1. John Mather's contributions to singularity theory

by Mark Goresky

1.1. John Mather played a pivotal role in the development of the intertwined theories of singularities of mappings and stratifications. He was involved in two significant projects: stability of smooth mappings, and the foundations of stratification theory.

In 1955 Hassler Whitney [19] defined a smooth mapping $f : M \to P$ between smooth manifolds to be C^∞ *stable* if, for any sufficiently nearby mapping $f' : M \to P$ there are diffeomorphism $\phi : M \to M$ and $\psi : P \to P$ that transform f' into f. Equivalently, f

$$\begin{array}{ccc} M & \xrightarrow{f} & P \\ \phi\big\downarrow & & \big\downarrow\psi \\ M & \xrightarrow{f'} & P \end{array}$$

Figure 1. C^∞ equivalence

should lie in an open orbit of the action of the diffeomorphism group $\mathrm{Diff}(M) \times \mathrm{Diff}(P)$ on the space $C^\infty(M, P)$ of smooth mappings (with respect to a suitable topology).

Whitney asked: Do stable mappings form an open and dense set in the space of all smooth proper mappings? For Morse functions $f : M \to \mathbb{R}$ and for mappings $f : \mathbb{R}^2 \to \mathbb{R}^2$, Whitney showed the answer is "yes". He believed the answer would always be "yes".

In 1960 René Thom [17] dropped a bombshell: he found a smooth proper mapping $\mathbb{R}^{16} \to \mathbb{R}^{16}$ that cannot be smoothly approximated by stable maps. This opened a Pandora's box of complications.

Despite efforts and partial results of many people, the answer was not known until 1968 when John Mather developed an enormous collection of techniques, resulting in six major papers ([4, 5, 6, 7, 8, 9]) that completely answered the question and in some sense killed the subject because there was nothing more to say.

Theorem 1. [9] *Stable mappings $M \to P$ form a dense subset of the space $C^\infty(M, P)$ if and only if $(\dim(M), \dim(P))$ is represented by a red dot as shown in Figure 2.*

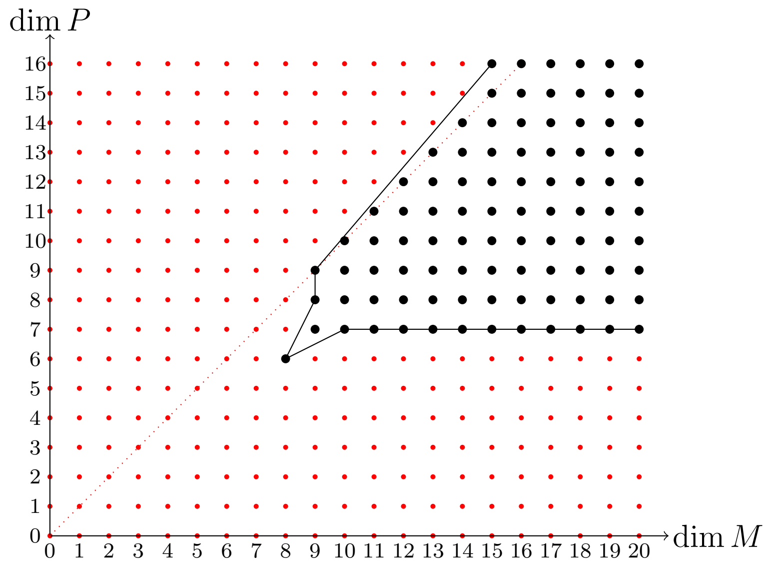

FIGURE 2. C^∞ stability range of dimensions

1.2. If $f \in \mathrm{Diff}(M, P)$ is stable, then near the identity, the diffeomorphism group $\mathrm{Diff}(M) \times \mathrm{Diff}(P)$ maps surjectively to a neighborhood of f so the map on the tangent space

$$T_I \mathrm{Diff}(M) \times T_I \mathrm{Diff}(P) \to T_f(C^\infty(M, P))$$

is surjective, which is to say that every vector field w "along" f is obtained from vector fields u, v on M, P so that $w = v \circ f + df \circ u$, as in the following diagram:

$$\begin{tikzcd}
TM \arrow[r, "df"] & TP \\
M \arrow[u, "u"] \arrow[r, "f"'] \arrow[ur, "w"] & P \arrow[u, "v"']
\end{tikzcd}$$

This necessary condition on vector fields is known as "infinitesimal stability". In [4, 5] Mather developed a far reaching generalization of the Weierstrass-Malgrange preparation theorem and used it to prove the converse: infinitesimal stability implies stability.

Infinitesimal stability provides an differential-geometric approach to the problem of stability but it requires solving nonlinear differential equations for the vector fields u, v. In [6] Mather introduced the notions of *finitely determined* mappings and *finite singularity type*. He showed how to replace the vector fields in the differential-geometric problem with their Taylor expansions to some finite order, thus converting the problem into a purely algebraic one.

