SIMPLICIAL DIFFERENTIAL FORMS WITH POLES

By J. P. Brasselet, M. Goresky[1] and R. MacPherson[2]

Introduction. This paper has two purposes: 1) to introduce and study a new class of differential forms on a simplicial complex and 2) to give an intuitive and elementary argument for the relation between intersection homology and L^2 cohomology. We will consider each of these purposes separately in this introduction.

Shadow forms. Consider a simplicial complex K. Whitney associated to each simplicial cochain ζ a differential form $\nu(\zeta)$ on K. Whitney's map ν from simplicial cochains to differential forms is a map of cochain complexes which induces the deRham isomorphism. We call the forms $\nu(\zeta)$ *Whitney forms*. They have been useful in several contexts, most notably integral geometry and rational homotopy theory.

In this paper, we introduce a generalization of Whitney forms called *shadow forms*. We associate to certain simplicial chains ξ with respect to the first derived subdivision K' of K a shadow form $\omega(\xi)$ on K. Let us illustrate the construction of shadow forms when K is a single n-simplex, which we now denote by Δ. In this case, $\omega(\xi)$ is defined for k-simplices ξ of Δ' which do not lie in the boundary of Δ (or for linear combinations of these). For example, if Δ is the 2-simplex, then either of the 1-simplices ξ_1 or ξ_2 pictured below has an associated shadow form.

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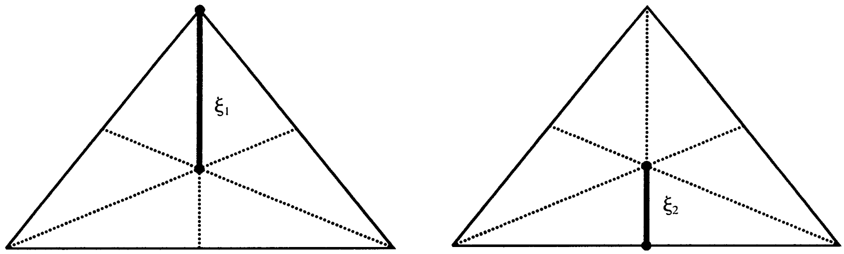
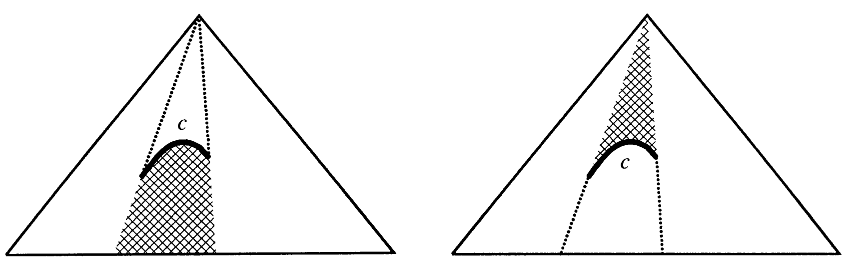
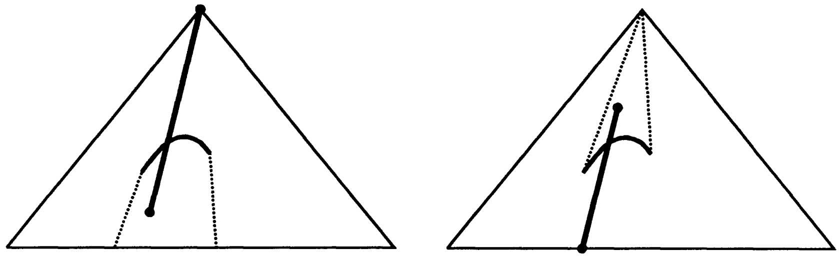


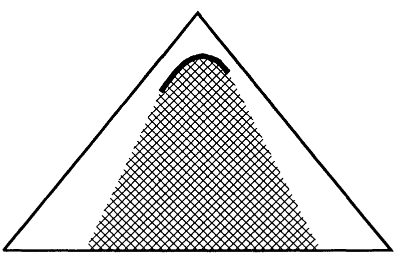
Given such a ξ, the associated shadow form will be an $(n - k)$-form $\omega(\xi)$ which we will describe by specifying its integral over $(n - k)$-chains c in K. The integral $\int_c \omega(\xi)$ is equal to the volume of a certain region $S_\xi(c)$ in Δ which we call the "shadow" cast by the chain c with respect to the simplex ξ. The following pictures show the shadows of a chain c with respect to ξ_1 and ξ_2 (as usual, the volume must be counted with appropriate signs and multiplicities).



In general, the shadow cast by c with respect to ξ can be defined as follows: For each point p in the interior of Δ, there is a canonical realization of the first derived subdivision Δ' such that p is the "barycenter" of Δ: the barycenter q of an $n - 1$ dimensional face F with opposite vertex v is determined by the condition that v, p, and q are colinear, and the barycenters of higher codimension faces are determined inductively. Denote by $\xi(p)$ the simplex corresponding to ξ in this subdivision. Then the shadow $S_\xi(c)$ cast by c with respect to ξ is the set of all points p for which the simplex $\xi(p)$ intersects c.



If the chain ξ is a union of dual cells of faces of Δ [Bra] [Mc], then $\omega(\xi)$ is precisely a Whitney form. For example, ξ_2 above is a dual cell so $\omega(\xi_2)$ is a Whitney form. In contrast to Whitney forms, however, general shadow forms may have poles near the faces of K of dimension $n - 2$ or less. For example, $\omega(\xi_1)$ has a pole near the top vertex of Δ. This may be seen from the fact that $\omega(\xi_1)$ integrated over a very small chain c near this vertex can produce a very big result.



In this paper, we study shadow forms on a simplex Δ by giving explicit formulas for them. Then their L^q norms (with respect to the usual Riemannian metric on the Euclidean space containing the simplex) can be written as explicit integrals. These integrals turn out to be classical: they are related to hypergeometric functions, Dirichlet integrals [WW] p. 253, and Eulerian integrals [WW] p. 258. By appropriate estimates, we determine for which combinatorial types of simplices ξ the forms $\omega(\xi)$ are L^q integrable.

Of course, we want to consider more general polyhedra than the simplex. We consider a simplicial complex K which is an n-dimensional oriented pseudomanifold. This means that K is the union if its n-simplices which are all oriented, and every $(n - 1)$-simplex is contained in exactly two n-simplices which induce opposite orientations on it. In this case, the associated shadow form $\omega(\xi)$ is defined whenever the chain $\xi \in C_i(K')$ satisfies a technical "transversality condition" (see Section 9). The definition of $\omega(\xi)$ amounts to the above construction in each simplex of K. The transversality condition guarantees that the resulting shadow form $\omega(\xi)$ extends continuously across the $n - 1$ dimensional faces of K. Then, ω gives a chain map from the chain complex $C_*^T(K')$ of "transverse" chains in K' to the cochain complex of shadow forms.

***L^2* cohomology and intersection homology.** There are a number of recent theorems which say that for certain classes of pseudomanifolds K, with an explicit Riemannian metric on the nonsingular part $K - \Sigma$, the intersection homology of K is the L^2 cohomology of K (see, for example, [B], [BC1], [BC2], [C1], [C2], [Ch1], [Ch2], [CGM], [G], [HP], [KK], [L], [N], [Sa], [SS], [Z1], [Z2], [Z3]). Intersection homology is most constructively defined by imposing an allowability condition on the simplicial chains which restricts the dimension of intersection of a chain with the singularities of K. The L^2 cohomology is defined by an integrability condition on differential forms.

Question. What do allowability conditions and integrability conditions have to do with each other?

Suppose K is an oriented pseudomanifold of dimension n and let $\Sigma = K_{(n-2)}$ denote the $n - 2$ skeleton. We may then consider the complex of transverse intersection chains in K' which satisfy the "allowability conditions" of intersection homology,

$$IC_i^T(K'; \mathbf{R}) = \{\xi \in C_i^T(K'; \mathbf{R}) \mid \dim(|\xi| \cap F) \leq i - [\lambda/2] - 1 \text{ and}$$
$$\dim(|\partial\xi| \cap F) \leq i - [\lambda/2] - 2 \text{ for any simplex } F \subset \Sigma\}$$

where $[x]$ denotes the greatest ∞teger less than or equal to x and λ denotes the codimension, $n - \dim(F)$, of the simplex F. The homology of this complex is the intersection homology of X (with "upper middle perversity" $p(\lambda) = [(\lambda - 1)/2]$) (see [GM2] and [GM3]).

If we linearly embed K in some Euclidean space, then this induces a metric on the manifold $K - \Sigma$, the "nonsingular part" of K. Then it makes sense to ask what the L^2 norm of a differential form on $K - \Sigma$ is. Our main result is

THEOREM 5.2. *For any chain ξ in $C_i^T(K'; \mathbf{R})$, the shadow form $\omega(\xi)$ is square integrable (i.e. $\omega(\xi) \in L^2$) if and only if ξ satisfies the allowability conditions for intersection homology (i.e. $\xi \in IC_i(K'; \mathbf{R})$).*

This theorem is proven by explicit computation of the L^2 norm of $\omega(\xi)$ as described earlier. It makes sense of the concept of the growth rate of a geometric chain and provides a direct answer to the question posed above.

Since ω is a chain map, this theorem gives a chain level proof of the following

COROLLARY 9.3. *The intersection homology of K is isomorphic to the cohomology of the complex of L^2 shadow forms on K.*

Since K with a linear embedding has conical singularities, it is a much deeper result due to Cheeger [Ch1], [Ch2] that the cohomology of the complex of all L^2 differential forms coincides with the middle intersection cohomology of X. We do not know if the methods here could be used to give a proof of Cheeger's theorem by successively refining the subdivision. However our construction provides a strong intuition behind his theorem.

