L^2 COHOMOLOGY IS INTERSECTION COHOMOLOGY

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§1. The statement.

Let $X = G/K$ be a Hermitian symmetric domain, and let $\Gamma \subset G$ be a torsion-free arithmetic subgroup. The symmetric space X may be metrized so that G acts by isometries on X and $\Gamma\backslash X$ is a complete Riemannian manifold. If $\Omega^i_{(2)}(\Gamma\backslash X)$ denotes the vectorspace of $\mathbb{R}$ valued smooth differential forms ω such that

$$\int (\omega \wedge *\omega) < \infty \text{ and } \int (d\omega \wedge *d\omega) < \infty$$

then $\Omega^*_{(2)}(\Gamma\backslash X)$ is a chain complex. The cohomology of this complex is the L^2 cohomology, $H^*_{(2)}(\Gamma\backslash X)$.

On the other hand the Baily-Borel Satake [BB] [S] compactification $\overline{\Gamma\backslash X}$ of $\Gamma\backslash X$ is a projective algebraic variety. Zucker's conjecture [Z1] is that integration of L^2 differential forms induces an isomorphism between the L^2 cohomology $H^*_{(2)}(\Gamma\backslash X)$ and the "middle perversity" intersection cohomology $IH^*(\overline{\Gamma\backslash X})$ of the Baily-Borel compactification.

More generally, one may start with a finite dimensional representation of G, restrict it to Γ, and obtain a local coefficient system E on $\Gamma\backslash X$. An appropriate inner product on the representation space induces a Hermitian metric on E. The cohomology of the complex of L^2 differential forms with coefficients in E is denoted $H^*_{(2)}(\Gamma\backslash X; E)$, and the intersection cohomology of the Baily- Borel compactification, with coefficients in E, is denoted $IH^*(\overline{\Gamma\backslash X}; E)$. Zucker's conjecture is that these are also isomorphic.

In his original paper on the subject [Z1], Zucker verified the conjecture for several important cases, including $G = GU(n, 1)$. The general $\mathbb{Q}$-rank 1 case was proven by A. Borel [B2]. Then Zucker verified several rank 2 and rank 3 cases [Z2] and the general $\mathbb{Q}$-rank 2 case was solved by A. Borel and W. Casselman [BC2]. Finally, in a dramatic turn of events, complete proofs of the general conjecture were discovered independently by E. Looijenga [L] and L. Saper and M. Stern [SS1], [SS2]. The two proofs are completely different.

From the point of view of the present book, the proof for the case of GU(2,1) has been known for many years. In this paper I will try to give some intuition behind Looijenga's proof (making this yet another in a long

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list [C1], [C2], [Z3], [Z4] of survey articles on the Zucker conjecture.) We will restrict to the case of trivial coefficient system E, although the general case is not significantly more difficult. The author would like to thank R. MacPherson, M. McConnell, S. Zucker, and an anonymous referee for their careful readings of the first version of this paper, and their many helpful suggestions.

§2. Why do we expect the Zucker conjecture to be true?

Let us suppose that Y is a complex d-dimensional projective variety with an isolated singularity at a point p. The link L of p is defined to be the intersection of Y with the boundary of a small ball $B_\delta(p)$ centered at p. It is a smooth manifold, and the closed neighborhood

$$U = Y \cap B_\delta(p)$$

is homeomorphic to the cone over L. The distance to p (in projective space) is the second coordinate of a homeomorphism,

$$U - \{p\} = L\times(0,\delta]$$

between the punctured neighborhood U and the cone over L minus its vertex.

If Y is the Baily-Borel compactification of $\Gamma\backslash X$, then the metric induced from the G-invariant metric on X is not the same as the distance as measured in projective space. In fact, the invariant metric is complete, and the point p is infinitely far away from every point in $\Gamma\backslash X$. If we represent the cone lines by their geodesic distance (which we denote by t), we obtain instead a homeomorphism

$$U - \{p\} \cong L\times[1,\infty)$$

which identifies $L\times\{1\}$ with the boundary of U and which puts the point p at $t = \infty$. In analogy with the complex 1-dimensional case, it is not unreasonable to hope that the metric on U is an exponential "horn" (in the sense of [Ch1]), i.e.

$$ds^2 = (e^{-at})^2 ds_L^2 + dt^2$$

where ds_L denotes distance as measured in the link coordinates. (But see § 4.)

