Invent. math. 73, 367–381 (1983)

*Inventiones mathematicae*


# A Decomposition Theorem for the Integral Homology of a Variety

J.B. Carrell[1] and R.M. Goresky[2]

University of British Columbia, Department of Mathematics,
Vancouver, B.C. Canada V6T 1Y4

## Introduction

In [2] it was shown (using the Weil conjectures) that the $\ell$-adic Betti numbers of a nonsingular projective algebraic variety $X$ (over an algebraically closed field) having a $G_m$ action can be recovered from those of the components $X_1, \ldots, X_r$ of the fixed point set $X^{G_m}$. It follows that for a smooth complex projective variety $X$ with algebraic $\mathbf{C}^*$ action, there exist canonical isomorphisms

$$\bigoplus_{j=1}^{r} H_{k-2m_j}(X_j; \mathbf{Q}) \cong \bigoplus_{j=1}^{r} H_{k-2m_j}(X_j^+; \mathbf{Q}) \cong H_k(X; \mathbf{Q}) \tag{1}$$

for $k=0,1,2,\ldots$ although such isomorphisms were not constructed explicitly. Here, $X = \bigcup X_j^+$ is the Bialynicki-Birula plus decomposition of $X$ (see [1]),

$$X_j^+ = \{x \in X \mid \lim_{\lambda \to 0} \lambda \cdot x \in X_j\}$$

and $m_j$ is the fibre dimension of the bundle

$$P_j\colon\ X_j^+ \to X_j; \qquad P_j(x) = \lim_{\lambda \to 0} \lambda \cdot x.$$

This homology basis formula (1) was motivated by a theorem of Frankel [7]. In [5], also motivated by [7], it was shown that (1) is valid for integer coefficients if $X$ is a compact Kaehler manifold with holomorphic $\mathbf{C}^*$ action having fixed points. In this paper we further extend (1) by allowing $X$ to be a (singular) compact complex space admitting a so called good decomposition, which is a generalization of the Bialynicki-Birula decomposition of a smooth complex projective variety with algebraic $\mathbf{C}^*$ action.

[1] Partially supported by the Natural Sciences and Engineering Research Council of Canada
[2] Partially supported by the Natural Sciences and Engineering Research Council of Canada and the Sloan Foundation

**Theorem 1.** *Suppose* $X=\bigcup_{j=1}^{r} W_j$ *is a "good decomposition"* (§1) *of* $X$ *into locally closed subsets* $W_1, \ldots, W_r$. *Then there are integers* $m_j$ *such that*

$$H_k(X;\mathbf{Z}) \cong \bigoplus_{j=1}^{r} H_{k-2m_j}(W_j;\mathbf{Z}). \tag{2}$$

The isomorphism (2) is explicitly constructed on the chain level (for a certain complex of "transverse" chains) and this leads to insight into certain phenomena such as restriction of (2) to subvarieties of $X$.

Theorem 1 is a partial generalization to singular spaces (and to integer coefficients) of the "glueing principle" [3] which arises from the Weil conjectures: certain algebraic decompositions of $X$ also decompose the homology and cohomology of $X$. Unfortunately, not every locally closed algebraic decomposition is "good", as we show in Example 5 (§3) where the homology basis formula (1) fails.

If $X$ is nonsingular, the isomorphism in Theorem 1 preserves $(p,q)$ types as in [5]. However, for singular $X$ we do not know whether this isomorphism is compatible with the weight filtration.

An important source for examples of good decompositions is among the invariant subvarieties of compact Kaehler manifolds with $\mathbf{C}^*$ action. In Theorem 4 of §2 we state a sufficient condition that the plus decomposition of an invariant subvariety be good. It is an open question whether formula (2) holds for each $\overline{X_j^+}$ when $X$ itself is smooth.

The plan of this paper is as follows: §1 concerns the definition of a good decomposition and the statements of the main results. In §2 we describe how a good decomposition can arise from a $\mathbf{C}^*$ action. §3 consists of examples. §4 contains the main technical tool of this paper (the 'plus construction'). The rest of the paper is devoted to the proofs of the theorems.

We would like to mention that Peter Orlik originally suggested to us that the homology basis formula might hold on singular varieties with $\mathbf{C}^*$ action. The final form of this paper has been greatly influenced by A. Bialynicki-Birula and we would like to thank him for several valuable discussions. We would also like to thank Andrew Sommese for his enlightening comments.

## § 1. Results on Good Decompositions

In this section, $X$ will denote a not necessarily irreducible but connected compact complex space.

**Definition.** A *good decomposition* of $X$ consists of a locally closed decomposition of $X$, say $X=\bigcup_{i=1}^{r} W_i$, where the subvarieties $W_i$, called *cells* of the decomposition, satisfy the following:

(1a) for each $i$, $1 \leqq i \leqq r$, there exists a holomorphic map $p_i$: $W_i \to X_i$ onto a compact complex space $X_i$ making $W_i$ a topologically locally trivial fibre bundle with affine fibres $\mathbf{C}^{m_i}$;

(1b) each $p_i$ extends meromorphically to $\overline{W_i}$ in the sense that

$$\Gamma_i = \text{topological closure } \{(x, p_i(x)) \mid x \in W_i\} \subset X \times X_i$$

is a closed subvariety of $X \times X_i$ containing the graph of $p_i$ as a Zariski open set.

(1c) if $X_i$ is singular, then it admits an analytic Whitney stratification such that if $A$ is a connected component of a stratum, then $g_i^{-1}(A)$ is irreducible, where $g_i\colon \Gamma_i \to X_i$ denotes the projection on the second factor; and

(1d) there exists a filtration of $X$ by closed subvarieties

$$\emptyset = Z_0 \subset \ldots \subset Z_r = X$$

such that (for some renumbering of $X_1, \ldots, X_r$), $Z_i - Z_{i-1} = W_i$ for $1 \leqq i \leqq r$.

The technical condition (1c) allows us to define homomorphisms

$$\mu_j\colon\ H_{k-2m_j}(X_j; \mathbf{Z}) \to H_k(X; \mathbf{Z})$$

with the following useful property:

For almost every cycle $z$ in $X_j$, $\mu_j(c\ell\, z)$ is (3)

is represented by $\overline{p_j^{-1}(z)}$ where $c\ell\, z$ denotes the homology class of $z$. The isomorphism (2) is the composition

$$\bigoplus H_{k-2m_j}(W_j; \mathbf{Z}) \to \bigoplus H_{k-2m_j}(X_j; \mathbf{Z}) \to H_k(X; \mathbf{Z})$$

defined by $\bigoplus_{j=1}^{r} \mu_j \circ p_{j*}$.