More generally, we investigate the shadow forms which have finite L^q norm and show that they are precisely those forms which are associated to chains satisfying an intersection homology restriction for a particular perversity ($p(\lambda) = \lambda/q$), which we call the L^q-perversity. It follows that the cohomology of the complex of L^q shadow forms is precisely the intersection homology with the L^q perversity: in fact the two chain complexes are isomorphic. For example, the cohomology of the complex of bounded (or L^∞) shadow forms is isomorphic to the ordinary cohomology, if K is normal. In general we are led to the following

Conjecture. If K is a pseudomanifold with conical singularities, $q \geq 2$, and the L^q cohomology of $K - \Sigma$ is finite dimensional, then it is isomorphic to the intersection homology of K with the L^q perversity.

Presumably the methods of [Ch1] could be used to prove this conjecture although they require careful L^q estimates for the homotopy operators.

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1. Differential forms on a simplex. Let $\Delta = \Delta^n$ denote the standard n-simplex

$$\Delta^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \mid 0 \le x_i \le 1, \Sigma x_i = 1\}.$$

We shall use the symbol $\Omega^k(\Delta)$ to denote the smooth differential k-forms on the interior of Δ. Recall that an (abstract) k-simplex σ in the first derived complex of Δ is a chain of incident faces $F_1 < F_2 < \cdots < F_{k+1}$. For any specific choice of barycentric subdivision of Δ, (where we denote the barycenter of a face F by the symbol $\hat{F}$), the realization of σ is the k-simplex spanned by the vertices $\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_{k+1}$. A choice of barycenter $p \in \operatorname{int}(\Delta)$ determines a barycentric subdivision of Δ: the barycenter $\hat{F}(p)$ of a face F is uniquely determined as the intersection

$$\hat{F}(p) = F \cap A$$

where A is the affine subspace spanned by the point p and the opposite face to F. This gives rise to a realization $\sigma(p)$ of σ, *i.e.* as the k-simplex which is spanned by $\hat{F}_1(p), \hat{F}_2(p), \ldots, \hat{F}_{k+1}(p)$. Throughout this paper we will suppose that $\sigma(p)$ is not contained in any proper face of Δ, *i.e.* that $F_{k+1} = \Delta$.

Fix the orientation of Δ corresponding to the ordered basis $\{e_2 - e_1, e_3 - e_1, \ldots, e_{n+1} - e_1\}$ of vectors in the affine space spanned by Δ (where $e_i \in \mathbf{R}^{n+1}$ is the standard unit vector with a 1 in the i^{th} position). Then Lebesgue measure on Δ is given by the Whitney form

$$\begin{aligned} \operatorname{dvol}(\Delta) &= W(\Delta) \\ &= n! \sum_{i=1}^{n+1} (-1)^{i+1} x_i dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}. \end{aligned}$$

Define the incidence manifold,

$$D_\sigma = \{(p, x) \in \operatorname{int}(\Delta) \times \operatorname{int}(\Delta) | x \in \sigma(p)\}$$

with inclusion $i : D_\sigma \hookrightarrow \Delta \times \Delta$. The manifold D_σ is $n + k$ dimensional and it fibers over Δ in two ways. The projection to the first factor has fiber $\sigma(p)$ which is oriented by the canonical ordering of the vertices of σ. We orient D_σ using the orientation of Δ followed by the orientation of σ. This induces an orientation on the fibres of the projection to the second factor by requiring that the orientation of D_σ agree with the orientation of Δ followed by the orientation of the fibre.

Definition. The shadow form $\omega(\sigma) \in \Omega^{n-k}(\Delta)$ is the differential form

$$\omega(\sigma) = \int_{\pi_2} i^* \pi_1^* (W(\Delta))$$

where π_1 and π_2 denote projections to the first and second factors, and where $\int_{\pi_2}$ denotes integration over the fibres of π_2 (see [GHV] p. 298 and [BT] p. 61).

This is a smooth differential form on $\mathrm{int}(\Delta)$ since $\pi_2 : \overline{D_\sigma} \cap \pi_2^{-1}(\mathrm{int}(\Delta)) \to \mathrm{int}(\Delta)$ is a smooth fibre bundle whose fibre $\pi_2^{-1}(x) \cap \overline{D_\sigma}$ is a smooth compact manifold with corners, and since the differential form $i^*\pi_1^* W(\Delta)$ is the restriction to D_σ of a smooth differential form $\pi_1^* W(\Delta)$ which is defined in a neighborhood of the closure $\overline{D_\sigma}$.

2. Currents on the simplex Δ. Let Ω_o^k denote the space of smooth k-forms on $\mathrm{int}(\Delta)$ with compact support, endowed with the C^∞ compact open topology ([F] Section 4.1.7). The space $\mathscr{C}^k$ of currents is the dual space, $\mathscr{C}^k = \mathrm{Hom}_c(\Omega_o^k, \mathbf{R})$ of continuous homomorphisms $\Omega_o^k \to \mathbf{R}$. There is an embedding $\Psi : \Omega^{n-k} \hookrightarrow \mathscr{C}^k$ defined by

$$\Psi(\omega)(\eta) = \int_\Delta \omega \wedge \eta.$$

We now give the dual construction of the form $\omega(\sigma)$. Fix a barycenter $p \in \mathrm{int}(\Delta)$ and consider the induced barycentric subdivision of Δ. Let σ denote an abstract k-simplex in the first derived complex of Δ. The corresponding realization $\sigma(p)$ defines a current $\phi(\sigma(p)) \in \mathrm{Hom}_c(\Omega_o^k, \mathbf{R})$ which associates to any smooth k-form η the number

$$\phi(\sigma(p))(\eta) = \int_{\sigma(p)} \eta.$$

We can average the current $\phi(\sigma(p))$ over all choices $p \in \Delta$ and so obtain a new current $\tilde{\phi}(\sigma)$,

$$\tilde{\phi}(\sigma)(\eta) = \int_{p\in\Delta}\left(\int_{\sigma(p)} \eta\right) \mathrm{dvol}(\Delta)$$

where $\mathrm{dvol}(\Delta)$ denotes the volume form on Δ. We will now show that this averaging procedure smooths the current $\sigma(p)$:

PROPOSITION 2.1. $\tilde{\phi}(\sigma) = \Psi(\omega(\sigma))$.

Proof. It suffices to show that, for any differential k-form with compact support, $\eta \in \Omega_o^k$, we have $\tilde{\phi}(\sigma)(\eta) = \Psi(\omega(\sigma))(\eta)$. But

$$\begin{aligned}
\Psi(\omega(\sigma))(\eta) &= \int_\Delta \omega(\sigma) \wedge \eta = \int_\Delta \left(\int_{\pi_2} i^*\pi_1^*(W(\Delta))\right) \wedge \eta \\
&= \int_{D_\sigma} i^*(\pi_1^*(W(\Delta)) \wedge \pi_2^*(\eta)) \\
&= \int_\Delta \left(\int_{\pi_1} i^*\pi_2^*(\eta)\right) \wedge W(\Delta) \\
&= \int_{p\in\Delta}\left(\int_{\sigma(p)} \eta\right) \wedge W(\Delta) = \int_{p\in\Delta}\left(\int_{\sigma(p)} \eta\right) \mathrm{dvol}(\Delta) \\
&= \tilde{\phi}(\sigma)(\eta)
\end{aligned}$$

by the projection formula (see [GHV] Proposition IX p. 303). The change of order of integration is justified since η has compact support. □

COROLLARY 2.2. *If σ is an abstract k-simplex in the first derived complex of Δ, then the differential of the associated shadow form is*

$$d\omega(\sigma) = \sum_\tau (-1)^{o(\tau)}\omega(\tau)$$

where the sum is taken over all codimension 1 *faces τ of σ such that* $\mathrm{int}(\tau) \subset \mathrm{int}(\Delta)$ *and where $o(\tau) = 0$ if the orientation of τ which is induced* *from that of σ coincides with the orientation which is determined from the ordering of the vertices of τ and $o(\tau) = 1$ otherwise.* □

3. Shadows. Imagine a piecewise linear or piecewise smooth chain c of dimension $n - k$ which is contained in the interior of Δ. Corresponding to an abstract k-simplex σ in the first derived complex of Δ (which is not contained in any proper face of Δ), we may associate the "shadow" of c which is "cast" by σ (as illustrated in the introduction),

$$S_\sigma(c) = \{p \in \Delta \mid c \cap \sigma(p) \neq \phi\}.$$

The differential form $\omega(\sigma)$ defined in Section 1 has the beautiful property that its integral over a chain c is precisely the volume of the shadow (counted with appropriate sign and multiplicity):

PROPOSITION 3.1. *For any chain $c \subset \operatorname{int}(\Delta)$ of dimension $n - k$ we have*

$$\int_c \omega(\sigma) = \text{volume}(S_\sigma(c)).$$

Proof. The proof is a direct computation:

$$\int_c \omega(\sigma) = \int_c \int_{\pi_2} i^*\pi_1^*(W(\Delta)) = \int_{\pi_2^{-1}(c)\cap D_\sigma} i^*\pi_1^*(W(\Delta))$$

$$= \int_{\pi_1(\pi_2^{-1}(c)\cap D_\sigma)} W(\Delta) = \text{vol}(\pi_1(\pi_2^{-1}(c) \cap D_\sigma))$$

where "vol" denotes volume counted with appropriate sign and multiplicity. □

4. Equations of shadow forms. In this section we give explicit formulas for the forms $\omega(\sigma)$ in terms of barycentric coordinates on the n-simplex Δ. First we recall the corresponding formula for Whitney's differential forms.