Now suppose that $\omega_L \in \Omega^k(L)$ is a smooth differential form on the link, and let $\pi^*(\omega_L)$ denote its pullback to $U-\{p\}$, where π denotes the projection $U - \{p\} \to L$. When is $\pi^*(\omega_L)$ square-integrable? At a point $(q,t) \in L\times[1,\infty)$ the L^2 pointwise norm of $\pi^*(\omega_L)$ is

$$\|\pi^*(\omega_L)(q,t)\|^2 = (e^{-at})^{-2k}\|\omega_L(q)\|^2$$

and the volume form is

$$d\text{vol} = (e^{-at})^{2d-1} d\text{vol}_L \wedge dt$$

(where 2d-1 is the real dimension of L). Thus,

$$\int_U \|\pi^*(\omega_L)\|^2 d\text{vol} = \int_L \|\omega_L\|^2 d\text{vol}_L \int\limits_{t=1}^{\infty} e^{2akt} e^{-at(2d-1)} dt$$

which is finite iff $2k - 2d + 1 < 0$, i.e. if $k < d - 1/2$. Since k and d are integers, we see that the differential form $\pi^*(\omega_L)$ is in L^2 iff $k < d$. By constructing appropriate homotopy operators, it is possible to complete this computation to a proof ([Ch1]) that the L^2 cohomology of U is

$$H^k_{(2)}(U) \cong \begin{cases} H^k(L) & \text{for} \quad k < d \\ 0 & \text{for} \quad k \geq d \end{cases}$$

On the other hand, the basic computation of the intersection cohomology of a cone is [GM1]

$$IH^k(U) \cong \begin{cases} H^k(L) & \text{for} \quad k < d \\ 0 & \text{for} \quad k \geq d \end{cases}$$

Furthermore, by general sheaf theoretic principles (see next section), it is only necessary to check the Zucker conjecture locally near each singular point. Thus, at least for metrical horns, the identification of the intersection cohomology with the L^2 cohomology is straightforward.

§3. Axiomatic characterization of intersection homology.

Suppose Y is a complex purely n dimensional algebraic variety, which is Whitney stratified by complex algebraic strata. Recall that for each stratum S, every point $x \in S$ has a neighborhood basis consisting of "good" neighborhoods of the form

$$U \cong D^{2s} \times \text{cone}(L)$$

where D^{2s} denotes a 2s-dimensional disk, ($s = \dim_{\mathbb{C}} S$) and where L is a compact stratified set, called the *link* of the stratum S. (If S is connected then the link is determined up to stratum preserving homeomorphism.) The intersection cohomology of such a neighborhood is

$$IH^k(U) \cong \begin{cases} IH^k(L) & \text{for} \quad k < \text{cod}_{\mathbb{C}} S \\ 0 & \text{for} \quad k \geq \text{cod}_{\mathbb{C}} S \end{cases}$$

and this property characterizes intersection cohomology, i.e.

Proposition [GM2]. *Suppose $\mathcal{F}^\bullet$ denotes a complex of fine sheaves on Y such that*

(1) *the restriction $\mathcal{F}^\bullet|Y^o$ to the nonsingular part of Y is quasi isomorphic to the constant sheaf, and*
(2) *for each stratum S, and for each point $x \in S$, and for any good neighborhood U of x,*

$$H^k(U;\mathcal{F}^\bullet) \cong \begin{cases} H^k(L;\mathcal{F}^\bullet|L) & \text{for} \quad k < \mathrm{cod}_{\mathbb{C}} S \\ 0 & \text{for} \quad k \geq \mathrm{cod}_{\mathbb{C}} S \end{cases}$$

Then the quasi-isomorphism in (1) extends canonically to a global quasi-isomorphism

$$\mathcal{F}^\bullet \cong \mathbf{IC}^\bullet(Y)$$

and in particular, the hypercohomology of $\mathcal{F}^\bullet$ is the intersection cohomology of Y.

Remarks.