*Remarks.* (a) Suppose $\pi\colon E \to Y$ is a holomorphic affine bundle where $Y$ is smooth and compact. If $X$ is any compactification of $E$ so that $\pi$ extends meromorphically to $X$ in the sense of (1b), then $X = (X - E) \cup E$ is a good decomposition of $X$ and formula (2) decomposes a homology class in $X$ into a component at $\infty$ plus a component along $Y$ which comes from the Thom isomorphism. Unfortunately (1b) cannot be relaxed, for the homology basis formula fails for the decomposition of the elliptic Hopf surface $H = (\mathbf{C}^2 - (0,0))/\mathbf{Z}$, where $n \in \mathbf{Z}$ acts via $n \cdot (w, z) = (2^n w, 2^n z)$, as $T \cup (H - T)$ where $T$ is the torus $T = \{(w, 0)\} \subset H$. $H - T$ is made into the trivial line bundle on $T$ via the map $\pi(w, z) = z$ but $\pi$ does not extend meromorphically to $H$.

(b) Condition (1d) is necessary in order to avoid certain spaces such as the union of three $\mathbf{P}^1$'s joined together so as to form a ring. This space can be decomposed as $\mathbf{C} \cup \mathbf{C} \cup \mathbf{C}$ but (2) obviously fails although (1a, b, c) hold. Of course (1d) implies that $W_1$ is closed and $\overline{W_r} = X$. In the terminology of $\mathbf{C}^*$ actions, $X_1$ is called the *sink* of $X$ and $X_r$ is called the *source* of $X$ (with respect to the plus decomposition).

(c) Formula (1) is also known to hold for algebraic $\mathbf{C}^*$ actions on a complete algebraic manifold $X$. Condition (1d) may fail however for such actions as was noticed first in [11] using torus embeddings (see also [12]) and later in [6] and [19] using quite different methods.

Theorem 1 has several interesting corollaries.

**Corollary 1.** *Let $X$ have a good decomposition, let $\{Z_j\}$ denote a filtration satisfying* (1d), *and let $Y$ be a closed subvariety of $X$ such that, for some $i$, $Y - (Y \cap Z_i)$ is a union of certain cells $W_j$, say*

$$Y - (Y \cap Z_i) = W_{j_1} \cup \ldots \cup W_{j_s}$$

*Then $Y = (Y \cap Z_i) \cup W_{j_1} \cup \ldots \cup W_{j_s}$ is a good decomposition of $Y$. In particular, if $J = \{j_1, \ldots, j_s\}$, then for every $k$,*

$$H_k(Y;\mathbf{Z})=H_k(Y\cap Z_i;\mathbf{Z})\oplus[\bigoplus_{j\in J}H_{k-2m_j}X_j;\mathbf{Z})]$$

*If $Y$ is, in fact, a union of cells in $X$, e.g. if $Y=Z_i$ for some $i$, then $H_{\bullet}(Y;\mathbf{Z})$ is a direct summand in $H_{\bullet}(X;\mathbf{Z})$.*

*Proof.* In fact, for the first part one only needs to show that the decomposition of $Y$ is filterable (i.e. satisfies (1d)), and this follows from induction using the fact that $X$ satisfies (1d). For the second part, just use Theorem 1. For the last assertion, use the fact that for every $a$, the diagram

$$\begin{array}{ccc}\bigoplus_{j\in J}H_{k-2m_j}(X_j;\mathbf{Z}) & \longrightarrow & H_k(Y;\mathbf{Z})\\ \big\downarrow & & \big\downarrow\\ \bigoplus_{a=1}^{s}H_{k-2m_a}(X_a;\mathbf{Z}) & \longrightarrow & H_k(X;\mathbf{Z})\end{array}$$

is commutative with respect to the natural maps, which follows from (3). □

There is a useful relative version of Theorem 1 for pairs $(X, Y)$ where $X$ has a good decomposition $X=\bigcup W_i$ and $Y$ is a closed subvariety of $X$ that is a union of certain $W_i$, say $Y=W_{j_1}\cup\ldots\cup W_{j_s}$ as in Corollary 1.

**Corollary 2.** *For each $k$,*

$$H_k(X,Y;\mathbf{Z})\cong\bigoplus_{i\notin J}H_{k-2m_i}(X_i;\mathbf{Z}),$$

*where $J=\{j_1,\ldots,j_s\}$.*

The next theorem was proved in [5] for $\mathbf{C}^*$ actions. We state it here for completeness.

**Theorem 2.** *Let $X$ be a compact Kaehler manifold with good decomposition $X=\bigcup_{i=1}^{r}W_i$ where all $X_i$ are smooth. Let $m_i$ denote fibre $\dim_{\mathbf{C}}p_i$. Then:*

(1) *for all $p$ and $q$,*

$$H^p(X,\Omega^q)\cong\bigoplus_{i=1}^{r}H^{p-m_i}(X_i,\Omega^{q-m_i});$$

(2) *there exists an exact sequence*

$$0\rightarrow K\rightarrow \mathrm{Pic}(X)\rightarrow \mathrm{Pic}(X_r)\rightarrow 0,$$

*where $X_r$ denotes the source of $X$ and $K$ denotes the $\mathbf{Z}$-module of divisors generated by the $\overline{W_i}$ of codimension one in $X$,*

(3) $\mathrm{Index}(X)=\sum_{i=1}^{r}\mathrm{Index}(X_i)$,

*and*

(4) $\pi_1(X)=\pi_1(X_r)$.

One of the pleasant features of good decompositions is that they are well behaved with respect to certain birational transformations.

**Theorem 3.** *Let $f$: $X \to X'$ be a holomorphic map of compact complex spaces where $X$ has a good decomposition $X = \bigcup W_i$, $1 \leqq i \leqq r$. Suppose $Y$ is a closed subvariety of $X$ so that $Y - (Y \cap W_1)$ is a union of cells of $X$, say $Y = (Y \cap W_1) \cup W_2 \cup \ldots \cup W_s$. Then if $f$: $X - Y \to X' - f(Y)$ is an isomorphism, the decomposition $X' = W_1' \cup W_2' \cup \ldots \cup W_{r-s}'$, where $W_1' = f(W_1 \cup Y)$ and $W_i' = f(W_{s+i})$ for $1 \leqq i \leqq r-s$, is a good decomposition of $X'$.*

**Corollary 3.** (i) *Let $f$: $X \to X'$ be a holomorphic surjective map blowing down a subvariety $Y$ of the sink $W_1$ of $X$. Then $X'$ has a good decomposition $W_1' \cup W_2' \cup \ldots \cup W_r'$ where each $W_i' = f(W_i)$.*

(ii) *If the subvariety of $Y$ of $X$ is a union of cells from $X$, say $Y = W_1 \cup \ldots \cup W_s$, then there exists a short exact sequence*

$$0 \to H_k(Y) \xrightarrow{\alpha} H_k(X) \oplus H_k(f(Y))$$

$$\xrightarrow{\beta} H_k(X') \to 0$$

*where $\alpha(x) = (i_*(x), 0)$ and $\beta(x, y) = f_*(x) + j_*(y)$, $i$: $Y \to X$ and $j$: $f(Y) \to X'$ denoting the inclusions. Moreover, $\alpha$ and $\beta$ split over $\mathbf{Z}$.*

This sequence bears some similarity to the exact homology sequence of a birational morphism [10].