Let e_i denote the standard unit vector with a 1 in the i^{th} position. For any ordered subset $S \subset \{1, 2, \ldots, n\}$ consisting of $r + 1$ numbers

$$S = \{i_1, i_2, \ldots, i_{r+1}\}$$

we associate the r-dimensional oriented face F of Δ whose vertices are $\{e_{i_1}, e_{i_2}, \ldots, e_{i_{r+1}}\}$ and whose orientation is given by this ordering of the vertices. Radial projection $\pi : \Delta \to F$ is then given by

$$\pi(x_1, x_2, \ldots, x_{n+1}) = (x_{i_1}, x_{i_2}, \ldots, x_{i_{r+1}})/\Sigma S$$

where $\Sigma S = \Sigma_{j=1}^{r+1} x_{i_j}$.

Definition. ([W] p. 139, [ST]) The Whitney form $W_\Delta(S)$ is the differential r-form on $\operatorname{int}(\Delta)$ which is given by

$$W_\Delta(S) = r! \sum_{j=1}^{r+1} (-1)^{j+1} x_{i_j} dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_{r+1}}$$

(for $r = 0$ we interpret this as $W_\Delta(\{x_{i_1}\}) = x_{i_1}$). If $S = \{1, 2, \ldots, n + 1\}$ (*i.e.* if $F = \Delta$) then $W_\Delta(S)$ is the volume form on Δ.

Remark. A closely related form on Δ is obtained by pulling back the volume form from F, and it is

$$\pi^*(\operatorname{dvol}(F)) = \frac{-1}{r!} \frac{W_\Delta(S)}{(\Sigma S)^{r+1}}.$$

Let σ be an abstract k-simplex in the first derived complex of Δ, which is not contained in any proper face of Δ and which is defined by the faces $F_1 < F_2 < \cdots < F_{k+1} = \Delta$. We obtain a decomposition of the set $\{1, 2, \ldots, n + 1\}$ into $k + 1$ disjoint subsets $S_1, S_2, \ldots, S_{k+1}$ such that for each α, $(1 \leq \alpha \leq k + 1)$, the face F_α has vertices $\{e_i | i \in S_1 \cup S_2 \cup \cdots \cup S_\alpha\}$.

Let $s_i = |S_i|$ denote the cardinality of S_i, and let $W(S_i) = W_\Delta(S_i)$ denote the Whitney form on Δ which corresponds to the face whose vertices are $\{e_j | j \in S_i\}$.

For fixed positive real numbers $v_1, v_2, \ldots, v_k$ whose sum is less than one, we define the function $\Phi(v_1, v_2, \ldots, v_k; s_1, s_2, \ldots, s_k)$ as follows:

$$\Phi = \frac{\int_{R(v)} \phi\, du_k du_{k-1} \cdots du_1}{\int_R \phi\, du_k du_{k-1} \cdots du_1}$$

where

$$\phi = (u_1)^{s_1-1}(u_2)^{s_2-1} \cdots (u_k)^{s_k-1}(1 - u_1 - u_2 - \cdots - u_k)^{s_{k+1}-1},$$

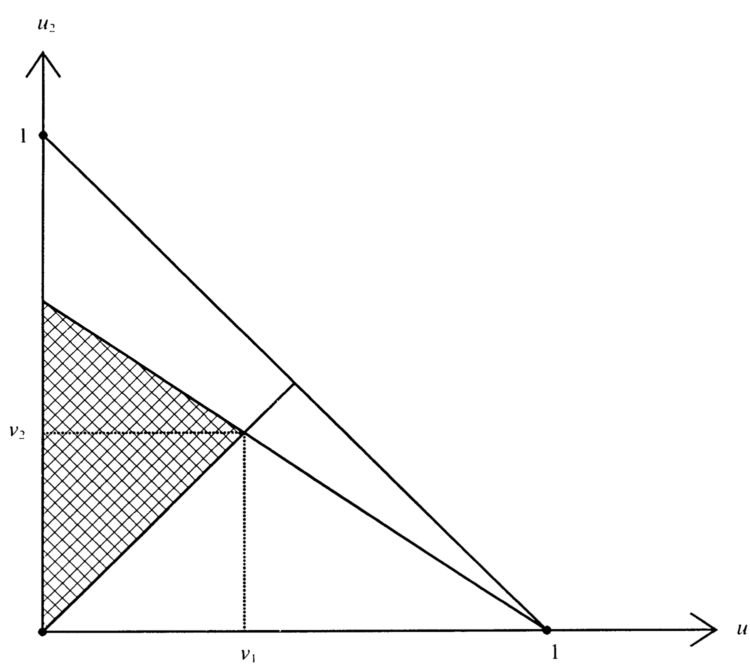
where $s_{k+1} = n + 1 - \sum_{j=1}^{k} s_j$, and where $R(v)$ is the subset of the simplex

$$R = \{(u_1, u_2, \ldots, u_k) \in \mathbf{R}^k \mid 0 \leq u_i \leq 1 \text{ and } u_1 + u_2 + \cdots + u_k \leq 1\}$$

given by

$$R(v) = \left\{(u_1, u_2, \ldots, u_k) \mid 0 \leq \frac{u_1}{v_1} \leq \frac{u_2}{v_2} \leq \cdots \leq \frac{u_k}{v_k} \leq \frac{1 - u_1 - u_2 - \cdots - u_k}{1 - v_1 - v_2 - \cdots - v_k}\right\}.$$

The shaded region in the following diagram is the subset $R(v)$:



The numerator of Φ is a Dirichlet integral which can be explicitly computed, but we do not know a general formula for it.[1] A simple formula approximating Φ is given in Proposition 6.1. The denominator is

$$\Phi_0 = \frac{\Gamma(s_1)\Gamma(s_2) \cdots \Gamma(s_{k+1})}{\Gamma(n + 1)}.$$

THEOREM 4.1. *The differential form* $\omega(\sigma)$ *is equal to*

$$(-1)^{s_2+2s_3+\cdots+ks_{k+1}}\eta(\sigma)\Phi(\Sigma S_1, \Sigma S_2, \ldots, \Sigma S_k; s_1, s_2, \ldots, s_k)$$

where $\eta(\sigma)$ *is the closed differential form of degree* $n - k$,

$$\eta(\sigma) = \frac{W(S_1)}{(\Sigma S_1)^{s_1}} \wedge \frac{W(S_2)}{(\Sigma S_2)^{s_2}} \wedge \cdots \wedge \frac{W(S_{k+1})}{(\Sigma S_{k+1})^{s_{k+1}}}.$$

Proof. We will explicitly perform the integration over the fibre as indicated in the definition (Section 1) of $\omega(\sigma)$,

$$\omega(\sigma) = \int_{\pi_2} i^*\pi_1^*(W(\Delta)).$$

We shall use the following notation: a point in $\Delta \times \Delta$ will be denoted

$$(x_1, x_2, \ldots, x_{n+1}, x_1', x_2', \ldots, x_{n+1}')$$

and in general a dot will be used to indicate a variable associated with the second factor. For each of the subsets S_i we define $u_i = \Sigma S_i = \Sigma\{x_j | j \in S_i\}$, $u_i' = \Sigma S_i'$, and $f^*W(u_1, u_2, \ldots, u_{k+1})$ to be the pullback of the volume form under the mapping $f : \Delta^n \to \Delta^k$ which is given by

$$f(x_1, x_2, \ldots, x_{n+1}) = (\Sigma S_1, \Sigma S_2, \ldots, \Sigma S_{k+1}) = (u_1, u_2, \ldots, u_{k+1}).$$

Remark. With this notation, the region $R(u')$ above is precisely the projection to the plane $u_{k+1}' = 0$ of the set $f\pi_1(\pi_2^{-1}(x') \cap D)$.

We will now decompose the form $W(\Delta)$ into a product and then compute its pullback to the incidence manifold D_σ.

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[1]Added in proof: A. Zelevinsky has recently shown us how to evaluate this integral by calculating the associated generating function.

LEMMA 4.2. *The restriction to* $D_\sigma = \{(x, x') \in \text{int}\,\Delta \times \text{int}\,\Delta \mid x' \in \sigma(x)\}$ *of the form* $\pi_1^* W(\Delta)$ *is equal to the restriction to* D_σ *of the form*

$$\eta(\sigma) = a_1 \pi_2^* \left[\frac{W(S_1') \wedge \cdots \wedge W(S_{k+1}')}{(u_1')^{s_1} \cdots (u_{k+1}')^{s_{k+1}}} \right]$$

$$\wedge\, \pi_1^* f^* W(u_{k+1}, \ldots, u_2, u_1) u_1^{s_1 - 1} u_2^{s_2 - 1} \cdots u_{k+1}^{s_{k+1} - 1}$$

where

$$a_1 = (-1)^{s_2 + 2s_3 + \cdots + ks_{k+1}} \frac{\Gamma(n+1)}{\Gamma(s_1)\Gamma(s_2) \cdots \Gamma(s_{k+1})}.$$