(1) It is a nontrivial fact [Z1], [Z2] that the sheaf of L^2 forms on the Baily-Borel compactification Y of $\overline{\Gamma\backslash X}$ is fine. Furthermore the "fine" hypothesis (or "soft" or "flabby") is essential to the axiomatic characterization.
(2) The sheaf of L^2 differential forms on $Y = \overline{\Gamma\backslash X}$ is not simply the pushforward to Y of the sheaf of L^2 differential forms on $\Gamma\backslash X$, and it is more accurately described as the "sheaf of locally L^2 differential forms" on Y. For example, any smooth differential form on $\Gamma\backslash X$ is a section of the sheaf of L^2 differential forms on $\Gamma\backslash X$ because, on any sufficiently small subset of $\Gamma\backslash X$, it is square-integrable. However, if such a smooth form grows too rapidly near a cusp, then it will not be in L^2 in a neighborhood of the cusp point.

§4. Why do we expect the Zucker conjecture to be false?

Unfortunately, algebraic varieties are not metrical horns. A more reasonable model for the local metric near a singular point is that of a warped product [B1] [Z1]. Let us suppose the link L decomposes as a product

$$L = L_1 \times L_2$$

and the factor L_2 shrinks twice as fast as the factor L_1, i.e.

$$ds^2 = (e^{-at})^2 ds^2_{L_1} + (e^{-2at})^2 ds^2_{L_2} + dt^2$$

Now consider a differential form $\omega_L = \omega_1 \otimes \omega_2$ which is a product of forms on the two factors. As in § 2, we compute the norm square of the pullback of ω_L to the neighborhood U,

$$\int_U \|\pi^*(\omega_L)\|^2 d\mathrm{vol} = C \int_{t=1}^{\infty} (e^{-at})^{-2k_1}(e^{-2at})^{-2k_2}(e^{-at})^{2d_1}(e^{-2at})^{2d_2-1} dt$$

where C is the constant

$$C = \int_{L_1} \|\omega_1\|^2 d\mathrm{vol}_1 \int_{L_2} \|\omega_2\|^2 d\mathrm{vol}_2$$

and where $k_i = \deg(\omega_i)$, $\dim_{\mathbb{R}}(L_1) = 2d_1$, and $\dim_{\mathbb{R}}(L_2) = 2d_2 - 1$. This integral is finite iff

$$k_1 + 2k_2 < d_1 + 2d_2 - 1$$

Thus the L^2 cohomology group $H^k_{(2)}(U)$ is no longer simply the cohomology of the link in degrees $k > d$ and 0 in degrees $k < d$, but it is a sum of groups $H^{k_1}(L_1) \otimes H^{k_2}(L_2)$ where $k_1 + k_2 = k$ and where k_1 and k_2 satisfy the above relation. In other words, the Zucker conjecture is simply false for this sort of metric, unless certain cohomology groups involved in this tensor product vanish.

The bad news is that warped products of this kind do appear in neighborhoods of a singular point, even in the case of GU(2,1). (As mentioned above, this case was completely understood by Zucker [Z1] (appendix)). The good news is that a link on which the metric behaves like a warped product is not actually a product, but is rather a **bundle** with base L_1 and fibre L_2, and this bundle is homologically nontrivial. The required vanishing of cohomology groups in the link occurs because the E^2 differentials in the Leray spectral sequence of this bundle are injective or surjective (and this in turn follows essentially from various local versions of the "hard" Lefschetz theorem).

Thus the proof of the Zucker conjecture will depend on a detailed understanding of the topology and geometry of the link, which we describe in the next section. In fact, Looijenga's proof of the Zucker conjecture may be summarized as follows:

(0) Reduce the problem to a local verification near each singular point $p \in Y = \overline{\Gamma \backslash X}$, using sheaf theory and the axiomatic characterization of intersection cohomology.
(1) Obtain as explicit as possible a description of the topology of the link L of the singular stratum S which contains the point p.
(2) Decompose the intersection cohomology of L into subspaces according to the rates of growth of representative differential forms.
(3) Realize this decomposition as the eigenspace decomposition of $IH^k(L)$ under the action of certain geometrically defined "local Hecke correspondences" $\Psi_a : L \to L$. Reduce the problem to proving that the intersection cohomology classes which must vanish are those of "weight" $\geq c$ and degree $k < c$ (where $c = \mathrm{codim}_{\mathbb{C}} S$).
(4) Resolve the singularities $\pi : \tilde{Y} \to Y$ of the Baily-Borel compactification, using [AMRT]. Apply the decomposition theorem [BBD] to

exhibit $IH^*(Y)$ as a subspace of $H^*(\tilde{Y})$ both locally and globally (i.e. $IH^*(L) \subset H^*(\pi^{-1}(L))$).