## §2. Good Decompositions Arising from C* Actions

In this chapter we assume $X$ is a closed invariant subvariety of some compact Kaehler manifold $Y$ with a holomorphic $\mathbf{C}^*$ action which has a nonempty fixed point set.

Let $X_1, \ldots, X_r$ denote the components of the fixed point set of $X$. The *Bialynicki-Birula plus decomposition* of $X$ is the union

$$X = \bigcup_{j=1}^{r} X_j^+ \quad \text{where } X_j^+ = \{x \in X \mid \lim_{\lambda \to 0} \lambda \cdot x \in X_j\}.$$

This decomposition exists by [18].

**Definition.** The $\mathbf{C}^*$ action on $X$ (which is induced from that on $Y$) is *good as* $\lambda \to 0$ if the associated Bialynicki-Birula plus decomposition is good.

*Remark.* Conditions (1b) and (1d) are always satisfied by the Bialynicki-Birula decomposition - see [4, 5] for the basic properties of this decomposition.

For good actions, $H_i(X_j) = H_i(X_j^+)$ so in this context, Theorem 1 becomes

**Theorem 1'.** *Suppose $X$ has a good $\mathbf{C}^*$ action with fixed point components $X_1, X_2, \ldots, X_r$. Then for all $k$,*

$$H_k(X) \cong \bigoplus_{j=1}^{r} H_{k-2m_j}(X_j)$$

*where $m_j$ is the fibre dimension of the affine space bundle $X_j^+ \to X_j$.*

**Definition.** The $\mathbf{C}^*$ action on $X$ is *singularity preserving* as $\lambda \to 0$ if there exists an equivariant Whitney stratification of $X$ such that for every stratum $A$, and for every $x \in A$, the limit $x_0 = \lim_{\lambda \to 0} \lambda \cdot x$ is also in $A$.

The role of the singularity preserving condition is in the following lemma.

**Lemma 1.** *Suppose $X$ admits a Whitney stratification which is singularity preserving as $\lambda \to 0$. Then condition* (1a) *is satisfied, i.e. $p_j$: $X_j^+ \to X_j$ is a locally trivial affine space bundle.*

**Theorem 4.** *Suppose $X$ admits a singularity preserving analytic Whitney stratification. Suppose that for any stratum $A$ of $X$ and for any fixed point component $X_j$, either $A \cap X_j = \emptyset$ or $\overline{A} \cap \overline{X_j^+} = \overline{(A \cap X_j)^+}$. Then the nonempty $A \cap X_j$ are the strata of a Whitney stratification of $X^{\mathbf{C}^*}$ which makes the $\mathbf{C}^*$ action good.*

Of course, one can also consider actions that are good as $\lambda \to \infty$, i.e. those for which the minus decomposition is good. For singular $X$, an action that is good as $\lambda \to 0$ may or may not be good as $\lambda \to \infty$ - see Examples (1), (2), (4).

## §3. Examples of Good Actions and Other Good Decompositions

*Example (1). A good action with no minus isomorphism.* Let $E$ be the cone (in $\mathbf{P}^3$) with vertex $p = (0,0,0,1)$ over an elliptic curve $Y \subset \mathbf{P}^2$. $E$ is invariant under the $\mathbf{C}^*$ action $\lambda \cdot (z_0, z_1, z_2, z_3) = (z_0, z_1, z_2, \lambda z_3)$ and $E^{\mathbf{C}^*} = Y \cup \{p\}$. $E$ is the Thom space of a negative line bundle on $Y$ and the plus isomorphism is the Thom isomorphism. In particular, $H_1(E) = 0$ which shows there is no minus isomorphism.

*Example (2). A good action with a singular source.* The $\mathbf{C}^*$ action on $\mathbf{C}^5$ which is given by

$$\lambda \cdot (z_1, z_2, z_3, z_4, z_5) = (z_1, z_2, \lambda z_3, z_4, z_5)$$

takes subspaces into subspaces and thus determines an action on the Grassman manifold $G_2(\mathbf{C}^5)$. The Schubert variety

$$X = \Omega(3,5) = \{P \in G_2(\mathbf{C}^5) | \dim_{\mathbf{C}}(P \cap \mathbf{C}^3) \geqq 1\}$$

is invariant. $X$ may be stratified with one singular stratum

$$\Sigma = \{P \in G_2(\mathbf{C}^5) | P \subset \mathbf{C}^3\}$$

and the action is singularity preserving as $\lambda \to 0$. $X^{\mathbf{C}^*}$ has two components

$$X_1 = \{P \in X | P \subset \langle e_1, e_2, e_4, e_5 \rangle \text{ and } \dim_{\mathbf{C}}(P \cap \langle e_1, e_2 \rangle) \geqq 1\},$$
$$X_2 = \{P \in X | P \supset \langle e_3 \rangle \text{ and } \dim_{\mathbf{C}}(P \cap \langle e_1, e_2, e_4, e_5 \rangle) = 1\}$$

where $e_i$ denote the standard basis vectors in $\mathbf{C}^5$ and $\langle\ \rangle$ denotes the span. Then $X_1 \cong \Omega(2,4)$, is singular, and has Betti numbers $1,0,1,0,2,0,1$. $X_2 \cong \mathbf{P}^3$

and has Betti numbers 1, 0, 1, 0, 1, 0, 1. $\Sigma \cap X_1 \cong$ a point and $\Sigma \cap X_2 \cong \mathbf{P}^1$. $X_1^+ \to X_1$ is a bundle with fibre $\cong \mathbf{C}^2$ while $X_2^+ = X_2$. $X_1$ can be stratified with one singular stratum (consisting of the singular point) and this renders the action good as $\lambda \to 0$.

