-^*} = 0.$$

This is an immediate consequence of the following well-known fact: if C_1, C_2 are closed convex cones whose union is convex, then $C_1^* \cup C_2^*$ is also convex and

$$(C_1 \cup C_2)^* = C_1^* \cap C_2^*$$

$$(C_1 \cap C_2)^* = C_1^* \cup C_2^*.$$

It is easy to see from the definitions of our two operations that $f^{**} = f$ and $f^{\wedge*\wedge*} = f$ for all $f \in C(X)$. (For the first equality, use (A.4), and for the second, use that $C^{**} = C$.) It follows that both of our operations are isomorphisms. These operations are best understood by placing them in a more general context (see [KS], [M]), in which $*$ comes from Verdier duality and $\wedge$ from the Fourier-Sato transformation, but this point of view is not needed for what follows.

It is interesting to calculate $(\xi_{\mathring{C}})^{\wedge}$. We claim that

$$(\xi_{\mathring{C}})^{\wedge} = (-1)^{\dim(C)} \xi_{-C^*}. \tag{A.6}$$

We will now give an elementary proof of this consequence of Lemma 3.7.10 (ii) of [KS]. Using (A.4), we see that (A.6) is equivalent to the equality

$$\sum_{F \in \mathscr{F}(C)} (-1)^{\dim(F)} \xi_F^{\wedge *} = (\xi_{-C^*})^*,$$

which in turn is equivalent to

$$\sum_{F \in \mathscr{F}(C)} (-1)^{\dim(F)} \xi_{F^*} = (-1)^{\dim(X)-\dim(C^*)} \xi_{-\mathring{C}^*}. \tag{A.7}$$

Let $\lambda \in X^*$. The value of the left-hand side of (A.7) at λ is

$$\sum_{F \in \mathscr{F}_\lambda} (-1)^{\dim(F)}, \tag{A.8}$$

where $\mathscr{F}_\lambda$ denotes the set of faces F of C such that λ is nonnegative on F. Let C_+ be the set

$$C_+ := \{x \in C \mid \lambda(x) \geqslant 0\}.$$

We claim that (A.8) is equal to

$$\sum_{F \in \mathscr{F}(C_+)} (-1)^{\dim(F)}. \tag{A.9}$$

If $\lambda = 0$, the claim is obvious, so we assume that $\lambda \neq 0$. Then $\mathscr{F}(C_+)$ contains $\mathscr{F}_\lambda$. Elements of $\mathscr{F}(C_+)$ that are not in $\mathscr{F}_\lambda$ arise in pairs $F \cap C_+$ and $F \cap \ker(\lambda)$, one pair for each face F of C such that $\ker(\lambda)$ meets $\mathring{F}$ but does not contain $\mathring{F}$. For such F

$$\dim(F \cap \ker(\lambda)) = \dim(F \cap C_+) - 1;$$

and therefore each such pair of faces contributes 0 to (A.9), and the claim follows.

It follows from (A.5) (applied to C_+) that (A.9) is equal to

$$\begin{cases} (-1)^{\dim(C_+)} & \text{if } C_+ \text{ is a linear subspace,} \\ 0 & \text{otherwise.} \end{cases}$$

But C_+ is a linear subspace if and only if λ is $\leqslant 0$ on C and $C \cap \ker(\lambda)$ is a linear subspace. If λ is $\leqslant 0$ on C, then $C \cap \ker(\lambda)$ is a face of C, and therefore $C \cap \ker(\lambda)$ is a linear subspace if and only if it is equal to the unique minimal face of C. Thus C_+ is a linear subspace if and only if $\lambda \in -\mathring{C}^*$, in which case $C_+ = C \cap \ker(\lambda)$ is the minimal face of C and thus has dimension equal to

$$\dim(X) - \dim(C^*).$$

This completes the proof of the equality (A.7).