(5) Observe that the local Hecke correspondence Ψ_a also acts locally on $\tilde{Y}$ and that it decomposes $H^*(\pi^{-1}(L))$ into eigenspaces as well.

(6) Apply [KK] or [CKS] to see that $H^k(\pi^{-1}(L))$ has no classes of weight $> k$, and so the same is true for $IH^k(L)$. Thus, for $k < c$, the group $IH^k(L)$ has no classes of weight $\geq c$.

§5. The five fold decomposition. (see [AMRT])

Any point p in a stratified complex n dimensional algebraic variety Y has a "good neighborhood" U with a 3-fold decomposition,

$$U \cong D^{2s} \times \mathrm{cone}(L)$$

(as in § 3). There are 2s dimensions along the stratum, 1 direction along the cone lines away from the stratum, and 2n-2s-1 directions around the stratum (in the link L). (Here s denotes the dimension of the stratum containing the point p.) It turns out that for a locally symmetric variety, the intersection L^o of the link L with the largest stratum Y^o of the space Y has a further 3-fold decomposition. There is a fibration $L^o \to B$, where B is a locally symmetric space (usually non-Hermitian) for some Lie group of lower rank, and the fibre of this bundle is a 2-step nilmanifold R, i.e. R is itself a (real) torus bundle over a (real) torus R_1.

Since the case of $GU(2,1)$ is particularly simple, and some of these factors are trivial, we will illustrate this decomposition for the Baily-Borel compactification of locally symmetric varieties associated to the group $Sp(2n, \mathbb{R})$. Recall that this compactification $Y = \overline{\Gamma\backslash X}$ is the quotient under Γ of a certain Satake compactification X^* of X, which is obtained from X by attaching "rational boundary components" X_P which are in turn indexed by maximal rational parabolic subgroups P of G. A neighborhood in X^* of such a rational boundary component X_P may be described as follows:

If ω denotes the chosen symplectic structure on $\mathbb{R}^{2n}$ then the parabolic P is the subgroup of G which fixes some rational isotropic subspace $F \subset \mathbb{R}^{2n}$. It also fixes the annihilator subspace,

$$F^\perp = \{v \in \mathbb{R}^{2n} | \omega(v, f) = 0 \quad \text{for all} \quad f \in F\}$$

and if we choose an appropriate symplectic basis of $\mathbb{R}^{2n}$ such that F and $F^\perp$ become coordinate subspaces, then the subgroup P may be written in the following block form:

$$P = \begin{bmatrix} M_1 & N_1 & N_2 \\ 0 & M_2 & N_1^* \\ 0 & 0 & M_1^* \end{bmatrix}$$

where M_1^* is determined by M_1, and N_1^* is determined by M_1, M_2, and N_1. We also consider the (one dimensional) center A of the Levi component M,

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}$$

and we write ${}_0M = M/A$ to obtain the Langlands decomposition,

$$P = {}_0MAN.$$

The action of P on the Satake compactification X^* normalizes the boundary component X_P (i.e. X_P is invariant under P) and each of the 5 groups ${}_0M_1 = M_1/A$, M_2, A, N_1, and N_2 act in different directions near X_P. The group

$$M_2 \cong Sp(F^\perp/F)$$

acts along the boundary component, the group A flows towards and away from the boundary (see below) and traces out the "cone lines", while the groups ${}_0M_1$ and N act along the link of the boundary component. These "directions" are preserved when we further divide out by the discrete group Γ, so we obtain

Theorem. *The stratum (or boundary component) $(\Gamma \cap P)\backslash X_P$ corresponding to the parabolic subgroup P is a locally symmetric space isomorphic to $\Gamma_1\backslash Sp(F^\perp/F)/K_1$. The link L of this stratum, when intersected with the largest stratum $Y^0 = \Gamma\backslash X$ of Y, fibres over the $Sl(F)$-locally symmetric space $\Gamma_2\backslash{}_0M_1/K_2$ and the fibre $\Gamma_3\backslash N$ is a two-step nilmanifold.*

(Here K_1, and K_2 are the maximal compact subgroups, and Γ_1, Γ_2, Γ_3 are the intersections of Γ with the appropriate subgroups M_2, ${}_0M_1$, and N respectively.)