1.3. Next, in [7], Mather solved the resulting algebraic problem. Suppose $f : M \to P$ with $f(x) = y$. The $\mathbb{R}$-algebra $\mathbf{C}_y$ of germs (at y) of C^∞ functions $P \to \mathbb{R}$ contains a maximal ideal $\mathfrak{m}_y$ of germs that vanish at y. Mather defined $Q(f) = \mathbf{C}_x/f^*(\mathfrak{m}_y)\mathbf{C}_x$ and $Q_k(f) = Q(f)/\mathfrak{m}^{k+1}$ where $\mathfrak{m}$ is the intersection of the maximal ideals in $Q(f)$. Mather proved, if f is stable (near x) then the isomorphism class of f is determined by the finite dimensional algebra $Q_{p+1}(f)$ where $p = \dim(P)$. He also characterized those $\mathbb{R}$-algebras that come from stable mappings, thus giving an "algebraic" solution to the problem of identifying stable mappings.

In [8] Mather used transversality techniques to address the density of stable mappings. Putting everything together he concluded that stable mappings are dense in the region described in Figure 2.

1.4. A central object in Mather's analysis is the *jet space*. Recall that $J^k_{x,y}(M, P)$ consists of equivalence classes of smooth functions $f : M \to P$, $f(x) = y$, two being equivalent if they agree to order k at $x \in M$. Allowing x, y to vary gives the vector bundle of k-jets, $J^k(M, P) \to M \times P$ whose elements may be thought of as finite approximations to smooth functions $f : M \to P$. Let $J^k_1(M, M)$ be the group of invertible k-jets. One might hope that the orbits of the action of $J^k_1(M, M) \times J^k_1(P, P)$ on $J^k(M, P)$ form a stratification of the jet space but this is only true for orbits of small codimension. Orbits of high codimension exist in continuous families, and this is the source of the non-density of C^∞ stable mappings.

1.5. Throughout this period, René Thom had been thinking that perhaps *topologically* stable maps might be dense, replacing the diffeomorphisms ϕ, ψ in Figure 1.1 with homeomorphisms. At first glance this may seem a hopeless task since a homeomorphism may be arbitrarily bad. But Thom developed a far reaching vision as to how this might be achieved: Develop the theory of stratifications that are *topologically* locally trivial along each stratum; show that the jet space has a natural stratification in which the aforementioned families of orbits combine into single strata; then show that transversality of the jet $J^k(f) : M \to J^k(M, P)$ to these strata, plus the "isotopy lemmas" implies that f is topologically stable. These ideas were outlined in §4 of [14], amplified in [15, 16].

Mather explained to the author [13] that he had a great deal of difficulty in making sense of Thom's outline and he did not understand Thom's definition of the stratification of the jet space. In the end, using his notions of finite singularity type, Mather was able to prove that topologically stable maps are dense. His proof differs from Thom's outline in that Mather does not use a stratification of the jet space, although he initially thought it would be necessary to do so. Mather published [10] an outline of his proof in 1973.

Mather later [11], citing his inspiration from Thom's outline, used his density theorem to construct a stratification of the jet space and showed that topological stability of $f : M \to P$ is implied by transversality of the jet mapping $M \to J^k(M, P)$ to this stratification.

1.6. In 1976 Gibson et. al. [2], using many of Mather's techniques and results, but following Thom's outline more closely, published a second proof, see also [1]. Gibson et. al. make use of a stratification of the jet space, possibly different from Mather's. They rely, in an essential way, on the foundational work on stratification theory in Mather's 1970 notes [12] which were available by then.

Mather's plan was to write a book explaining his proof of the density of topologically stable mappings, but only the first chapter [12] was completed. This chapter, however, had an incredible influence on singularity theory and is still the best source on the foundations of stratification theory. In this chapter Mather modfies Thom's definitions slightly and provides a complete proof of the isotopy lemma that was sketched by Thom in [10].

1.7. Stratification theory. Can an analytic set exhibit fractal behavior? If $f : X \to Y$ is a proper analytic mapping between analytic sets, do the fibers fall into finitely many distinct homotopy types? These questions were partially put to rest when Lojasiewicz triangulaged [3] semi-analytic sets. However, analytic and algebraic sets appear to have natural decompositions into more manageable pieces.

Whitney [18, 20, 21] made various attempts to decompose algebraic sets into unions of smooth manifolds. He found two problems, illustrated by examples. The first example is an algebraic subset of Euclidean space such that no decomposition into smooth manifolds will be locally trivial in the C^1 sense. The variety

$$xy(y-x)(y-zx)=0$$

shown in Figure 3 consists of four "sheets" meeting along the z axis. Three of the sheets are simply a product with the z axis, but the fourth sheet twists around the axis. Any differentiable flow in the ambient Euclidan space, parallel to the z axis, that preserves the first three sheets cannot preserve the fourth, because the derivative at any point on the z axis is determined by the cross-ratio. So the homogeneity that is apparent in this example can only be realized by a *continuous* flow.