We say that an integer-valued function on $X \times X^*$ is *biconic* if it is a finite $\mathbb{Z}$-linear combination of characteristic functions $\xi_{C\times D}$, where C (respectively, D) is a closed convex polyhedral cone in X (respectively, X^*). Of course, the functions ψ_C considered earlier are biconic functions on $X \times X^*$. In fact we are now going to generalize ψ_C by associating to any conic function f on X a biconic function ψ_f on $X \times X^*$. If f is the characteristic function ξ_C of a closed convex polyhedral cone C, we define ψ_f to be the function ψ_C defined earlier. Since any conic function f can be written as a $\mathbb{Z}$-linear combination

$$f = \sum_C n_C \xi_C$$

for a finite set of closed convex polyhedral cones C and integers n_C, we are then forced to define ψ_f by

$$\psi_f := \sum_C n_C \psi_C.$$

Of course, we must show that ψ_f depends only on f, and not on the way in which we write it as a $\mathbb{Z}$-linear combination of characteristic functions of cones. This can be done directly, but instead we will use the two operations introduced earlier to give a shorter proof. For any biconic function ψ and any $\lambda \in X^*$, the function $\psi(\cdot, \lambda)$ on X is conic, and therefore we can apply the operation $*$ to ψ in the variable $x \in X$ (holding λ fixed); this operation sends $\xi_{C\times D}$ to $(-1)^{\dim(C)}\xi_{\mathring{C}\times D}$, which is again biconic. It follows that applying $*$ to ψ in the variable $x \in X$ yields a biconic function. Similarly, applying $*$ to ψ in the variable $\lambda \in X^*$ yields a biconic function. Any biconic function is conic on $X \times X^*$, and so we may apply the operation $\wedge$ on $X \times X^*$ to any biconic function ψ on $X \times X^*$. Since the dual of $X \times X^*$ is $X^* \times X$, which we may identify with $X \times X^*$ by switching the order of the two factors, we may regard $\wedge$ as an operation taking biconic functions on $X \times X^*$ to biconic functions on $X \times X^*$. (It is clear that $\hat{\psi}$ is biconic, not just conic.)

For a biconic function ψ on $X \times X^*$, we denote by ψ' the biconic function on $X \times X^*$ obtained by applying $*$ in the variable $\lambda \in X^*$ to the biconic function $\hat{\psi}$.

We will use the operation $\psi \mapsto \psi'$ to show that $f \mapsto \psi_f$ is well defined. It is enough to show that

$$\sum_C n_C(\psi_C)'$$

depends only on f. We need to find a simple expression for $(\psi_C)'$; by definition it is

$$\sum_F \xi_{\mathring{F}\times F^\perp}.$$

But for any $x \in \mathring{F}$ we have

$$F^\perp = C^* \cap \{\lambda \in X^* \mid \lambda(x) = 0\}.$$

Therefore $(\psi_C)'(x, \lambda)$ is 0 unless $\lambda(x) = 0$, in which case it is equal to

$$\sum_F \xi_{\mathring{F}\times C^*}(x, \lambda) = \xi_{C\times C^*}(x, \lambda).$$

Our problem has been reduced to the following: for $(x, \lambda) \in X \times X^*$ such that $\lambda(x) = 0$ we must show that

$$\sum_C n_C \xi_{C\times C^*}(x, \lambda) \tag{A.10}$$

depends only on f. In fact, we will show more: for such (x, λ), the expression (A.10) is equal to

$$[f^* \xi_{X_+}]^*(x), \tag{A.11}$$

where X_+ is the set

$$X_+ = \{x \in X \mid \lambda(x) \geqslant 0\}.$$

Since this statement is linear in f, we may assume without loss of generality that $f = \xi_C$. Then (A.10) is equal to $\xi_{C\times C^*}(x, \lambda)$, which in turn is equal to

$$\begin{cases} \xi_C(x) & \text{if } C \subset X_+, \\ 0 & \text{otherwise.} \end{cases}$$