Remark. The fibration $L \cap Y^0 \to \Gamma_2\backslash{}_0M_1/K_2$ extends to a map from the whole link to a certain Satake compactification of $\Gamma_2\backslash{}_0M_1/K_2$, however this extended map is no longer a fibration: it is instead a stratified map whose fibres over various strata in $\overline{\Gamma_2\backslash{}_0M_1/K_2}$ can be described in a similarly explicit way (see [Z2] § 3.21 and [Z5]). In the rank 1 case, $L = L \cap Y^0$ (i.e. the singular strata of Y never interact with the topology of L). This is the technical point which makes the rank 1 case simpler. In the next section we will restrict our attention to the rank 1 case (which includes $GU(n, 1)$).

§6. The local Hecke operators.

To summarize the notation so far, we assume that $X = G/K$ is a $\mathbb{Q}$-rank 1 Hermitian symmetric space, $\Gamma \subset G$ is a torsion free arithmetic subgroup which acts on a particular Satake compactification X^* of X. The quotient

$Y = \Gamma\backslash(X^*)$ is the Baily-Borel compactification of the locally symmetric space $\Gamma\backslash X$. Each singular stratum (or boundary component) Y_P of Y is indexed by a rational parabolic subgroup $P \subset G$. Let $P = {}_0MAN$ be a Langlands decomposition of P, so A is the center of the Levi factor $M = {}_0MA$. There are two actions of A on Y which are locally defined near Y_P and which fix Y_P. They are induced , respectively, from left and right multiplication by A on $X = G/K$.

In order to define an action from the right, we notice that $X = G/K = P/(P \cap K) = P/(M \cap K)$. Since A commutes with M it gives a well defined right action on $P/(M \cap K)$, i.e.

$$x(M \cap K) \cdot a = xa(M \cap K)$$

This "geodesic action" [BS] defines the cone lines in a neighborhood of the stratum X_P. It commutes with the (left) action of $\Gamma_P = \Gamma \cap P$, and therefore defines cone lines in a neighborhood of the stratum $\Gamma_P\backslash X_P$ in the Baily-Borel compactification Y of $\Gamma\backslash X$.

Let L denote the link of a singular stratum Y_P of the Baily-Borel compactification of $\Gamma\backslash X$. In the rank 1 case, the link L is completely contained in the largest stratum $\Gamma\backslash X$. We would like to decompose $H^i(L) = IH^i(L)$ into subspaces according to the rate of growth of representative differential forms, so that the contribution of each growth rate may be studied independently. For example, one might try defining

$$\Omega^i_\ell(L) = \{\omega \in \Omega^i(L) | \frac{\|\lambda^*(\omega)\|}{\|\omega\|} = \lambda^\ell\}$$

where $\lambda \in A$ acts on differential forms by the geodesic action. It follows from the computations illustrated in §2 and §4 that if L is the link of a stratum Y_P and if $\omega \in \Omega^i_\ell(L)$ then its pullback to a neighborhood of Y_P is in L^2 iff $\ell < \mathrm{codim}_{\mathbb{C}} Y_P$.

One would like to decompose $H^i(L)$ by finding representative closed (or harmonic) forms in $\Omega^i_\ell(L)$. In order to make sense of such ideas it is necessary to observe that (by [E]) we may first restrict to N-invariant differential forms on L. These decompose under the geodesic action of A into growth rate subspaces and this decomposition passes to cohomology. Looijenga observes that it is possible to recover this "growth rate" decomposition of $H^i(L)$ by considering the effect of a left "action" of A on X.