From Theorem 1, the Betti numbers of $X$ can be read off from the following table:

| dim | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\beta_i$ | 1 | 0 | 1 | 0 | 1+1 | 0 | 1+1 | 0 | 2 | 0 | 1 |

$X_2^- \to X_2$ is not a fibration and the homology basis formula fails if we use the minus homomorphism.

*Example (3). The Springer fibre for* $SL(n, \mathbf{C})$ ([17, 20]). Let $F(V)$ denote the flag manifold of $V = \mathbf{C}^n$ and let $u \in SL(n, \mathbf{C})$ be a fixed unipotent matrix. The variety $X_u$ in $F(V)$ consisting of flags fixed by $u$ admits a good decomposition into affines, by [17]. $X_u$ also admits $\mathbf{C}^*$ actions induced by the one parameter subgroups of the centralizer of $u$. If the Young diagram associated to $u$ as in [17] has $r$ rows with $\lambda_i$ cases in the $i$-th row and $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_r$, then a $\mathbf{C}^*$ action on $X_u$ is obtained as follows: fix integers $\alpha_1, \ldots, \alpha_r$ and for all $i$ such that

$$\sum_{j=1}^{s-1} \lambda_j \leqq i \leqq \sum_{j=1}^{s} \lambda_j$$

set

$$\lambda \cdot e_i = \lambda^{\alpha_s} e_i \qquad (s = 1, \ldots, r).$$

This action defines a one parameter subgroup of the centralizer of $u$, hence a $\mathbf{C}^*$ action on $X_u$. Unfortunately, the affine decomposition of [17] is only the plus decomposition associated to a $\mathbf{C}^*$ action with isolated fixed points in special cases, as the following lemma shows.

**Lemma.** (a) *If* $\alpha_1, \ldots, \alpha_r$ *are all distinct, then* $X_u^{\mathbf{C}^*}$ *is an isolated set having* $\binom{n}{\lambda_1, \ldots, \lambda_r}$ *points.*

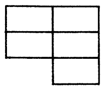
(b) $X_u$ *has a good action with isolated fixed points precisely in the following cases:* (i) $r=1$ *or* $r=2$, *and* (ii) $r$ *is arbitrary but* $\lambda_2 = \lambda_3 \ldots = \lambda_r = 1$ *and* $\alpha_1 > \alpha_2 > \ldots > \alpha_r$. *In the case* $r=1$ *or* $r=2$, *the action is good both as* $\lambda \to 0$ *and* $\lambda \to \infty$. *In case* (ii), *the action is only good as* $\lambda \to 0$.

(c) *If the diagram of u has more entries in the first column than the second, then any* $\mathbf{C}^*$ *action with* $\alpha_1 = \alpha_2 = \ldots = \alpha_{r-1} > \alpha_r$ *is good as* $\lambda \to 0$ *(but fails to have isolated fixed points).*

*Outline of the Proof.* The first part of part (a) follows from the fact that the maximal torus in the centralizer of $u$ has the same fixed point set as the one parameter group being considered, so that $X_u^{\mathbf{C}^*}$ must be finite. To see the second part of (a) it is easiest to consider an example. Suppose that the diagram of $u$ is  Then we may suppose that $u$ has been chosen so that ($u$

$-I$): $e_2 \to e_1 \to 0$, $e_4 \to e_3 \to 0$. Hence the flags fixed by both $u$ and $\mathbf{C}^*$ are of the form $(F_0, F_1, F_2, F_3, F_4)$ where $F_i$ is generated by $i$ of the vectors $e_1, e_2, e_3, e_4$ and $(u-I) F_i \subset F_{i-1}$. Therefore the fixed flags can be uniquely labelled by the permutations of $1,2,3,4$ where 1 preceeds 2 and 3 preceeds 4. The fixed flags are (1 2 3 4), (1 3 2 4), (1 3 4 2), (3 1 2 4), (3 1 4 2), (3 4 1 2). Obviously, in the general case, this procedure leads to exactly $\binom{n}{\lambda_1, \ldots, \lambda_r}$ fixed points.

The proofs of (b) and (c) of the lemma are done by induction, using the construction of Spaltenstein in [17, Lemma, p. 453] which is, in fact, $\mathbf{C}^*$ equivariant. As soon as the diagram for $u$ contains the sub-diagram



(4)

any action on $X_u$ with isolated fixed points fails to be good both as $\lambda \to 0$ and $\lambda \to \infty$. The action on $X_u$ associated to (4) with $\alpha_1 > \alpha_2 > \alpha_3$ has a singular three dimensional plus cell and a singular two dimensional minus cell. In the case (b) where $u$ has two Jordan blocks ($r=2$) or is of one hook type, the plus decomposition decomposes $X$ into a union of affines (if the fixed points are isolated), giving us another proof of a special case of the result of Spaltenstein [17].

*Example (4). Blowing down a good action.* Let $\mathbf{F}_n$ denote the rational ruled surface $\mathbf{P}(\mathbf{C} \oplus \mathcal{O}(n))$, and let $C$ denote the exceptional curve with $C^2 = -n$. $\mathbf{F}_n$ is a compactification of $\mathbf{C}^* \times \mathbf{C}^*$, i.e. a torus embedding. The map $f$: $\mathbf{F}_n \to \mathbf{F}_n'$ blowing $C$ down to a point is equivariant with respect to any $\mathbf{C}^*$ action on $\mathbf{F}_n$ and by Theorem 3 any $\mathbf{C}^*$ action on $\mathbf{F}_n$ induces a good one on $\mathbf{F}_n'$. Notice that if $\mathbf{F}_n^{\mathbf{C}^*}$ is finite, then the action on $\mathbf{F}_n'$ is not singularity preserving either as $\lambda \to 0$ or $\lambda \to \infty$, although it is good in both directions.

The rest of this section is devoted to showing that some condition such as the graph condition (1c) is needed for Theorem 1 to hold.