By definition f^* equals $(-1)^{\dim(C)}\xi_{\mathring{C}}$, and therefore the product $f^*\xi_{X_+}$ equals

$$(-1)^{\dim(C)}\xi_{\mathring{C}\cap X_+}.$$

There are three cases. If C is contained in X_+, then $\overset{\circ}{C} \cap X_+ = \overset{\circ}{C}$ and $[f^*\xi_{X_+}]^* = \xi_C$, so that (A.11) agrees with (A.10) in this case. If C is not contained in X_+ but is contained in X_-, where

$$X_- = \{x \in X \mid \lambda(x) \leqslant 0\},$$

then $\overset{\circ}{C} \cap X_+$ is empty and therefore (A.11) is 0, which again agrees with (A.10). If C is contained in neither X_+ nor X_-, then we are in the situation considered during the proof that $*$ is well defined, and $\overset{\circ}{C}$ is the disjoint union of $\overset{\circ}{C}_+$, $\overset{\circ}{C}_-$, and $\overset{\circ}{C}_0$, where $C_+ = C \cap X_+$, $C_- = C \cap X_-$, and $C_0 = C \cap \ker(\lambda)$. Therefore $\overset{\circ}{C} \cap X_+$ is the disjoint union of $\overset{\circ}{C}_+$ and $\overset{\circ}{C}_0$, and it follows that

$$\begin{aligned}[f^*\xi_{X_+}]^* &= (-1)^{\dim(C)}[\xi_{\overset{\circ}{C}_+} + \xi_{\overset{\circ}{C}_0}]^* \\ &= \xi_{C_+} - \xi_{C_0}.\end{aligned}$$

For x such that $\lambda(x) = 0$, we have

$$(\xi_{C_+} - \xi_{C_0})(x) = \xi_C(x) - \xi_C(x) = 0,$$

which shows that (A.11) again agrees with (A.10). This concludes the proof that ψ_f is well defined.

The following proposition is an easy consequence of the fact that ψ_f is well defined.

PROPOSITION A.4. *Let C be a closed convex polyhedral cone and suppose that $\overset{\circ}{C}$ is the disjoint union of the relative interiors $\overset{\circ}{C}_1, \dots, \overset{\circ}{C}_r$ of r closed convex polyhedral cones $C_1, \dots, C_r$. Then*

$$\psi_C = \sum_{i=1}^{r} (-1)^{\dim(C)-\dim(C_i)} \psi_{C_i}.$$

Proof. Indeed, our hypothesis is that

$$\xi_{\overset{\circ}{C}} = \sum_{i=1}^{r} \xi_{\overset{\circ}{C}_i}.$$

Applying the operation $*$, we find that

$$(-1)^{\dim(C)}\xi_C = \sum_{i=1}^{r} (-1)^{\dim(C_i)} \xi_{C_i}.$$

Applying the map $f \mapsto \psi_f$, we then get the desired equality.

Proposition A.5. *For any conic function f on X, the quantity $\psi_f(x,\lambda)$ vanishes unless $\lambda(x) \leqslant 0$.*

Proof. The conic function f can be written as a $\mathbb{Z}$-linear combination of characteristic functions of closed convex polyhedral cones C that are simplicial as cones in their linear spans. Therefore it is enough to prove the proposition in the case that $f = \xi_C$ for such a cone C. By (A.2) we may assume that C spans X and hence that C is simplicial in X. Then the proposition follows easily from Lemma A.1.

The function ψ_C is equal to ψ_f for the conic function $f = \xi_C$. We can also use the *open* cone $\mathring{C}$ to obtain an equally useful variant $\varphi_{\mathring{C}}$ of ψ_C by putting

$$\varphi_{\mathring{C}} := \psi_g,$$

where g is the conic function $\xi_{\mathring{C}}$.