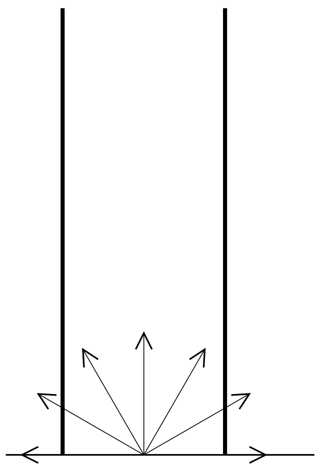
The subgroup A acts on $X = G/K$ from the left by isometries (in fact, all of G acts from the left by isometries), but it does not commute with the action of Γ. However there is a discrete subset D of A consisting of "sufficiently divisible elements" for which $a \in D$ implies that $a\Gamma_P \subset \Gamma_P a$. Left multiplication by such an $a \in D$ defines a self-map Φ_a on a neighborhood of the boundary component $Y_P = \Gamma_P\backslash X_P$ because it is possible to identify a neighborhood of Y_P with (a neighborhood of infinity in $\Gamma_P\backslash X)\cup(Y_P)$. Thus, Looijenga defines

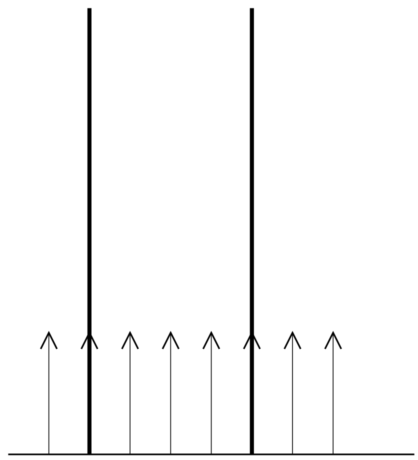
$$\Phi_a(\Gamma_P x) = \Gamma_P a x$$

In summary, conjugating by sufficiently divisible elements $a \in D$ induces a well defined finite covering map $\Psi_a : L \to L$ (called the local Hecke correspondence) on the link L of the stratum $\Gamma_P\backslash X_P$ by

$$\Psi_a(x) := \Phi_a(x) \cdot a^{-1}$$

For $SL_2(\mathbb{R})$ these two actions of A on the upper halfplane are depicted:





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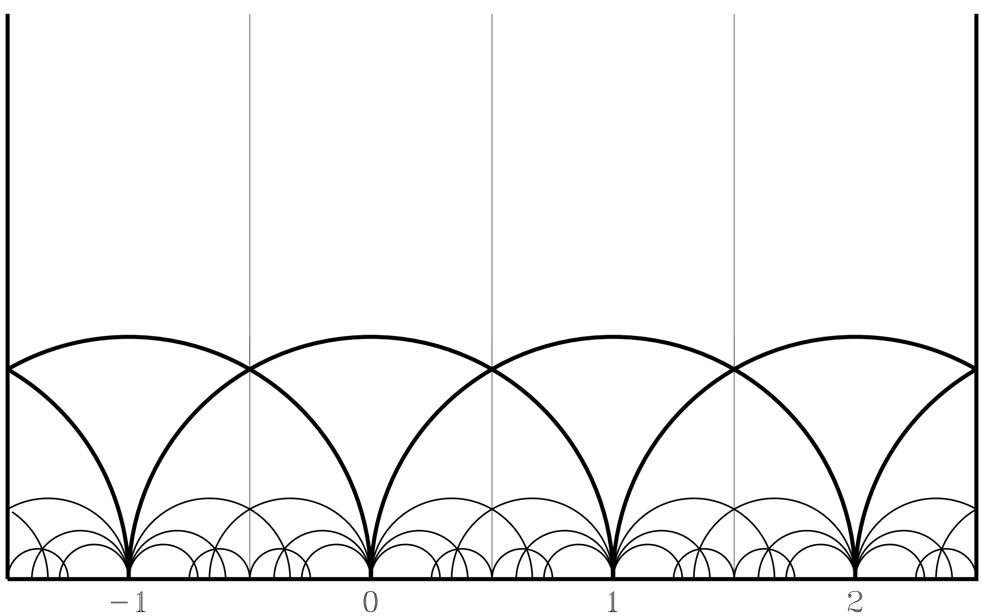

FUNDAMENTAL DOMAINS FOR $SL_2(\mathbb{Z})$

Keeping in mind the fundamental domains for $SL_2(\mathbb{Z})$ as illustrated in diagram 2, we see that, after dividing by Γ, the left action by an element

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in A$$

induces a covering of a neighborhood of the cusp point provided λ is an odd integer (which is the meaning of "sufficiently divisible", in this case).