This lemma will complete the proof of Theorem 4.1 because

$$\omega(\sigma) = \int_{\pi_2} i^* \pi_1^* W(\Delta) = \int_{\pi_2} \eta(\sigma)$$

$$= a_1 \int_{\pi_2} \pi_2^*(\eta(\sigma)) \wedge \pi_1^* f^* W(u_{k+1}, \ldots, u_2, u_1) u_1^{s_1 - 1} u_2^{s_2 - 1} \cdots u_{k+1}^{s_{k+1} - 1}$$

and the value of this integral, at a point $x' \in \Delta$ is

$$a_1 \eta(x') \int_{\pi_2^{-1}(x') \cap D_\sigma} \pi_1^* f^* W(u_{k+1}, \ldots, u_2, u_1) u_1^{s_1 - 1} u_2^{s_2 - 1} \cdots u_{k+1}^{s_{k+1} - 1}$$

$$= a_1 \eta(x') \int_{\pi_1(\pi_2^{-1}(x') \cap D_\sigma)} f^* W(u_{k+1}, \ldots, u_2, u_1) u_1^{s_1 - 1} u_2^{s_2 - 1} \cdots u_{k+1}^{s_{k+1} - 1}$$

$$= a_1 \eta(x') \int_{\pi_1(\pi_2^{-1}(x') \cap D_\sigma)} f^*(\text{dvol}(\Delta^k)) u_1^{s_1 - 1} u_2^{s_2 - 1} \cdots u_{k+1}^{s_{k+1} - 1}$$

$$= (-1)^{s_2 + s_3 + \cdots + s_{k+1}} \eta(x') \Phi(u'; s_1, \ldots, s_k)$$

by the above remark. □

Proof of Lemma 4.2. The lemma follows immediately from two calculations:

Calculation 1:

$$W(\Delta) = a_1 W(S_1) \wedge \cdots \wedge W(S_{k+1}) \wedge \frac{f^* W(\Sigma S_{k+1}, \ldots, \Sigma S_2, \Sigma S_1)}{(\Sigma S_1)(\Sigma S_2) \cdots (\Sigma S_{k+1})}.$$

Calculation 2:

$$i^*((\Sigma S_i)^{s_i} \pi_2^* W(S_i')) = i^*((\Sigma S_i')^{s_i} \pi_1^* W(S_i)).$$

Calculation 1 is proven by induction on k, using the following formulas which can be verified directly:

(a) $$W(S_1 \cup S_2) = (-1)^{s_2} \frac{\Gamma(s_1 + s_2)}{\Gamma(s_1)\Gamma(s_2)} W(S_1) \wedge W(S_2) \wedge \frac{f^* W(\Sigma S_2, \Sigma S_1)}{(\Sigma S_1)(\Sigma S_2)}$$

(b) $$W(\Sigma S_2, \Sigma S_1) = W(\Sigma S_2 + \Sigma S_1, \Sigma S_1).$$

Calculation 2 is proven by induction on $s = |S_i|$. By relabelling the coordinate axes, we can assume that $S = S_i = \{1, 2, \ldots, s\}$. Writing $u = \Sigma S = x_1 + x_2 + \cdots + x_s$ it suffices to show that

$$u^s \pi_2^* W(u', x_2', \ldots, x_s') = (u')^s \pi_1^* W(u, x_2, \ldots, x_s)$$

on the incidence manifold D_σ, since $W(x_1, x_2, \ldots, x_s) = W(u, x_2, \ldots, x_s)$. The manifold D_σ is contained in the plane

$$\frac{x_1}{x_1'} = \frac{x_2}{x_2'} = \cdots = \frac{x_s}{x_s'} = \frac{u}{u'}.$$

We conclude

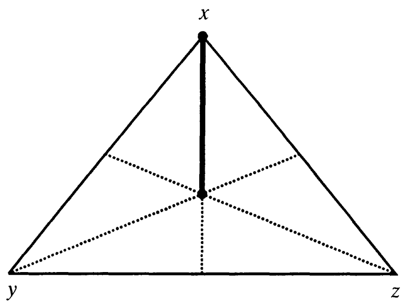
$$u\,dx_s' - x_s\,du' = u'\,dx_s - x_s'\,du.$$

Multiplying by uu' we have

$$u^2 \pi_2^* W(u', x_s') = (u')^2 \pi_1^* W(u, x_s)$$

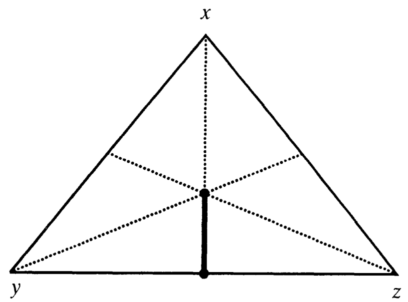
when restricted to D_σ. This is the case $s = 2$ of calculation 2. The general case is easily checked by induction, using part (a) above. □

Examples. If an abstract k-simplex σ is defined by the sequence of groups of indices $S_1, S_2, \ldots, S_{k+1}$ then we will write either $\sigma = (S_1)(S_2) \cdots (S_{k+1})$ or $\sigma = (\{x_i\}_{i \in S_1})(\{x_i\}_{i \in S_2}) \cdots (\{x_i\}_{i \in S_{k+1}})$.



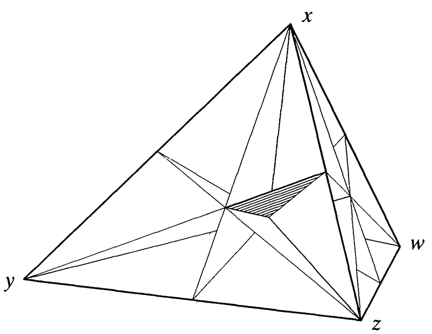
$\sigma = (x)(yz)$

$$\omega(\sigma) = \frac{x(2-x)}{(1-x)^2}(ydz - zdy)$$



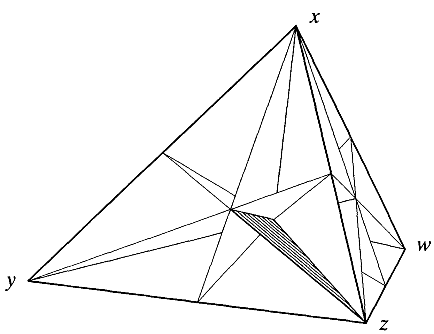
$\sigma = (yz)(x)$

$$\omega(\sigma) = ydz - zdy$$



$\sigma = (xz)(y)(w)$

$$\omega(\sigma) = \frac{y}{(y+w)}(xdz - zdx)$$



$\sigma = (z)(xy)(w)$

$$\omega(\sigma) = \frac{z(2-z)}{(1-z)^2}(xdy - ydx)$$

5. Growth rates of shadow forms. In this section we study the relationship between the growth rate of a shadow form $\omega(\sigma)$ and the allowability properties of σ (in the sense of intersection homology). We fix the standard orientation of $\mathbf{R}^n$ and orient Δ by requiring that the projection $\Delta \to \mathbf{R}^n$ to the first n coordinates is orientation preserving. The standard inner product on $\mathbf{R}^{n+1}$ induces an inner product on the tangent spaces of Δ and therefore also on the differential forms $\Omega^k(\Delta)$ which are defined on the interior of Δ. Let $*$ denote the associated Hodge star operator. Thus we may define the q-norm $\|\omega\|_q$ of a differential form ω, where $1 \leq q \leq \infty$ by

$$\|\omega\|_q = \left(\int_\Delta f^{q/2}\, \mathrm{dvol}(\Delta)\right)^{1/q}$$

where $\omega \wedge *\omega = f\, \mathrm{dvol}(\Delta)$. A differential form ω is said to be of type L^q if $\|\omega\|_q < \infty$.

Recall [GM1] that a perversity is an integer valued function $\bar{p}$ defined on the positive integers such that $p(0) = p(1) = p(2) = 0$ and $p(\lambda) \leq p(\lambda + 1) \leq p(\lambda) + 1$. A k-dimensional simplex σ in the first barycentric subdivision of Δ is $(\bar{p}, k)$-*allowable* iff for each proper face F of Δ we have

$$\dim(\sigma \cap F) \leq k - \lambda + p(\lambda)$$

where $\lambda = n - \dim(F)$.

Definition. If σ is a k-simplex in the first barycentric subdivision of Δ whose vertices are the barycentres of the faces $F_1 < F_2 < \cdots < F_{k+1} = \Delta$, we define the profile p_σ of σ to be the following integer valued function:

$$p_\sigma(0) = 0$$

$$p_\sigma(\lambda) = \lambda - (k - i + 1) \text{ if } \lambda \text{ is one of the special values,}$$

$$\lambda = s_{k+1} + s_k + \cdots + s_{i+1}.$$

If λ is between two special values, say $\lambda = \lambda_0 + t$ where $\lambda_0 = s_{k+1} + s_k + \cdots + s_{i+1}$ and $1 \leq t < s_i$, define $p_\sigma(\lambda) = p_\sigma(\lambda_0) + (t - 1)$.

LEMMA 5.1. *The profile p_σ is the smallest perversity $\bar{p}$ for which σ is $(\bar{p}, k)$-allowable.*

Proof. This is a simple counting argument. □

THEOREM 5.2. *Let σ be an abstract k-simplex in the first derived complex of Δ, let p_σ denote the profile of σ and let $\omega(\sigma)$ denote the shadow form corresponding to σ. Fix a real number q with $1 \le q \le \infty$. Then $\omega(\sigma)$ is in L^q iff $p_\sigma(\lambda) < \lambda/q$ for all integers λ with $1 \le \lambda \le n + 1 - s_1$.*

The proof will occupy Section 6, Section 7, and Section 8. In order to estimate $\|\omega(\sigma)\|_q = \|\Phi\eta(\sigma)\|_q$ we will estimate separately $|\Phi|^q$ and $\|\eta(\sigma)\|_q$.

6. Estimates on $|\Phi|$. Fix nonnegative integers $s_1, s_2, \ldots, s_k, s_{k+1}$ such that $s_{k+1} = n + 1 - s_1 - s_2 - \cdots - s_k$. Fix positive real numbers v_i, $(1 \le i \le k)$ such that $0 < v_i < 1$ and $\Sigma_{i=1}^k v_i < 1$.