*Example (5).* In this example we consider a variety $X$ with a decomposition satisfying (1a), (1b), (1d) but not (1c) for which the homology basis formula is false. Let $Y$ denote the nodal curve or "pinched torus" obtained by identifying two points of $\mathbf{P}^1$. Thus the singularity of $Y$ is a normal crossing at the singular point $y_0$ of $Y$. Let $X$ denote the space obtained from $Y \times \mathbf{P}^1$ by blowing up $(y_0, \infty)$ where $\infty = [0,1]$. Let $f$: $X \to \mathbf{P}^1$ be the composition $X \to Y \times \mathbf{P}^1 \to \mathbf{P}^1$ and define a decomposition of $X$ by

$$W_1 = f^{-1}(\infty)$$
$$W_2 = f^{-1}(\mathbf{P}^1 - \infty).$$

$W_2$ is the trivial line bundle on $Y$ and $W_1 = Z_1$ consists of three copies of $\mathbf{P}^1$ joined at points so as to form a cycle (i.e. ring). Condition (1c) fails for the projection $W_2 \to Y$ for $g_2^{-1}(y_0)$ is a union of three $\mathbf{P}^1$'s meeting at a point. The homology basis formula fails because $H_3(W_1) \oplus H_1(W_2) = \mathbf{Z}$, while $H_3(X) = 0$. In fact the vanishing of $H_3(X)$ can most easily be seen by observing that $X$

admits a good action. Namely the action $\lambda\cdot(y,[z_0,z_1])=(y,[\lambda z_0,z_1])$ on $Y\times\mathbf{P}^1$ induces a good action on $X$ whose source is the $\mathbf{P}^1\subset W_1$ corresponding to the desingularization of $Y$, whose sink is $Y\times[1,0]$, and whose remaining fixed component is an isolated fixed point whose plus cell is $\mathbf{C}$. The plus cell over the source is $\mathcal{O}(-2)$. Since $H^1(\mathbf{P}^1)=0$, $H_3(X)=0$ too.

## §4. Preliminary Remarks on the Plus Construction

The main step in the proof of Theorem 1 is the construction of wrong way homomorphisms $H_k(X_j;\mathbf{Z})\to H_{k+2m_j}(\overline{W}_j;\mathbf{Z})$. The definition of these maps, when $X_j$ is smooth, goes as follows ($\mathbf{Z}$ coefficients understood). We have the sequence

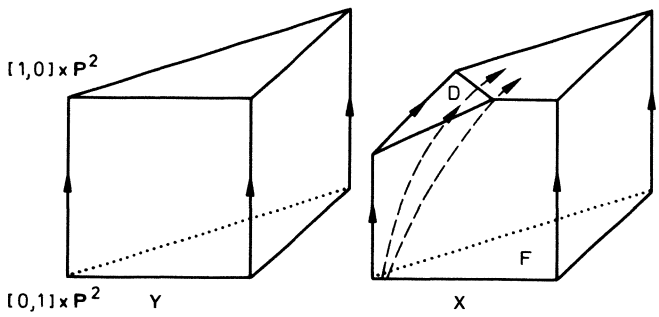
$$H_k(X_j)\xrightarrow{PD} H^{2b_j-k}(X_j)\to H^{2b_j-k}(\tilde{\Gamma}_j)$$

$$\xrightarrow{PD} H_{k+2m_j}(\tilde{\Gamma}_j)\to H_{k+2m_j}(\overline{W}_j)$$

where $PD$ stands for Poincaré duality, $\tilde{\Gamma}_j$ denotes a resolution of singularities of $\Gamma_j$, and $b_j=\dim_{\mathbf{C}} X_j$. However, our aim is to describe these wrong way maps on the chain level even when $X_j$ is singular. To do this, we describe a sub-complex $C_*^T(X_j)$ of the complex of singular chains on $X_j$ which admits a wrong way chain map $C_i^T(X_j)\to C_{i+2m_j}(\Gamma_j)$. If $X_j$ is smooth, then the first step will be to filter $X_j$ by subvarieties $X_{jk}=\{x\in X_j|\dim_{\mathbf{C}} g_j^{-1}(x)\geqq k+m_j\}$. The smoothness implies that the codimension of $X_{jk}$ in $X_j$ is at least $k+1$. The chains that lift nicely under the plus construction to $\Gamma_j$ are those "transverse" to the stratification $\{X_{jk}-X_{jk+1}\}$ of $X_j$.

There are two reasons such transverse cycles must be used. First, not all cycles on $X_j$ lift to cycles on $\Gamma_j$ (see Example (5)). The other reason is that homologous cycles in $X_j$ may lift to nonhomologous cycles in $\overline{W}_j$. Such behavior is due to the existence of envelopes at infinity, as the following example shows.

*Example (6).* Let $Y$ denote $\mathbf{P}^1\times\mathbf{P}^2$. Blow up the point $[1,0]\times[1,0,0]$ on $Y$ to obtain $X$. The $\mathbf{C}^*$ action $(\lambda,[z_0,z_1],[y_0,y_1,y_2])\to([z_0,\lambda z_1],[y_0,y_1,y_2])$ on $Y$ lifts to $X$, and the plus decomposition associated to this action on $X$ is a good decomposition. Let $D\subset X$ denote the exceptional divisor. $D$ is the "envelope" of limits of orbits. Take $F=[0,1]\times\mathbf{P}^2$. Then $D\subset\overline{F^+}-F^+$ but $D$ is not contained in the fiberwise compactification of $F^+$. In fact $D$ is isomorphic to a subvariety of the fiber of the projection $g\colon\Gamma\to F$ over $[0,1]\times[1,0,0]$ as in Example (5) where $\Gamma$ is the closure of the graph defined in §1. The naive definition of the plus homomorphism is to lift the cycle $\xi$ in $F$ to $\overline{\xi^+}$ and show that this yields a cycle in $X$. To see why homologous cycles in $F$ may lift to nonhomologous cycles, in $X$, let $X=[0,1]\times[1,0,0]$ and let $y$ be any other point of $F$. Then $\overline{y^+}\sim\overline{x^+}+z$, where $z$ is a nontrivial 2-cycle in $D$, so $\overline{x^+}$ and $\overline{y^+}$ are nonhomologous. Note that in the plus isomorphism for $X$, the homology class of $z$ comes from the sink of $X$; in the diagram below, this class arises by




intersecting $D$ and the top face of $X$. Instead, we will show in §5 how to lift transverse cycles (e.g. $y$) in $F$ to $X$.

## §5. The Plus Construction

Throughout this section, $W_j$ will denote one cell of a good decomposition of $X$, and $A$ a connected stratum of $X_j$ satisfying (1c). Homology groups are understood to have integer coefficients. We will define the homomorphism $H_i(X_j) \to H_{i+2m}(X)$, where $m$ is the complex dimension of the fibre of the affine space bundle $p: W_j \to X_j$. This construction associates to *almost* every oriented cycle $\xi \subset X_j$ the oriented cycle $\tilde{\xi} = \pi_* g^{-1}(\xi)$ where $g: \Gamma \to X_j$ and $\pi: \Gamma \to X$ are the projections and $\Gamma$ denotes the graph $\Gamma_j$ of (1b). The surprising fact is that $\tilde{\xi}$ is a cycle, i.e. that $\partial \tilde{\xi} = 0$.