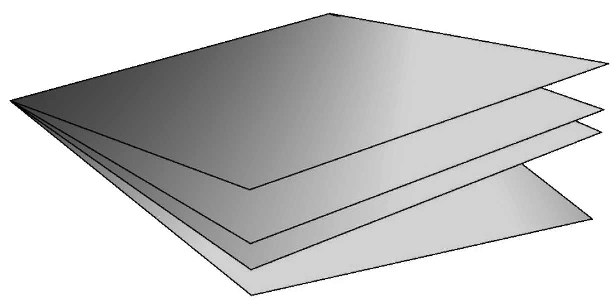

FIGURE 3. $xy(y-x)(y-zx)=0$

The second example shows that the naive approach to stratifying an algebraic set does not work. One might hope to stratify an algebraic variety by starting with the nonsingular part, then throwing it away and continuing by induction. Whitney's second example, is illustrated in Figure 4. The nonsingular part consists of the two dimensional surfaces. When we remove this part, what remains is a smooth line. However one point on this line is special, because of the way the two dimensional part twists around the line. Whitney proposed conditions A and B as criteria that would (correctly) force the origin to be considered as a separate stratum.

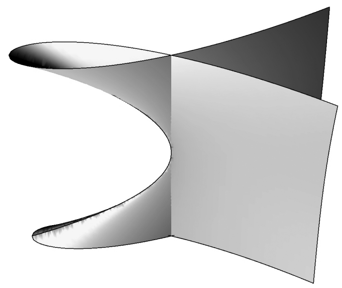

FIGURE 4. $y^2 = x^3 - x^2z^2$

In [14], R. Thom proposed a program to solve both of Whitney's problems: a way to prove that a stratification satisfying the Whitney conditions would be *topologically* locally trivial.

1.8. Thom's outline was very difficult to follow and Mather felt that the first step in proving his own results on topological stability required a complete proof of Thom's proposal, that a Whitney stratified set is topologivally locally trivial ([13]). Mather's 1970 notes accomplish this in great detail.

The proof, approximately following Thom, first requires the construction of a *system of control data*: a "tubular neighborhood" T_X of each stratum X together with a projection $\pi_X : T_X \to X$ to the stratum, and a function $\rho_X : T_X \to [0, \epsilon)$ measuring the "distance" from the stratum. The various (π_X, ρ_X) are required to satisfy certain compatibility conditions between strata.

Assuming Whitney's conditions, and using a delicate construction of tubular neighborhoods and a double induction, Mather shows how to construct such a system of control data. Mather once mentioned that he had an additional 20 pages of notes to verify the domains and ranges of the various π, ρ agree, but he asked "Does anyone care?"

1.9. Given a system of control data on a stratified set, the next step in the solution to Whitney's problems is the construction of homeomorphisms that do not come from ambient smooth mappings. A *controlled vector field* is a collection $\{V_X\}$ of vector fields, tangent to the strata, such that $(\pi_X)_*(V_Y) = V_X$ and $V_Y(\rho_X) = 0$ whenever $X \subset \overline{Y}$ are strata. In other words, the vector field along Y preserves the distance to X. Such a vector field may fail to be continuous but it has a continuous flow that is smooth on each stratum, see Figure 5. The result is the first isotopy lemma of Thom and Mather,

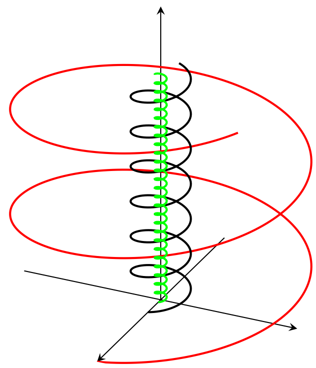

FIGURE 5. Controlled flow of $\frac{1}{r^2}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial z}$

Theorem 2. [12] *Let W be a Whitney stratified subset of some smooth manifold P, let $f : P \to M$ be a smooth mapping whose restriction to W is proper and suppose that the restriction $f|X : X \to M$ to each stratum X is a submersion. Then $f : W \to M$ is a locally trivial fibration.*

1.10. If X is a stratum in W then applying the isotopy lemma to $(\pi_X, \rho_X) : T_X \to [0, \epsilon)$ gives the long sought **local structure theorem** for stratified spaces: *any point $x \in X$ in a stratified space W has a basis of neighborhoods homeomorphic, by a stratum preserving homeomorphism, smooth on each stratum, to the product $B_x(\epsilon) \times c(L_x)$ where $B_x(\epsilon)$ is an open ball in X, and $c(L_x)$ is the cone over the link, $L_x = \pi_X^{-1}(x) \cap \rho_X^{-1}(\epsilon')$ of the stratum X at $x \in X$.* (The link L_x is canonically stratified by its intersection with the strata of W.)

In this sense, the stratification is topologically locally trivial and it may be understood inductively in terms of the topology of the link. This key point is the basis for countless applications of the theory.

1.11. Mather's notes have had an enormous impact on the mathematical literature. Intersection homology theory and stratified

Morse theory could not have been developed without Mather's foundational work. Stratification has become a standard technique in algebraic geometry, topology and even in number theory. Although Mather's notes were not published until 40 years after they were written and distributed, they have been cited hundreds of times and remain the single most complete and accessible approach to stratification theory. It is ironic that Mather's unpublished notes have had a greater influence on subsequent mathematical developments than his monumental work on the stability of mappings.

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