PROPOSITION 6.1. *The function $\Phi(v_1, \ldots, v_k; s_1, \ldots, s_k)$ is bounded as follows*:

$$J \le \Phi \le a_2 J$$

where

$$a_2 = \frac{\Gamma(n+1)}{\Gamma(s_1)\Gamma(s_2)\cdots\Gamma(s_{k+1})} \frac{1}{(s_1)(s_1+s_2)\cdots(s_1+s_2+\cdots+s_{k+1})}$$

and where

$$J = v_1^{s_1}\left(\frac{v_2}{1-v_1}\right)^{s_2} \cdots \left(\frac{v_k}{1-v_1-v_2-\cdots-v_{k-1}}\right)^{s_k}.$$

Proof. To compute the numerator of Φ, use the change of variables given by $t_1 = u_1/v_1$ and $t_i = (1 - v_1 - v_2 - \cdots - v_{i-1})((u_i/v_i) - (u_{i-1}/v_{i-1}))$ for $2 \le i \le k$. The inequalities in the definition of the domain of integration imply that $t_i \ge 0$ and $\Sigma t_i \le 1$, and this defines a new domain of integration. Solving for u_i,

$$u_i = v_i\left(\frac{t_i}{1-v_1-\cdots-v_{i-1}} + \frac{t_{i-1}}{1-v_1-\cdots-v_{i-2}} + \cdots + \frac{t_2}{1-v_1} + t_1\right)$$

we obtain the bound

$$\frac{v_i}{1 - v_1 - v_2 - \cdots - v_{i-1}} \leq u_i \leq \frac{v_i(t_1 + \cdots + t_i)}{1 - v_1 - v_2 - \cdots - v_{i-1}}$$

and

$$1 - t_1 - t_2 - \cdots - t_k \leq 1 - u_1 - u_2 - \cdots - u_k \leq 1.$$

The volume form becomes

$$du_k du_{k-1} \cdots du_1$$
$$= v_1\left(\frac{v_2}{1 - v_1}\right) \cdots \left(\frac{v_k}{1 - v_1 - v_2 - \cdots - v_{k-1}}\right) dt_k dt_{k-1} \cdots dt_1$$

so we conclude

$$J \int t_1^{s_1 - 1} \cdots t_k^{s_k - 1}(1 - t_1 - \cdots - t_k)^{s_{k+1} - 1} dt \leq \Phi\Phi_0$$
$$\leq J \int t_1^{s_1 - 1}(t_1 + t_2)^{s_2 - 1} \cdots (t_1 + \cdots + t_k)^{s_k - 1} dt$$

where the integrals are taken over the region $t_i \geq 0$ and $\Sigma t_i \leq 1$. The integral in the lower limit was previously defined as the "Dirichlet integral" and has the value Φ_0, while the integral in the upper limit can be evaluated explicitly as $1/(s_1)(s_1 + s_2) \cdots (s_1 + s_2 + \cdots + s_k)$. □

7. Estimates on $\|\eta(\sigma)\|$. The Whitney form $W_\Delta(S)$ is the pullback to $\Delta = \Delta^n$ of a differential form on $\mathbf{R}^{n+1}$. We wish to compute $*W_\Delta(S)$ by working with differential forms on the ambient Euclidean space. For this purpose we will use the following two lemmas.

LEMMA 7.1. *Let $i : \Delta \hookrightarrow \mathbf{R}^{n+1}$ denote the inclusion of the standard n simplex into Euclidean space (see Section 1) and let $\theta \in \Omega^r(\mathbf{R}^{n+1})$ be a differential form. Let*

$$dL = \frac{1}{\sqrt{n + 1}} \sum_{i=1}^{n+1} dx_i$$

denote the differential form of unit norm, whose kernel is parallel to the simplex Δ, *then*

$$*i^*(\theta) = \frac{(-1)^{n-r}}{\sqrt{n+1}} i^* * (\theta \wedge dL).$$

Proof. We have an orientation preserving orthogonal decomposition as Riemannian manifolds, $\mathbf{R}^{n+1} \cong H \times L$, where L is the line spanned by the vector $(1, 1, 1, \ldots, 1)$ and H is the affine subspace containing Δ. Let π_H and π_L denote projections to H and L respectively. Then

$$\Omega^r(\mathbf{R}^{n+1}) \cong \pi_H^*\Omega^r(H) \otimes \pi_L^*\Omega^0(L) \oplus \pi_H^*\Omega^{r-1}(H) \otimes \pi_L^*\Omega^1(L).$$

With respect to this decomposition, the operations $*$ and i^* are given by

$$i^*(\pi_H^*\alpha_1 \wedge \pi_L^*\beta_1 + \pi_H^*\alpha_2 \wedge \pi_L^*\beta_2) = \beta_1(1, 1, \ldots, 1)\alpha_1$$

and

$$*(\pi_H^*\alpha_1 \wedge \pi_L^*\beta_1 + \pi_H^*\alpha_2 \wedge \pi_L^*\beta_2)$$
$$= \pi_H^*(*\alpha_1) \wedge \pi_L^*(*\beta_1) + (-1)^{n-r+1}\pi_H^*(*\alpha_2) \wedge \pi_L^*(*\beta_2).$$

The result follows immediately. □

LEMMA 7.2. *Suppose F is a r dimensional face of the n-simplex,* Δ^n. *By relabelling coordinates if necessary, let us assume that F is spanned by the vertices* $\{e_1, e_2, \ldots, e_{r+1}\}$. *We denote by W(F) the Whitney form on F: it is the pullback to F of the r-form on* $\mathbf{R}^{n+1}$ *which is given by*

$$\tilde{W}(F) = r! \sum_{j=1}^{r+1} (-1)^{j+1} x_j dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{r+1}.$$

Then

$$(1) \qquad *h^*\tilde{W}(F) = (-1)^r r! \sum_{j=1}^{r+1} x_j dx_j$$

where $h : \mathbf{R}^{r+1} \to \mathbf{R}^{n+1}$ *is the inclusion of the first* $r + 1$ *coordinates.*

$$(2)\quad \tilde{W}(F) \wedge \sum_{j=1}^{r+1} dx_j = (-1)^r r! \left(\sum_{j=1}^{r+1} x_j \right) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{r+1}$$

$$(3)\quad \tilde{W}(F) \wedge \sum_{j=1}^{r+1} x_j dx_j = (-1)^r r! \left(\sum_{j=1}^{r+1} x_j^2 \right) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{r+1}.$$

Proof. The proof is straightforward. □

PROPOSITION 7.3. *Suppose* $S_1, S_2, \ldots, S_{k+1}$ *forms a partition of the set* $\{1, 2, \ldots, n+1\}$ *as in Section* 4, *and let*

$$\theta = W(S_1) \wedge W(S_2) \wedge \cdots \wedge W(S_{k+1})$$

be the corresponding differential $(n-k)$*-form on the n-simplex* Δ. *Then*

$$\theta \wedge *\theta = f \,\mathrm{dvol}(\Delta)$$

where f is bounded above and below, $a_3 F \le f \le a_4 F$, *with*

$$F = (\Sigma S_1)^2 (\Sigma S_2)^2 \cdots (\Sigma S_{k+1})^2$$

$$a_3 = \frac{(k+1)}{(s_1 s_2 \cdots s_{k+1})} \frac{(\Gamma(s_1)\Gamma(s_2) \cdots \Gamma(s_{k+1}))^2}{(n+1)}$$

$$a_4 = (k+1) \frac{(\Gamma(s_1)\Gamma(s_2) \cdots \Gamma(s_{k+1}))^2}{(n+1)}.$$

Proof. We shall show that

$$\theta \wedge *\theta = a_5 \sum_{i=1}^{k+1} (\Sigma S_1^2)(\Sigma S_2^2) \cdots (\Sigma S_i)^2 \cdots (\Sigma S_{k+1}^2) \,\mathrm{dvol}(\Delta)$$

where $\Sigma S_i^2 = \Sigma\{x_{i_j}^2 | i_j \in S_i\}$ and where $a_5 = (\Gamma(s_1)\Gamma(s_2) \cdots \Gamma(s_{k+1}))^2/(n+1)$. The conclusion will then follow from the inequalities of Hölder and Minkowski:

$$\frac{1}{s} (\Sigma S)^2 \le (\Sigma S^2) \le (\Sigma S)^2.$$

Clearly θ is the pullback (to Δ) of a differential form $\tilde{\theta} \in \Omega^*(\mathbf{R}^{n+1})$ which is given by the same formula. By Lemma 7.1 we have,

$$\theta \wedge *\theta = i^*(\tilde{\theta}) \wedge *i^*(\tilde{\theta}) = \frac{(-1)^k}{\sqrt{n+1}} i^*(\tilde{\theta} \wedge *(\tilde{\theta} \wedge dL)).$$

By abuse of notation we write

$$\tilde{\theta} = W(S_1) \wedge W(S_2) \wedge \cdots \wedge W(S_{k+1})$$

and

$$dL = (d\Sigma S_1 + d\Sigma S_2 + \cdots + d\Sigma S_{k+1})/\sqrt{n+1}$$

from which we see

$$\tilde{\theta} \wedge dL = \frac{1}{\sqrt{n+1}} \left(\sum_{i=1}^{k+1} (-1)^{t_i} W(S_1) \wedge W(S_2) \wedge \cdots \wedge [W(S_i) \wedge d\Sigma S_i] \wedge \cdots \wedge W(S_{k+1}) \right)$$

where $t_i = n - k + i + s_{i+1} + s_{i+2} + \cdots + s_{k+1}$. Since this differential form on $\mathbf{R}^{n+1}$ has been written as a product of forms on orthogonal subspaces of dimensions $s_1, s_2, \ldots, s_{k+1}$, the Hodge star operator acts on each factor separately according to equation (1) of Lemma 7.2. The result now follows directly from equations (2) and (3) in Lemma 7.2. □

COROLLARY 7.4. *The pointwise norm of*

$$\eta(\sigma) = \frac{\theta}{(\Sigma S_1)^{s_1} \cdots (\Sigma S_{k+1})^{s_{k+1}}}$$

is bounded above and below by a constant times the following quantity

$$(\Sigma S_1)^{1-s_1}(\Sigma S_2)^{1-s_2} \cdots (\Sigma S_{k+1})^{1-s_{k+1}}. \qquad \square$$

8. Integration of ω. By Proposition 6.1 and Corollary 7.4 we see that $\|\omega(\sigma)\|^q$ is bounded above and below by a constant times the integral of the q^{th} power of the following quantity:

$$\frac{(\Sigma S_1)(\Sigma S_2)\cdots(\Sigma S_{k+1})}{(1-\Sigma S_1)^{s_2}(1-\Sigma S_1-\Sigma S_2)^{s_3}\cdots(1-\Sigma S_1-\Sigma S_2-\cdots-\Sigma S_k)^{s_{k+1}}}.$$

We will integrate this quantity over the simplex Δ.