Composing this with the geodesic (right) action by λ^{-1} gives a covering $\Psi_\lambda : L \to L$ of the link of the cusp point: it wraps the link around itself λ^2 times.

Since divisible elements of A act from the left by isometries, they induce maps on differential forms (in a neighborhood of Y_P) which do not change the norm of the form, although they do change the cohomology class of the form. Since A acts from the right by isotopies, the induced map on differential forms does not change the cohomology class, but it does change the norm. Thus we have:

Proposition [L]. *The decomposition of $H^i(L)$ into weight spaces under the action of conjugation by (sufficiently divisible) elements of A coincides with the decomposition of $H^i(L)$ into subspaces of given growth rate which is obtained from considering the geodesic action of A on the N-invariant differential forms.*

The significance of this proposition is that the study of the growth rates of differential forms has been replaced by the study of eigenspaces (in the cohomology of the link) of the local Hecke operators.. For example, if $\omega \in \Omega^i(L)$ is an N- invariant eigenform of the local Hecke operator, i.e. if $\Psi_\lambda(\omega) = \lambda^\ell \omega$, then $\omega \in \Omega^i_\ell(L)$ so its pullback to the neighborhood of Y_P will be L^2 iff $\ell < c = \mathrm{codim}_{\mathbb{C}} Y_P$. The danger is that there may be cohomology classes in $H^i(L)$ of weight $\ell \geq c$ (which are therefore represented by differential forms $\omega \in \Omega^i(L)$ which are not L^2) but which give rise to local intersection cohomology ($i < n$) and we must show this never happens. (There is another danger: that there may be differential forms which are L^2 but which are not in intersection cohomology ($i > n$). This case is dual to the previous case and need not be considered separately). The nonexistence of cohomology classes of degree $< c$ and weight $\geq c$ follows from Looijenga's "purity" theorem,

Proposition [L]. *("purity") The A-weights on $IH^i(L)$ are all $\leq i$.*

As mentioned in § 4 above, this proposition is proven using the decomposition theorem (as applied to the toroidal resolution) and the purity theorem of [KK] or [CKS].

§7. How do we compute the weights?

In general, the A-weights of the action on the cohomology of the link follow from Kostant's theorem [K] on N-cohomology. However, in particular cases, the weights can be easily determined by calculating the effect of conjugation by $\lambda \in A$. For example, in the case of $Sp(2n,\mathbb{R})$ we have

$$\lambda \begin{pmatrix} m_1 & n_1 & n_2 \\ 0 & m_2 & n_1^* \\ 0 & 0 & m_1^* \end{pmatrix} \lambda^{-1} = \begin{pmatrix} m_1 & \lambda n_1 & \lambda^2 n_2 \\ 0 & m_2 & \lambda n_1^* \\ 0 & 0 & m_1^* \end{pmatrix}$$

Thus A acts with weight 0 on M, weight 1 on N_1, and weight 2 on N_2.

The case of $GU(n,1)$ is somewhat simpler since M is compact and the singularities are isolated. The link L of each singular point p is a compact 2n-1 dimensional 2- step nilmanifold $L = N \cap \Gamma \backslash N$, which fibers over an abelian variety $E = \Gamma \cap N_1 \backslash N_1$ (of complex dimension $d_1 = n-1$), with fibre $S^1 = \Gamma \cap N_2 \backslash N_2$. The local Hecke operator Ψ_λ acts with weight 1 on the base E and with weight 2 on the fiber. The Leray spectral sequence for $H^*(L)$ has E_2 term

$$E_2^{pq} = H^p(E; H^q(S^1))$$

on which Ψ_λ acts with weight $p+2q$. By the calculation in § 4, we see that a class in E_2^{pq} is represented by an L^2 differential form if its weight is $w = p+2q < d_1+1$, whereas a class in E_2^{pq} gives rise to a local intersection cohomology class if

$$p+q \leq d_1 + 1/2$$

Thus there is a single offending group $H^{d_1-1}(E;\, H^1(S^1))$ which is allowable in intersection cohomology but which is not represented by an L^2 differential form, since it has weight d_1+1. So in this case the validity of the Zucker conjecture amounts to the statement that $E_2^{d_1-1,1}$ does not contribute to $H^{d_1}(L)$, i.e. that the differential d_2 is injective on $E_2^{d_1-1,1}$.