Lemma A.6. *There is an equality*

$$\varphi_{\mathring{C}}(x,\lambda) = \sum_{F\in\mathscr{F}} (-1)^{\dim(F)} \xi_{\mathring{C}+\operatorname{span}(F)}(x)\xi_{F^*}(\lambda).$$

The cone $\mathring{C} + \operatorname{span}(F)$ is the relative interior of the cone $C + \operatorname{span}(F)$, whose faces are in one-to-one correspondence with the faces G of C containing F, via the map $G \mapsto G + \operatorname{span}(F)$. Applying (A.4) to $C + \operatorname{span}(F)$, we see that the right-hand side of the equality we are trying to prove is equal to

$$\sum_F (-1)^{\dim(F)} \sum_{\{G\in\mathscr{F}\mid G\supset F\}} (-1)^{\dim(C)-\dim(G)} \xi_{G+\operatorname{span}(F)}(x)\xi_{F^*}(\lambda). \tag{A.12}$$

Applying (A.4) to C, we see that

$$\varphi_{\mathring{C}} = \sum_{G\in\mathscr{F}} (-1)^{\dim(C)-\dim(G)} \psi_G,$$

and by writing out the definition of ψ_G, we see that $\varphi_{\mathring{C}}(x,\lambda)$ is equal to the expression (A.12). This proves the lemma.

Note that the expression for $\varphi_{\mathring{C}}(x,\lambda)$ given by Lemma A.6 is almost the same as the expression for $\psi_C(x,\lambda)$ given by its definition; the only difference is that the factor

$$\xi_{C+\operatorname{span}(F)}(x)$$

appearing in the definition of $\psi_C(x,\lambda)$ is replaced by

$$\xi_{\mathring{C}+\operatorname{span}(F)}(x)$$

in the expression for $\varphi_{\mathring{C}}(x, \lambda)$. Since $\mathring{C} + \mathrm{span}(F)$ is the relative interior of $C + \mathrm{span}(F)$, we conclude that for any conic function f on X, the biconic function ψ_{f^*} associated to f^* is obtained from ψ_f by applying the operation $*$ to ψ_f in the first variable. Moreover, we conclude that

$$\text{(A.13)} \qquad \varphi_{\mathring{C}}(x, \lambda) = \psi_C(x, \lambda) \qquad \text{if } x \text{ is } C\text{-regular}.$$

It is no surprise that $\varphi_{\mathring{C}}$ behaves just about the same way as ψ_C. For example, $\varphi_{\mathring{C}}$ satisfies the obvious analog of (A.2) (replace $C, \tilde{C}$ in (A.2) by their relative interiors). Now suppose that C is a simplicial cone in X and use the same notation as in Lemma A.1. We need one more bit of notation: put

$$\mathring{I}_x = \{i \in I \mid x_i > 0\}.$$

Then it is easy to see that $\varphi_{\mathring{C}}$ satisfies the following analog of Lemma A.1.

LEMMA A.7. *The number $\varphi_{\mathring{C}}(x, \lambda)$ is 0 unless the subsets $\mathring{I}_x$ and I_λ of I are complementary, in which case*

$$\varphi_{\mathring{C}}(x, \lambda) = (-1)^{|I_\lambda|},$$

where $|I_\lambda|$ denotes the cardinality of I_λ.

B. Combinatorial lemma of Langlands. In this appendix we generalize a combinatorial lemma of Langlands [A2, Lemma 6.3]. Let X be a finite-dimensional real vector space, and let $(\cdot, \cdot)$ be a positive definite symmetric bilinear form on X, with associated metric

$$d(x, y) = (x - y, x - y)^{1/2}.$$

Let C be a closed convex polyhedral cone in X. Let $x \in X$. Since C is closed, there exists a point $x_0 \in C$ that is closest to x, and since C is convex, the point $x_0 \in C$ is unique. Again by convexity this closest point can be characterized as the unique point $x_0 \in C$ such that

$$d(x, x_0) \leqslant d(x, x_0 + r(y - x_0))$$

for all $y \in C$ and all real numbers $r \in [0, 1]$. For fixed $y \in C$, the truth of the inequality above for all $r \in [0, 1]$ is equivalent to the inequality

$$(x - x_0, y - x_0) \leqslant 0.$$

Thus, since C is a cone, $x_0 \in C$ is characterized by the property that

$$(x - x_0, z) \leqslant 0$$

for all $z \in Z$, where Z is the set of all elements in X of the form $y - rx_0$ for some $y \in C$ and some positive real number r.