Let $S=\{i_j\}\subset\{1,2,\ldots,n+1\}$ be a subset of s elements which corresponds to a face F of Δ as in Section 4. For simplicity of notation we will write $y_1, y_2, \ldots, y_s$ in place of $x_{i_1}, x_{i_2}, \ldots, x_{i_s}$. Let

$$\Sigma S = \sum_{t=1}^{s} y_t.$$

We will use the symbol $\int_{\Sigma S=0}^{A} f\,dV(S)$ to denote the following integral

$$\int_{y_1=0}^{A}\int_{y_2=0}^{A-y_1}\cdots\int_{y_s=0}^{A-y_1-y_2-\cdots-y_{s-1}} f\,dy_s dy_{s-1}\cdots dy_1.$$

LEMMA 8.1. *Assuming $q\ge 1$ as above, the function*

$$J(A)=\int_{\Sigma=0}^{A}(\Sigma S)^q(A-\Sigma S)^{r-1}dV(S)$$

converges iff $r>0$, and in this case there are constants a_6, and a_7 such that

$$a_6A^{q+r+s-1}\le J(A)\le a_7A^{q+r+s-1}.$$

Proof. Use the change of variables $y_s = At - y_1 - y_2 - \cdots - y_{s-1}$ (so $\Sigma S = At$) in the first integral (let us call it J_1), and set $\Sigma S' = y_1 + y_2 + \cdots + y_{s-1}$ to obtain

$$J_1=\int_{y_s=0}^{A-\Sigma S'}(\Sigma S)^q(A-\Sigma S)^{r-1}dy_s = A^{q+r}\int_{t=(\Sigma S')/A}^{1} t^q(1-t)^{r-1}dt$$

which diverges if $r\le 0$. Now suppose $r>0$. Since t^q is monotonically increasing on $[0,1]$, the integral J_1 is bounded above and below by

$$\frac{1}{r}(\Sigma S')^q(A - \Sigma S')^r \le J_1 \le A^{q+r}\int_0^1 t^q(1 - t)^{r-1}dt.$$

This last integral is an "Eulerian integral" and has the value $\Gamma(q + 1)\Gamma(r)/(q + r + 1)$ ([WW]). Therefore $J(A)$ is bounded as follows:

$$\frac{1}{r}\int_{\Sigma'=0}^{A}(\Sigma S')^q(A - \Sigma S')^r dV(S') \le J(A)$$

$$\le A^{q+r}\frac{\Gamma(q + 1)\Gamma(r)}{\Gamma(q + r + 1)}\int_{\Sigma'=0}^{A} 1\, dV(S')$$

from which we conclude by induction that

$$\frac{A^{q+r+s-1}}{\Gamma(r + s - 1)}\int_0^1 t^q(1 - t)^{r+s-2}dt \le J(A)$$

$$\le A^{q+r}\frac{\Gamma(q + 1)\Gamma(r)}{\Gamma(q + r + 1)}A^{s-1}\frac{1}{\Gamma(s)}.$$

This completes the proof and in fact we see that we may take

$$a_6 = \frac{\Gamma(q + 1)}{\Gamma(q + r + s)} \quad \text{and} \quad a_7 = \frac{\Gamma(q + 1)\Gamma(r)}{\Gamma(q + r + 1)\Gamma(s)}. \qquad \square$$

Proof of Theorem 5.2. We must decide whether the following integral converges:

$$\int_{\Sigma_1=0}^{1}(\Sigma_1)^q(1 - \Sigma_1)^{-qs_2}\int_{\Sigma_2=0}^{1-\Sigma_1}(\Sigma_2)^q(1 - \Sigma_1 - \Sigma_2)^{-qs_3}\cdots$$

$$\int_{\Sigma_{k-1}=0}^{1-\Sigma_1-\cdots-\Sigma_{k-2}}(\Sigma_{k-1})^q(1 - \Sigma_1 - \cdots - \Sigma_{k-1})^{-qs_k}$$

$$\int_{\Sigma_k=0}^{1-\Sigma_1-\cdots-\Sigma_{k-1}}(\Sigma_k)^q(1 - \Sigma_1 - \cdots - \Sigma_k)^{q-qs_{k+1}}\int_{\Sigma'_{k+1}=0}^{1-\Sigma_1-\cdots-\Sigma_k} 1\, d\Delta'$$

where Σ_k denotes $\Sigma S_k = \Sigma\{x_j | j \in S_k\}$, where $\Sigma'_{k+1} = \Sigma_{k+1} - x_{n+1}$, and where $d\Delta' = d\Sigma'_{k+1}\Sigma_k \cdots d\Sigma_1$. (In other words, no integration is performed over the last coordinate y_{n+1} because the sum of all the coordinates is 1).

The value of the last integral is

$$\frac{1}{\Gamma(s_{k+1})}(1 - \Sigma S_1 - \Sigma S_2 - \cdots - \Sigma S_k)^{s_{k+1}-1}.$$

We now perform the remaining integrals successively, using Lemma 8.1. Thus the ΣS_k integral converges iff $q - qs_{k+1} + s_{k+1} > 0$, and in this case it is bounded above and below by a constant times

$$(1 - \Sigma S_1 - \cdots - \Sigma S_{k-1})^{q(2-s_{k+1})+s_{k+1}+s_k-1}.$$

Similarly the ΣS_{k-1} integral converges iff $-qs_k + q(2 - s_{k+1}) + s_{k+1} + s_k > 0$, and so on. We conclude by induction that $\omega(\sigma)$ is in L^q iff the following conditions hold:

$$s_{k+1} - 1 < (s_{k+1})/q$$

$$s_k + s_{k+1} - 2 < (s_k + s_{k+1})/q$$

$$\cdots$$

$$(*) \qquad s_{i+1} + \cdots + s_{k+1} - (k - i + 1) < (s_{i+1} + \cdots + s_{k+1})/q$$

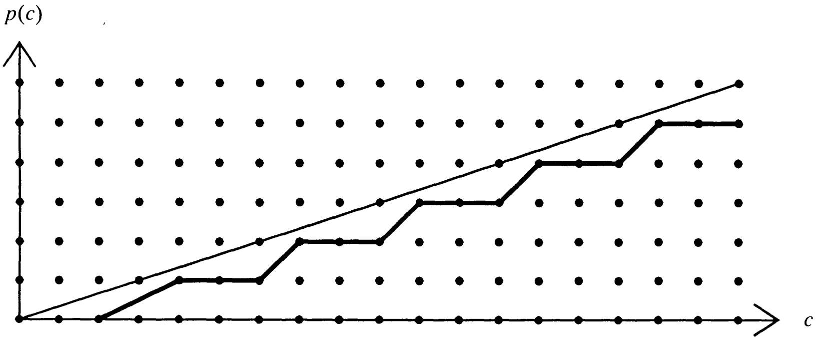
$$\cdots$$

$$n + 1 - s_1 - k < (n + 1 - s_1)/q.$$

Finally we claim the conditions (*) hold iff the profile p_σ of σ satisfies

$$(**) \qquad p_\sigma(\lambda) < \lambda/q$$

for all λ with $1 \leq \lambda \leq n + 1 - s_1$. First note that condition (*) is equivalent to the conditions $p_\sigma(\lambda) < \lambda/q$ for every "special value" $\lambda = s_{i+1} + s_{i+2} + \cdots + s_{k+1}$ because for these values the profile satisfies $p_\sigma(\lambda) = \lambda - (k - i + 1)$. But it is easy to see that $p_\sigma(\lambda) < \lambda/q$ for the special values of λ iff $p_\sigma(\lambda) < \lambda/q$ for all values of λ because the special values are the only points of concavity on the graph of $p_\sigma(\lambda)$. □



L^3 Growth Rates

9. Globalization. In this section we consider a triangulated pseudomanifold K and we show how the L^2 forms $\omega(\sigma)$ of the preceding sections may be patched together to form a chain complex whose homology is the intersection homology of K. Our goal is to associate to any $(\overline{m}, i)$-allowable chain $c \in IC_i(K')$ a collection of L^2 differential forms on the n-simplices of K which agree on the $n - 1$ dimensional simplices. Unfortunately there is a technical complication introduced by the possibility that c might be contained in the $n - 1$ skeleton of K, for in this case we have no way of constructing corresponding differential forms on the n-simplices of K. We will rule out this possibility by assuming the chain c is "transverse" to the $n - 1$ skeleton of K.

Throughout this section, K will denote a purely n dimensional simplicial complex linearly embedded in $\mathbf{R}^n$, which is also an oriented pseudomanifold, stratified by the simplices of K. Let K' denote the barycentric subdivision of K, with respect to some fixed choice of barycenters. Let $K_{(j)}$ denote the j-skeleton of K and $\Sigma = K_{(n-2)}$ denote the "singularity set" of K.