Let $\Gamma_A = g^{-1}(\bar{A})$. Since the projection $\Gamma_A \to \bar{A}$ is analytic, there is a proper subvariety $\Sigma_A \subset \bar{A}$ such that $g^{-1}(\bar{A} - \Sigma_A) \to \bar{A} - \Sigma_A$ is a topologically locally trivial fibre bundle whose fibre is an analytic compactification of $\mathbf{C}^m$. Define

$$A_0 = \Sigma_A,$$
$$A_k = \{x \in \bar{A} \mid \dim_{\mathbf{C}} g^{-1}(x) \geqq m + k\} \quad \text{for } k \geqq 1.$$

Then we have the following key fact based on the graph condition.

**Proposition.** *For any $k \geqq 0$ the complex codimension of $A_k$ in $A$ is at least $k+1$.*

*Proof.* Since $A$ is connected, $\Gamma_A$ is irreducible. If the proposition were false for some $k$, then $g^{-1}(A_k)$ would be a proper subvariety of codimension 0 in $\Gamma_A$, which is impossible. □

*Remark.* $\Gamma_A$ is, in fact, the closure of the graph of $p_j | p_j^{-1}(A)$ in $X \times X_j$.

Let $C_*(X_j)$ be the complex of subanalytic chains on $X_j$ (see [9]). For each chain $\xi \in C_i(X_j)$, let $|\xi|$ denote the support of $\xi$.

**Definition.** A subanalytic chain $\xi \in C_i(X_j)$ is *transverse* if, for every stratum $A$ of $X_j$ and for every $k \geqq 0$,

$$\dim_{\mathbf{R}}(|\xi| \cap A_k) \leqq i - 2k - 2$$

and

$$\dim_{\mathbf{R}}(|\partial\xi| \cap A_k) \leqq i-2k-3$$

where

$$A_k \quad \text{is defined in §4.}$$

The subcomplex of transverse chains will be denoted $C_*^T(X_j)$.

**Lemma.** *The inclusion $C_*^T(X_j) \to C_*(X_j)$ induces an isomorphism on homology.*

*Proof.* For any subanalytic cycle $\xi$ there is a triangulation of $X_j$ such that $|\xi|$ and each $A_k$ are subcomplexes. By [14], $\xi$ can be approximated by a P.L. cycle $\xi'$ such that for each stratum $A$, $|\xi'| \cap A$ is in general position with respect to each $A_k \cap A$. Thus, $\xi'$ is "transverse" and it may be further approximated by a nearby subanalytic cycle $\xi''$ which is still transverse. Therefore every homology class has a transverse representative. A relative version of this argument shows that any two transverse representations of a given homology class can be made homologous by a transverse chain. □

**Definition.** For any transverse chain $\xi \in C_i^T(X_j)$, let $g^*(\xi) \in C_{i+2m}(\Gamma)$ denote the chain whose support is $|g^*(\xi)| = g^{-1}(|\xi|)$ and whose orientation is the product of the orientation of $\xi$ with the canonical orientation of the fibres of $g$.

**Proposition.** *If $\xi \in C_i^T(X_j)$ then $\partial g^*(\xi) = g^*(\partial\xi)$ so $g^*$ induces a map on homology (which we also denote by $g^*$: $H_i(X_j) \to H_{i+2m}(\Gamma)$).*

*Proof.* We shall denote by $\Sigma$ the set of nongeneric points in the image of $g$,

$$\Sigma = \bigcup_A \Sigma_A$$

where the union runs over all strata $A$ of $X_j$. It follows that $g^{-1}(X_j - \Sigma) \to (X_j - \Sigma)$ is a topological fibre bundle with a compact oriented fiber of real dimension $2m$. But $g^{-1}(|\xi|)$ is an oriented subanalytic subset of $\Gamma$ with real dimension $i+2m$ and

$$\dim_{\mathbf{R}} g^{-1}(|\xi| \cap \Sigma) \leqq i+2m-2,$$
$$\dim_{\mathbf{R}} g^{-1}(|\partial\xi| \cap \Sigma) \leqq i+2m-3$$

(which can be verified by counting $\dim g^{-1}(|\xi| \cap A_k)$. Therefore $\partial g^*(\xi)$ has no contribution from $g^{-1}(|\xi| \cap \Sigma)$. This implies $\partial g^*(\xi) = g^*(\partial\xi)$. □

**Definition.** The homomorphism

$$\mu_j\colon\ H_i(X_j) \to H_{i+2m}(X)$$

is the composition of the homomorphisms

$$g^*\colon\ H_i(X_j) \to H_{i+2m}(\Gamma)$$

and

$$H_{i+2m}(\Gamma) \to H_{i+2m}(X)$$

which is induced by projection to the second factor.

It is clear from the construction that $|\xi| = \text{closure } p^{-1}(|\xi|)$ whenever $\xi$ is a transverse chain in $F$. It is also clear that the homomorphism $\mu_j$ factors through $H_{i+2m}(\overline{W_j})$.

Our construction can be used to produce wrong way homomorphisms in other circumstances. For example, if $g\colon Z \to W$ is a surjective morphism between normal compact algebraic varieties such that all the fibres of $g$ have dimension $k$, then $g$ induces $g^*\colon H_i(W) \to H_{i+2k}(Z)$. The appropriate dimension estimates follow from the isomorphism $H_i(X) \cong IH_i^{\bar{t}}(X)$ of [8].

## §6. Proof of Theorem 1

The decomposition is filterable, i.e., there exists a filtration of $X$ by closed invariant subvarieties,

$$\emptyset = Z_0 \subset Z_1 \subset \ldots \subset Z_r = X$$

such that $Z_i - Z_{i-1} = W_i$. Consider the following exact sequences, where $\alpha$, $\beta$, $\gamma$ are the plus homomorphisms of the previous section:

$$\begin{array}{ccccccccc}
H_{q+1}(Z_k, Z_{k-1}) & \xrightarrow{\partial} & H_q(Z_{k-1}) & \longrightarrow & H_q(Z_k) & & \longrightarrow & H_q(Z_k, Z_{k-1}) & \xrightarrow{\partial} \\
 & & \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} & \nwarrow{\scriptstyle\gamma} & & \uparrow{\scriptstyle\delta} & \\
0 \longrightarrow & & \bigoplus_{j<k} H_{q-2m_j}(X_j) & \longrightarrow & \bigoplus_{j \leqq k} H_{q-2m_j}(X_j) & & \longrightarrow & H_{q-2m_k}(X_k) & \longrightarrow 0
\end{array}$$

The decomposition $Z_{k-1} = W_1 \cup \ldots \cup W_{k-1}$ is good. Hence by induction on $k$, $\alpha$ is an isomorphism. We must show $\delta$ is an isomorphism. Now $Z_k - X_k$ deformation retracts to $Z_{k-1}$ because:

(a) $X_k$ has a tubular neighbourhood $V$ in $Z_k$ so $Z_k - X_k$ deforms to $Z_k - V$,

(b) one can use the affine bundle structure on $W_k$ to deform $Z_k - V$ into some regular neighbourhood $U$ of $Z_{k-1}$ (by compactness), and

(c) $U$ can be chosen to deformation retract to $Z_{k-1}$. Thus one sees that $\delta$ is the Thom isomorphism since

$$H_q(Z_k, Z_{k-1}) \cong H_q(Z_k, Z_k - X_k) \cong H_q(W_k, W_k - X_k)$$

by excision. The diagram commutes, so $\partial = 0$. Theorem 1 now follows from the five lemma and induction on $k$. $\square$

To prove Corollary 2, consider the sequences

$$\begin{array}{ccccccc}
0 \longrightarrow & \bigoplus_{j \in J} H_{k-2m_j}(X_j; \mathbf{Z}) & \longrightarrow & \bigoplus_{j} H_{k-2m_j}(X_j; \mathbf{Z}) & \longrightarrow & \bigoplus_{j \notin J} H_{k-2m_j}(X_j; \mathbf{Z}) & \longrightarrow 0 \\
 & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & \\
0 \longrightarrow & H_k(Y; \mathbf{Z}) & \xrightarrow{\iota_*} & H_k(X; \mathbf{Z}) & \rightarrow & H_k(X, Y; \mathbf{Z}) & \longrightarrow 0
\end{array}$$

where the maps $\alpha$ and $\beta$ are the isomorphisms of Theorem 1 and the homomorphism $\gamma$ is induced by an obvious relativization of the plus homomorphism. Now the top sequence is exact and from this and the fact that the bottom sequence contains part of the long homology sequence of the pair $(X, Y)$, it follows that the bottom sequence is also exact, due to the fact that $\iota_*$: $H_k(Y; \mathbf{Z}) \to H_k(X; \mathbf{Z})$ is injective for all $k$. Hence, by the five lemma, $\gamma$ is an isomorphism. □

## § 7. Proof of Theorem 3

Note first that the decomposition $X' = f(Y \cup W_1) \cup f(W_{s+1}) \cup \ldots \cup f(W_r)$ consists of locally closed affine bundles provided we let $f(Y \cup W_1)$ be the sink of $X'$. Moreover, since $Y \cup W_1$ is closed and a union of plus cells in $X$, one can show directly that the decomposition of $X'$ is filterable. Hence conditions (1a, b, d) are satisfied. To show (1c), let us suppose for simplicity that $W = W_t$, $W' = f(W_t)$ and $Z = X_t$. Also, let $\Gamma$ and $\Gamma'$ denote, respectively, the closures of the graphs of $p$: $W \to Z$ and $p'$: $W' \to Z$. We must verify that $\Gamma' | A$ is irreducible for any connected component of a stratum for the Whitney stratification of $Z$ satisfying (1c) for $\Gamma$. Now the point of condition (1c) is that for any such $A$,

$$\Gamma | \bar{A} = \overline{\text{graph } (p | A)} \subset X \times Z \tag{5}$$

For the right hand side of (5) is always an irreducible component of $\Gamma | \bar{A}$. Using the isomorphism $f_*$: graph $(p|A) \to$ graph $(p'|A)$ induced by $f$, we obtain a morphism $f_*$ of $\Gamma | \bar{A}$ onto an irreducible variety containing $\Gamma' | A$. Hence $\Gamma' | A$ must also be irreducible. □

Corollary 3(i) follows immediately from Theorem 3. The proof of Corollary 3(ii) is identical with the proof of Corollary 2, so it will be omitted.

## § 8. Proof of Theorem 4

Suppose $X$ has a Whitney stratification that is singularity preserving as $\lambda \to 0$, and let $F$ denote a fixed point component. To show that for any $x \in X$, $p^{-1}(x)$ is equivariantly biholomorphic to some $\mathbf{C}^m$, let $x \in A \cap F$ for some stratum $A$ of $X$. By the singularity preserving condition, it follows that $x^+ = \{y \in X \colon y_0 = x\} \subset A$. But as $A$ is nonsingular, the result follows from [4]. The linearized $\mathbf{C}^*$ action on $T_x A$ gives rise to subspaces $(T_x A)^-$, $(T_x A)^0$, $(T_x A)^+$ generated by all vectors of negative, zero, or positive weight. These subspaces may be locally identified with $T_x(x^- \cap A)$, $T_x(F \cap A)$ and $T_x(x^+ \cap A)$ by exponentiation. Therefore $F \cap A$ is a manifold and the fibres of $p^{-1}(F \cap A) \to F \cap A$ form a smooth affine space bundle as in [4] or [1].

Now suppose $y \in \bar{A} \cap F$. We shall show $\dim_{\mathbf{C}}(y^+) = \dim_{\mathbf{C}}(x^+)$. By the upper semicontinuity principle applied to $\bar{A} \cap F$, we have $\dim_{\mathbf{C}} x^+ \leqq \dim_{\mathbf{C}} y^+$. To get the reverse inequality, suppose $B$ is the stratum containing $y$ and choose a sequence $x_i \in A \cap F$ such that $x_i \to y$. By choosing a subsequence if necessary we may assume the $T_{x_i}A$ converge to some limiting plane $\tau$ and the subspaces $(T_{x_i}A)^-$, $(T_{x_i}A)^0$, and $(T_{x_i}A)^+$ converge to subspaces of $\tau$. By continuity, $\tau$ is $\mathbf{C}^*$ invariant. Since the weights are integers, $\lim(T_{x_i}A)^0 \subset \tau^0$ and $\lim(T_{x_i}A)^{\mp} \subset \tau^{\mp}$. Thus these inclusions are equalities. Whitney's condition $A$ says $T_y B \subset \tau$, so $(T_y B)^+ \subset \tau^+$. Hence

$$\dim_{\mathbf{C}} y^+ = \dim_{\mathbf{C}}(T_y B)^+ \leqq \dim_{\mathbf{C}} \tau^+ = \dim_{\mathbf{C}}(T_x A)^+ = \dim_{\mathbf{C}} x^+$$

which establishes the reverse inequality. In fact, we have shown that $(T_y B)^+ = \tau^+ = \lim(T_{x_i}A)^+$ so $F^+ \to F$ is (topologically) locally trivial near $y$. This completes the proof of Lemma 1. □

Let $F$ be any fixed point component. We claim that $F$ can be Whitney stratified with the manifolds $A^* = A \cap F$ where $A$ runs over the strata of $X$. We check the Whitney conditions for a pair of such manifolds $(A^*, B^*) = (A \cap F, B \cap F)$ at a point $y \in B^*$. Let $x_i \in A^*$ and $y_i \in B^*$ converge to $y$ and suppose the secant lines $\overrightarrow{x_i y_i}$ converge to some line $\ell$ and suppose the tangent planes $T_{x_i}A^*$ converge to some $\tau^*$. We must show that $\ell \subset \tau^*$. We may suppose the $T_{x_i}A^*$ converge to some limiting plane $\tau$ as above, so $\tau^0 = \lim(T_{x_i}A)^0 = \lim T_{x_i}A^* = \tau^*$. Whitney's condition applied to $X$ guarantees that $\ell \subset \tau$ and in fact $\ell \subset \tau^0$ because $x_i$ and $y_i$ are in $F$. Thus $\ell \subset \tau^*$.