Let F be the unique face of C such that x_0 lies in the relative interior $\mathring{F}$ of F. We claim that Z is equal to $C + \operatorname{span}(F)$. Clearly Z is contained in $C + \operatorname{span}(F)$. Moreover, in order to prove the reverse inclusion, it is enough to show that $-F$ is contained in Z. Let $x_1 \in F$. Since $x_0 \in \mathring{F}$, there exist $x_2 \in F$ and a positive real number r such that

$$x_1 - x_0 = -r(x_2 - x_0),$$

which shows that $-x_1 \in Z$, as desired.

We use the inner product $(\cdot, \cdot)$ to identify X with its dual. In particular, we now view the dual cone C^* as subset of X itself:

$$C^* = \{x \in X \mid (x, y) \geqslant 0 \text{ for all } y \in C\}.$$

As in Appendix A, we denote by $F^\perp$ the face $C^* \cap \operatorname{span}(F)^\perp$ of C^* determined by the face F of C. Of course, $F^\perp$ is equal to

$$(C + \operatorname{span}(F))^*.$$

We conclude that $x_0 \in \mathring{F}$ is the point of C closest to x if and only if

$$x - x_0 \in -F^\perp,$$

in which case x_0 is the orthogonal projection of x on $\operatorname{span}(F)$, and $x - x_0$ is the orthogonal projection of x on $\operatorname{span}(F)^\perp$. In particular, the set of all points $x \in X$ such that the point x_0 in C closest to x lies in $\mathring{F}$ is equal to

$$\mathring{F} \oplus (-F^\perp).$$

We have written $\oplus$ rather than $+$ in order to emphasize that the cones $\mathring{F}$ and $-F^\perp$ lie in the complementary subspaces $\operatorname{span}(F)$ and $\operatorname{span}(F)^\perp$, respectively. Let $\xi_{\mathring{F} \oplus (-F^\perp)}$ denote the characteristic function of the subset $\mathring{F} \oplus (-F^\perp)$ of X. We conclude that

$$\text{(B.1)} \qquad 1 = \sum_{F \in \mathscr{F}} \xi_{\mathring{F} \oplus (-F^\perp)}(x)$$

for all $x \in X$, where $\mathscr{F}$ denotes the set of faces of C.

The equality (B.1) is a (generalization of) a simple special case of Langlands's combinatorial lemma. We will now use this special case to prove the general case.

We continue to identify X with its dual, so that the function ψ_C on $X \times X^*$ (see Appendix A) becomes a function on $X \times X$. For any subspace Y of X, we denote by p_Y the orthogonal projection map from X onto Y. Now we can state our generalization of the combinatorial lemma of Langlands. (Take C to be simplicial, and use Lemma A.1 to recover the usual form of the lemma.)

LEMMA B.1. *For $x, y \in X$, there is an equality*

$$\sum_{F \in \mathscr{F}} \psi_{C+\operatorname{span}(F)}(-y, -p_{\operatorname{span}(F)^\perp}(x)) \cdot \xi_{\overset{\circ}{F}}(p_{\operatorname{span}(F)}(x)) = (-1)^{\dim(C)} \xi_{\overset{\circ}{C}}(y).$$

Proof. The faces of $C + \operatorname{span}(F)$ are precisely the cones $G + \operatorname{span}(F)$, where G ranges through the set of faces of C containing F; note that for such G

$$\operatorname{span}(G + \operatorname{span}(F)) = \operatorname{span}(G)$$

$$\dim(G + \operatorname{span}(F)) = \dim(G)$$

$$(C + \operatorname{span}(F)) + \operatorname{span}(G + \operatorname{span}(F)) = C + \operatorname{span}(G).$$