Definition. A chain $c \in C_i(K')$ is *transverse* if $\dim(|c| \cap K_{(n-1)}) \leq i - 1$ and if, for each $n - 1$ dimensional simplex F in K, we have $\dim(|\partial c| \cap F) \leq i - 2$. The transverse chains (with $\mathbf{R}$ coefficients) form a complex, which we denote by $C_*^T(K'; \mathbf{R})$.

Remark. If $c = \Sigma a_j \sigma_j$ is a transverse i-chain then each σ_j is contained in some n-dimension simplex Δ of K, and is not contained in any proper face of Δ. Consequently σ_j is the kind of simplex which was considered in Section 1 and for which we were able to associate a shadow form.

Now fix a perversity $\bar{p}$. Define $I^{\bar{p}}C_i(K'; \mathbf{R})$ to be the group of all chains $\xi \in C_i(K'; \mathbf{R})$ in the first barycentric subdivision of K, such that ξ is $(\bar{p}, i)$-allowable, and $\partial\xi$ is $(\bar{p}, i-1)$-allowable with respect to the stratification of K which is given by the simplices of K. Define the complex of allowable transverse chains $I^{\bar{p}}C_*^T(K'; \mathbf{R})$ to be the intersection $I^{\bar{p}}C_*(K'; \mathbf{R}) \cap C_*^T(K'; \mathbf{R})$.

Proposition 9.1. *The homology of the complex $I^{\bar{p}}C_*^T(K'; \mathbf{R})$ is the intersection homology $IH_*^{\bar{p}}(K; \mathbf{R})$.*

Proof. The proof is surprisingly delicate since the chains are restricted to those which come from the first barycentric subdivision. See [GM3] for a proof without the transversality assumption. The addition of the transversality condition is handled as follows: Any chain $\xi \in IC_i(K')$ is a sum of i-dimensional simplices of two types: those which are contained in an $(n-1)$-dimensional simplex of K, and those which are not. This gives rise to a canonical direct sum decomposition of $IC_i(K')$ into two subgroups, which we denote as follows:

$$IC_i(K') \cong IC_i(K' - K_{(n-1)}) \oplus IC_i(K'_{(n-1)}).$$

The boundary homomorphism preserves the second subgroup but not the first, so we have

$$\partial(\xi_1, \xi_2) = (\partial_1\xi_1, \partial_2\xi_1 + \partial\xi_2)$$

where $\partial_1\xi_1$ (resp. $\partial_2\xi_1$) is the component of $\partial\xi_1$ which lies in the first (resp. second) subgroup. Clearly,

$$IC_i^T(K') = IC_i(K' - K_{(n-1)}) \cap \ker(\partial_2).$$

However, $\partial_2|_{IC_i(K' - K_{(n-1)})}$ is surjective since each allowable $(i-1)$-simplex in $K'_{(n-1)}$ is the face of an allowable i-simplex in K'. It is a simple exercise in homological algebra to verify that the surjectivity of ∂_2 implies that the subcomplex IC_*^T has the same homology as the full complex. □

Definition. A differential form on $K - \Sigma$ is a collection of smooth forms ω_Δ defined on each n dimensional simplex Δ of K, which agree on the interior of each $n - 1$ dimensional simplex, *i.e.* whenever $F = \Delta_1 \cap \Delta_2$ is an $n - 1$ dimensional face of two n-simplices, then

$$\omega_{\Delta_1}\big|_{F^o} = \omega_{\Delta_2}\big|_{F^o}.$$

Definition. Let $\tilde{\Omega}^i(K)$ denote the vectorspace of all linear combinations of shadow forms $\omega(\sigma)$ such that σ is an $n - i$ dimensional simplex in K'. Let Ω^i denote the subspace of $\tilde{\Omega}^i$ consisting of those linear combinations which agree on each $n - 1$ dimensional simplex of K. Let $\Omega^i_{(q)}$ denote the subspace of Ω^i consisting of forms ω such that $\omega \in L^q$ and $d\omega \in L^q$.

THEOREM 9.2. *The operation* ω *which associates to each transverse chain* $c = \Sigma a_j \sigma_j \in C_i^T(K')$ *the differential form* $\omega(c)$

$$\omega(c)_\Delta = \Sigma a_j \omega_\Delta(\sigma_j) \cdot \chi_\Delta(\sigma_j)$$

(*where* $\omega_\Delta(\sigma)$ *is the shadow form defined in Section* 1, *and where* χ_Δ *is the characteristic function of* Δ, *i.e. it takes the value* 1 *on* σ *iff* $\sigma \subset \Delta$) *induces a chain isomorphism*

$$\omega : C_i^T(K'; \mathbf{R}) \to \Omega^{n-i}(K).$$

Furthermore if $1 \leq q \leq \infty$ *and if* $\bar{p} = \bar{p}(q)$ *is the largest perversity function such that* $p(\lambda) < \lambda/q$ *for all* λ, *then* ω *restricts to a chain isomorphism*

$$I^{\bar{p}}C_i^T(K'; \mathbf{R}) \to \Omega^{n-i}_{(q)}(K).$$

Proof. The mapping ω is an isomorphism of groups, by definition. It commutes with the boundary because of Stokes' theorem (see [BT] p. 62). □

COROLLARY 9.3. *If* $\bar{p}$ *and* q *are related as above, then the cohomology of the complex* $\Omega^*_{(q)}(K)$ *is naturally isomorphic to the intersection homology* $IH^{\bar{p}}_*(K; \mathbf{R})$.

10. Products of shadow forms. For differentiable manifolds, the wedge of differential forms corresponds, by deRham-Poincaré to the intersection of cycles. In this section we relate the wedge of shadow forms $\omega(\sigma) \wedge \omega(\sigma')$ to the shadow form of the intersection $\omega(\sigma \cap \sigma')$. For this, we need the following notations and definitions:

Notation. Let σ be an abstract k-simplex in the first derived complex of Δ. For $p \in \operatorname{int}(\Delta)$ we denote by $[\sigma(p)]$ the affine plane spanned by $\sigma(p)$. If σ' is another abstract simplex, of dimension k' in the first derived complex of Δ, we denote the intersection dimensions by:

$$q = \dim([\sigma(p)] \cap [\sigma'(p)]) \quad \text{and} \quad r = \dim(\sigma(p) \cap \sigma'(p)).$$

These dimensions do not depend of the choice of $p \in \operatorname{int}(\Delta)$ and we have $q \geq r$ and $q \geq k + k' - n$.

Definition. The simplices σ and σ' are affine transverse iff $q = k + k' - n$, *i.e.*:

$$\operatorname{codim}([\sigma(p)] \cap [\sigma'(p)]) = \operatorname{codim}[\sigma(p)] + \operatorname{codim}[\sigma'(p)].$$

Definition. The simplices σ and σ' are topologically transverse iff $r = k + k' - n$, *i.e.*:

$$\operatorname{codim}(\sigma \cap \sigma') = \operatorname{codim}(\sigma) + \operatorname{codim}(\sigma').$$

Denote by $S_1, S_2, \ldots, S_{k+1}$ and $S'_1, S'_2, \ldots, S'_{k'+1}$ respectively the groups of indices corresponding to σ and σ' as in Section 4.

THEOREM 10.1. 1) *The simplices σ and σ' are affine transverse iff $\omega(\sigma) \wedge \omega(\sigma') \neq 0$.*

2) *If the simplices σ and σ' are affine transverse and topologically transverse (i.e. $q = r = k + k' - n$) then*:

$$\omega(\sigma) \wedge \omega(\sigma') = \pm \frac{\Psi_\sigma \Psi_{\sigma'}}{\Psi_{\sigma \cap \sigma'}} \left(\prod_{i=1}^{n+1} x_i \right) \omega(\sigma \cap \sigma')$$

where, for $\sigma = (S_1)(S_2) \cdots (S_{k+1})$,

$$\Psi_\sigma = \frac{\Phi(\Sigma S_1, \Sigma S_2, \ldots, \Sigma S_k; s_1, s_2, \ldots, s_k)}{(\Sigma S_1)^{s_1} (\Sigma S_2)^{s_2} \cdots (\Sigma S_{k+1})^{s_{k+1}}}.$$

Note that $\sigma \cap \sigma' = (T_1)(T_2) \cdots (T_{r+1})$ where the groups of indices T_j are described as follows: Set $r_0 = r_0' = 0$ and define by induction r_j and r_j' as the minimal numbers such that:

$$\bigcup_{\alpha=r_j+1}^{r_{j+1}} S_\alpha = \bigcup_{\beta=r_j'+1}^{r_{j+1}'} S_\beta'.$$

Then this union is precisely T_j.

The rest of this section is devoted to the proof of Theorem 10.1. Define a bipartite graph $G(\sigma, \sigma')$, as follows:

The vertices are labelled $S_1, S_2, \ldots, S_{k+1}, S_1', S_2', \ldots, S_{k'+1}'$. The edges connecting S_α and S_β' are labelled by the indices they have in common.

LEMMA 10.2. *The graph $G(\sigma, \sigma')$ has $q + 1$ connected components.*

Proof. Suppose the vertices $S_{\alpha_1}, S_{\alpha_2}, \ldots, S_{\alpha_j}, S_{\beta_1}', S_{\beta_2}', \ldots, S_{\beta_{j'}}'$, form a connected component of the graph. Then we have a minimal expression

$$(*) \qquad S_{\alpha_1} \cup S_{\alpha_2} \cup \cdots \cup S_{\alpha_j} = S_{\beta_1}' \cup S_{\beta_2}' \cup \cdots \cup S_{\beta_{j'}}'.$$

We denote this union by L, and we associate to it a vertex in the intersection $[\sigma(p)] \cap [\sigma'(p)]$, namely the barycenter of the face $\langle x_i \rangle_{i \in L}$.