Now suppose $\Gamma_{A \cap F}$ denotes the closure in $X \times F$ of $\{(a, p(a)) | a \in A \cap F^+\}$. We will verify (1c) for the above stratification of $F$. To do so, it suffices to show $g^{-1}(A \cap F) = \Gamma_{A \cap F}$ since $\Gamma_{A \cap F}$ is irreducible. Let $w \in g^{-1}(A \cap F)$. Clearly $w = (g(w), \pi(w))$ where $g(w) \in g^{-1}(A \cap F)$. Clearly $w = (g(w), \pi(w))$ where $g(w) \in A \cap F$ and $\pi(w) \in \overline{F^+}$. We claim that there exists a chain of orbits from $g(w)$ to $\pi(w)$ which is completely contained in $\overline{A} \cap \overline{F^+}$. To see this, note that $w$ is a limit of points $(p(x_n), x_n)$ in $\Gamma$ such that $g(w) = \lim p(x_n)$. We may, by [13, Corollary to Theorem 1.1] choose a subsequence so that $\lim \mathbf{C}^* \cdot x_n$ exists and is a chain $\overline{\mathcal{O}}_1 \cup \ldots \cup \overline{\mathcal{O}}_t$ of closures of orbits, i.e. $(\overline{\mathcal{O}}_i)_\infty = (\overline{\mathcal{O}}_{i+1})_0$ for $1 \leqq i \leqq t-1$. Clearly $(\overline{\mathcal{O}}_1)_0 = g(w)$ and $\pi(w)$ lies in some $\overline{\mathcal{O}}_\ell$. The singularity preserving assumption implies that each $\overline{\mathcal{O}}_k \subset \overline{A}$. In fact if $y \in \overline{A} \cap F$ then $\overline{y^+} \subset \overline{A}$. (For if $B$ is the stratum of $X$ containing $y$, then $y^+ \subset B \subset \overline{A}$ by the frontier condition). It follows by induction that $\overline{\mathcal{O}}_1 \cup \ldots \cup \overline{\mathcal{O}}_t \subset \overline{A}$. Hence $\pi(w) \in \overline{A} \cap \overline{F^+} = \overline{(A \cap F)^+}$ (since $A \cap F \neq \emptyset$). This means $w = (g(w), \pi(w)) \in \Gamma_{A \cap F}$ which proves that $g^{-1}(A \cap F) \subset \Gamma_{A \cap F}$. The reverse inclusion is clear. This completes the proof that the action is good and the proof of Theorem 4. □

## References


1. Bialynicki-Birula, A.: Some theorems on actions of algebraic groups. Annals of Mathematics **98**, 480–497 (1973)
2. Bialynicki-Birula, A.: On fixed points of torus actions on projective varieties. Bull. de l'Acad. Polon des Sciences **32**, 1097–1101 (1974)
3. Bialynicki-Birula, A., Sommese, A.J.: Complete quotients of torus actions. Transactions of American Math. Society in press (1983)

4. Carrell, J.B., Sommese, A.J.: **C*** actions. Math. Scand. **43**, 49–59 (1978)
5. Carrell, J.B., Sommese, A.J.: Some topological aspects of **C*** actions on compact Kaehler manifolds. Comment. Math. Helvetici **54**, 567–582 (1979)
6. Carrell, J.B., Sommese, A.J.: Filtrations of complex manifolds with a **C*** action. Math. Scand. in press (1983)
7. Frankel, T.: Fixed points and torsion on Kaehler manifolds. Ann. of Math. **70**, 1–8 (1959)
8. Goresky, M., MacPherson, R.: Intersection homology theory. Topology **19**, 135–162 (1980)
9. Hardt, R.: Topological properties of subanalytic sets. Trans. Amer. Math. Soc. **211**, 57–70 (1975)
10. Hartshorne, R.: On the deRham cohomology of algebraic varieties. Publ. Math. I.H.E.S. n°45, 1–99 (1975)
11. Jurkiewicz, J.: An example of algebraic torus action which determines the non-filterable decomposition, Bull. de l'Acad. Polon. des Sciences **24**, 667–674 (1976)
12. Konarski, J.: Some pathological actions of $k^*$. Lecture Notes in Mathematics, vol. 956. Berlin-Heidelberg-New York: Springer
13. Lieberman, D.: Compactness of the Chow scheme: applications to automorphisms and deformations of Kaehler manifolds. Seminaire Francois Norguet. Lecture Notes in Mathematics, vol. 670. Berlin-Heidelberg-New York: Springer
14. McCrory, C.: Stratified general position. Algebraic and geometric topology. Lecture Notes in Mathematics, vol. 664, pp. 142–146. Berlin-Heidelberg-New York: Springer
15. Mather, J.: Stratifications and mappings. Dynamical systems Peixote, M.M., (ed.). New York: Academic Press 1973
16. Milnor, J., Stasheff, J.: Characteristic classes. Annals of mathematics study no. 76. Princeton, N.J.: Princeton University Press 1974
17. Spaltenstein, N.: The fixed point set of a unipotent transformation on the flag manifold. Indag. Math. **38**, 542–456 (1976)
18. Sommese, A.J.: Extension theorems for reductive group actions on compact Kaehler manifolds. Math. Ann. **218**, 107–116 (1975)
19. Sommese, A.J.: Some examples of **C*** actions. Lectures Notes in Mathematics, vol. 956. Berlin-Heidelberg-New York: Springer
20. Vargas, J.A.: Fixed points under the action of unipotent elements of $SL_n$ in the flag variety. Boletin de la Sociedad Matematica Mexicana **24**, 1–15 (1979)


Oblatum 10-XI-1982