Therefore by the definition of $\psi_{C+\operatorname{span}(F)}$, the left-hand side of the equality in the lemma is equal to

$$\sum_{F \in \mathscr{F}} \sum_{G \in \mathscr{F}: G \supset F} (-1)^{\dim(G)} \xi_{C+\operatorname{span}(G)}(-y) \cdot \xi_{(G+\operatorname{span}(F))^*}(-p_{\operatorname{span}(F)^\perp}(x)) \cdot \xi_{\overset{\circ}{F}}(p_{\operatorname{span}(F)}(x)),$$

which, by interchanging the order of summation, we rewrite as

$$\sum_{G \in \mathscr{F}} (-1)^{\dim(G)} \xi_{C+\operatorname{span}(G)}(-y) \cdot \sum_{F \in \mathscr{F}(G)} \xi_{(G+\operatorname{span}(F))^*}(-p_{\operatorname{span}(F)^\perp}(x)) \cdot \xi_{\overset{\circ}{F}}(p_{\operatorname{span}(F)}(x)),$$

where $\mathscr{F}(G)$ denotes the set of faces of G. The second sum is 1 by (B.1) (applied to G). Therefore the double sum reduces to

$$\sum_{G \in \mathscr{F}} (-1)^{\dim(G)} \xi_{C+\operatorname{span}(G)}(-y).$$

Applying equality (A.7) to C^*, we see that this last expression is equal to

$$(-1)^{\dim(C)} \xi_{\overset{\circ}{C}}(y).$$

This completes the proof of the lemma.

There are two special cases of Lemma B.1 that are worth noting. First, if $y \in \overset{\circ}{C}$, it is easy to see that Lemma B.1 reduces to the equality (B.1). Second, if $y \in -C$, then Lemma B.1 reduces to the following result.

COROLLARY B.2. *There is an equality*

$$\sum_{F\in\mathscr{F}}(-1)^{\dim(F^{\perp})}\xi_{(F^{\perp})^{\circ}\oplus\mathring{F}} = \begin{cases}(-1)^{\dim(X)-\dim(C)} & \text{if } C \text{ is a linear subspace,}\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Indeed, if $x\in C$, then $\xi_{(F^{\perp})^*}(x)=1$ for all $F\in\mathscr{F}$, and therefore

$$\begin{aligned}\psi_C(x,\lambda) &= \sum_{F\in\mathscr{F}}(-1)^{\dim(F)}\xi_{F^*}(\lambda)\\ &= (-1)^{\dim(X)-\dim(C^*)}\xi_{\mathring{C}^*}(-\lambda).\end{aligned}$$

(We used the equality (A.7).) Applying this to $C+\operatorname{span}(F)$ instead of C, we see that if $y\in -C$, then

$$\psi_{C+\operatorname{span}(F)}(-y,\lambda) = (-1)^{\dim(X)-\dim(F^{\perp})}\xi_{(F^{\perp})^{\circ}}(-\lambda).$$

Moreover, if $y\in -C$, then

$$\xi_{\mathring{C}}(y) = \begin{cases}1 & \text{if } C \text{ is a linear subspace,}\\ 0 & \text{otherwise.}\end{cases}$$

Therefore Lemma B.1 does reduce to Corollary B.2 when $y\in -C$.

COROLLARY B.3. *The sum*

$$\sum_{F\in\mathscr{F}}(-1)^{\dim(F)}\xi_{((F^{\perp})^*)^{\circ}\oplus(F^*)^{\circ}}(x)$$

is 0 *unless* C *is a linear subspace and* $x=0$. *Here the dual cone* $(F^{\perp})^*$ *is taken inside the subspace*

$$\operatorname{span}(F)^{\perp} = \operatorname{span}(F^{\perp}),$$

and the dual cone F^* *is taken inside the subspace* $\operatorname{span}(F)$.

This corollary can be derived from the previous one by applying the operation $*$ and then the operation $\wedge$. Note that the formula

$$\sum_{\{P_3:P_3\supset P_1\}}(-1)^{\dim(A_1/A_3)}\tau_1^3(H)\hat{\tau}_3(H) = 0$$

of [A2, p. 940] is the special case of this corollary in which the cone C is the closed chamber in $\mathfrak{A}_{M_1}/\mathfrak{A}_G$ determined by P_1 (still using Arthur's notation).

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GORESKY AND MACPHERSON: SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540, USA

KOTTWITZ: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637, USA