The vertices obtained in this way are clearly linearly independent. They span the plane $[\sigma(p)] \cap (\sigma'(p)]$ because every vertex in this plane is described by an expression (*) but not necessarily a minimal such expression. In summary, there are $q + 1$ such vertices, each of which is associated to a connected component of $G(\sigma, \sigma')$. □

LEMMA 10.3. (a) *If $S_1 \cap S_2 = \{x_i\}$ then*:

$$W(S_1) \wedge W(S_2) = \pm \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1 + s_2 - 1)} x_i W(S_1 \cup S_2).$$

(b) *If* $\operatorname{card}(S_1 \cap S_2) \geq 2$ *then*:

$$W(S_1) \wedge W(S_2) = 0.$$

Proof. The proof is straightforward (the case $S_1 \cap S_2 = \emptyset$ was the formula (a) in the proof of Lemma 4.2). □

To prove Theorem 10.1 we will calculate the product $W(S_1) \wedge \cdots \wedge W(S_{k+1}) \wedge W(S'_1) \wedge \cdots \wedge W(S'_{k'+1})$ by repeated application of Lemma 10.3 within each connected component of the graph $G(\sigma, \sigma')$.

Proof of the Theorem 10.1. (1) The simplices σ and σ' are affine transverse iff the graph $G(\sigma, \sigma')$ does not contain a cycle, since both statements are equivalent to the equality:

$$\#(\text{edges}) = \#(\text{vertices}) - \#(\text{connected components}).$$

However, if some connected component contains a cycle, then Lemma 10.3(b) implies that the corresponding wedge of differential forms vanishes.

(2) If σ and σ' are affine transverse and topologically transverse, the differential part of $\omega(\sigma) \wedge \omega(\sigma')$ is the product of Whitney forms:

$$W(S_1) \wedge W(S_2) \wedge \cdots \wedge W(S_{k+1}) \wedge W(S'_1) \wedge W(S'_2) \wedge \cdots \wedge W(S'_{k'+1})$$

$$= \pm \frac{\Gamma(s_1) \cdots \Gamma(s_{k+1}) \cdot \Gamma(s'_1) \cdots \Gamma(s'_{k'+1})}{\Gamma(t_1) \cdots \Gamma(t_{r+1})} \left(\prod_{i=1}^{n+1} x_i \right) W(T_1) \wedge W(T_2) \wedge \cdots \wedge W(T_{r+1})$$

where $s_i = |S_i|$, $s'_i = |S'_i|$ and $t_j = |T_j|$. This gives the differential part of $\omega(\sigma \cap \sigma')$ since in this case, the groups of indices L_j and T_j coincide up to reordering. The formula is then a consequence of Theorem 4.1. $\square$

Remarks. 1) Even if σ and σ' are affine transverse, it is not true that $\omega(\sigma) \wedge \omega(\sigma')$ and $\omega(\sigma \cap \sigma')$ are proportional (by a function), but there is a differential $q - r$-form η such that $\omega(\sigma \cap \sigma') = \omega(\sigma) \wedge \omega(\sigma') \wedge \eta$.

2) The differential $n - r$-form $\omega(\sigma \cap \sigma')$ is never 0.

Examples (*in* Δ^3). 1) Let $\sigma = (x_1x_2)(x_3)(x_4)$ and $\sigma' = (x_4)(x_1x_3)(x_2)$. Then $\sigma \cap \sigma' = \{p\}$ and $[\sigma] \cap [\sigma']$ is the line spanned by $(\hat{x}_4)(\widehat{x_1x_2x_3})$. In this case, σ and σ' are affine transverse, but not topologically transverse. We have: $r = 0$ and $q = k + k' - n = 1$; $L_1 = (1, 2, 3)$, $L_2 = (4)$. The differential forms are $\omega(\sigma \cap \sigma') = W(\Delta)$ and

$$\omega(\sigma) \wedge \omega(\sigma') = \frac{x_1x_3x_4(2 - x_4)}{(x_3 + x_4)(1 - x_4)^2} W(x_1, x_2, x_3).$$

2) Let $\sigma = (x_1x_2)(x_3)(x_4)$ and $\sigma' = (x_4)(x_1x_2x_3)$. Then $\sigma \cap \sigma' = \{p\}$ and $[\sigma] \cap [\sigma']$ is the line spanned by $(\hat{x_4})(\widehat{x_1x_2x_3})$. In this case, σ and σ' are topologically transverse, but not affine transverse. We have: $r = k + k' - n = 0$ and $q = 1$; $L_1 = (1,2,3)$, $L_2 = (4)$. The differential forms are $\omega(\sigma \cap \sigma') = W(\Delta)$ and $\omega(\sigma) \wedge \omega(\sigma') = 0$.

3) Let $\sigma = (x_1x_2)(x_3)(x_4)$ and $\sigma' = (x_4)(x_1x_2)(x_3)$. Then $\sigma \cap \sigma' = \{p\}$ and $[\sigma] \cap [\sigma']$ is the plane spanned by $(\widehat{x_1x_2})(\hat{x_3})(\hat{x_4})$. In this case, σ and σ' are neither topologically transverse nor affine transverse. We have: $r = 0$, $k + k' - n = 1$ and $q = 2$; $L_1 = (1, 2)$, $L_2 = (3)$ and $L_3 = (4)$. The differential forms are $\omega(\sigma \cap \sigma') = W(\Delta)$ and $\omega(\sigma) \wedge \omega(\sigma') = 0$.

4) Let $\sigma = (x_1)(x_2, x_3)(x_4)$ and $\sigma' = (x_1, x_2)(x_3)(x_4)$. Then $\sigma \cap \sigma' = (x_1, x_2, x_3)(x_4)$ and $[\sigma] \cap [\sigma']$ is the line spanned by $(\widehat{x_1x_2x_3})(\hat{x_4})$. In this case, σ and σ' are topologically transverse and affine transverse. We have: $r = k + k' - n = q = 1$; $L_1 = (1, 2, 3)$ and $L_2 = (4)$. The differential forms are $\omega(\sigma \cap \sigma') = W(x_1, x_2, x_3)$ and

$$\omega(\sigma) \wedge \omega(\sigma') = \frac{x_1x_2x_3(2 - x_1)}{(1 - (x_1 + x_2))(1 - x_1)^2} W(x_1, x_2, x_3).$$

11. Whitney forms. Whitney's form $W(S)$ may be recovered as sums of shadow forms corresponding to the dual cone decomposition of K. Fix an n dimensional simplex Δ and let F denote an r dimensional face of Δ. The dual cell $D(F)$ consists of the union of all abstract simplices σ in the first derived complex of Δ which are spanned by vertices $\hat{F}$, $\hat{F}_1$, . . . , $\hat{F}_r$ with $F < F_1 < \cdots < F_r$.

PROPOSITION 11.1. *The Whitney form $W(F)$ is equal to the sum of the shadow forms*

$$W(F) = \Sigma\{\omega(\sigma) | \sigma \text{ is an } n - r \text{ dimensional simplex in } D(F)\}.$$

Remarks. The Sullivan-Whitney theorem [Su] [W] that the cohomology of the complex of Whitney forms is isomorphic to the cohomology of K is thus equivalent to the fact ([Bra]) that the homology of the chain complex of dual cones is isomorphic to the cohomology of

K. Whitney forms are bounded (*i.e.* in L^∞); however there are bounded forms which are not Whitney forms. In fact the cohomology of the complex of bounded shadow forms is precisely the perversity 0 intersection homology of K, which coincides with the cohomology of K if K is normal [GM1].

Proof. For notational simplicity we set $k = n - r$. By renumbering the vertices of Δ so that F is spanned by $\{e_1, e_2, \ldots, e_{r+1}\}$, and using the notation of Section 4, each k dimensional simplex σ in the dual cone $D(F)$ is associated to a decomposition of the set $\{1, 2, \ldots, n+1\}$ into $k+1$ nonempty subsets $S_1 \cup S_2 \cup \cdots \cup S_{k+1}$ with $S_1 = \{1, 2, \ldots, r+1\}$ (corresponding to F) and with $s_j = |S_j| = 1$ for $2 \le j \le k+1$. Thus the k simplices σ in $D(F)$ are in one to one correspondence with permutations of the set $\{r+2, r+3, \ldots, n+1\}$. A permutation τ corresponds to the choices $S_j = \{\tau(r+j)\}$. Let us denote this simplex by σ_τ. We have

$$\omega(\sigma_\tau)$$

$$= (-1)^{k(k+1)/2}\Phi(\Sigma S_1, x_{\tau(r+2)}, \ldots, x_{\tau(n+1)}; r+1, 1, 1, \ldots, 1)\,\frac{W(S_1)}{(\Sigma S_1)^r}.$$

In particular, the forms $\omega(\sigma_\tau)$ for various choices of τ are all proportional to $W(S_1)$. The factor of proportionality $\Phi = \Phi_\tau$ can be evaluated using the change of variables given in Section 4, and it equals

$$\Phi_\tau = (\Sigma S_1)^r \frac{x_{\tau(r+2)}}{(1 - \Sigma S_1)} \frac{x_{\tau(r+3)}}{(1 - \Sigma S_1 - x_{\tau(r+2)})}$$

$$\cdots \frac{x_{\tau(n+1)}}{(1 - \Sigma S_1 - x_{\tau(r+2)} - \cdots - x_{\tau(n)})}.$$

Summing over all $(n - r)!$ permutations τ gives

$$\sum_\tau \Phi_\tau = (\Sigma S_1)^r$$

which completes the proof. □

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