We will see in § 8 that (for $GU(n,1)$) this follows directly from the decomposition theorem, which is the main ingredient in the "purity" theorem.

§8. The Decomposition Theorem.

This section is little more than a repeat of the survey articles [GM3], [CGM] to which we refer the reader for further details.

Suppose $\pi : Y \to X$ is a proper algebraic map. Choose stratifications of Y and X,

$$X = X_0 \cup X_1 \cup X_2 \cup \cdots \cup X_r$$

so that π becomes a stratified map, i.e. it takes strata to strata. Let $\mathbf{IC}^\bullet(Y)$ denote the complex of sheaves on Y which gives the intersection cohomology.

Theorem [BBD]. *There is an isomorphism (in the derived category of sheaves on X),*

$$R\pi_*(\mathbf{IC}^\bullet(Y)) \cong \bigoplus_{i=0}^{r} \bigoplus_{j} \mathbf{IC}^\bullet(\overline{X_i}; L_j^i)[\ell_j^i]$$

where ℓ_j^i are integers, L_j^i are local systems on the strata X_i, and where $[t]$ denotes a shift by t. If we further stipulate that the L_j^i are irreducible local systems, then the terms in this decomposition are uniquely determined by the map π and the choice of the stratification.

This theorem has global and local consequences. The global consequence is obtained by applying hypercohomology and it says that the intersection cohomology of Y is a direct sum of intersection cohomology groups of closures of strata of X (with twisted coefficients and shifts). The local consequences are obtained by applying stalk cohomology at a point $x \in X$. If Y is nonsingular (or even rationally smooth) then we may replace $\mathbf{IC}^\bullet(Y)$ with the constant sheaf, so we have:

$$H^a(\pi^{-1}(x)) \cong \bigoplus_{i=0}^{r} \bigoplus_{j} IH_x^{a-\ell_j^i}(\overline{X_i}; L_j^i)$$

where IH_x denotes the local (or stalk) cohomology at the point x. In particular a stratum X_i can make a nonzero contribution in this formula only when $x \in \overline{X_i}$.

There is also a relative "hard" Lefschetz theorem: fix a stratum X_i and let $Loc(\ell)$ denote the direct sum of all the local systems on X_i which appear with shift ℓ in the decomposition theorem. Then there is a relative Lefschetz operator

$$\Lambda : Loc(\ell) \to Loc(\ell + 2)$$

and it induces an isomorphism

$$\Lambda^r : Loc(N - r) \cong Loc(N + r)$$

where $N = \dim_{\mathbb{C}}(Y)$. For example, if Y is an arbitrary projective variety and X is a point, this is precisely the hard Lefschetz theorem for the intersection cohomology of Y.

Example 1. Suppose X and Y are nonsingular and π is a smooth fibration with fibre F. The stratifications of X and Y may be taken to consist of a single stratum each. The local consequence says that for each $x \in X$, there is a decomposition

$$H^m(F) \cong \bigoplus_{j} H_x^{m-\ell_j}(X; L_j)$$

Since X is a manifold, the local cohomology groups vanish except in degree 0, i.e. when $m = \ell_j$, and in this case the local cohomology group is the stalk of L_j at the point x. Thus the local systems L_j which appear in this formula are the cohomology groups of the fibres. By numbering them appropriately, we may take $L_j = H^j(F)$. The global statement now reads,

$$H^m(Y) \cong \bigoplus_j H^{m-j}(X; H^j(F))$$

which is the main consequence of Deligne's degeneration theorem [D].

Example 2. If Y is nonsingular but X is arbitrary, then by applying the preceding argument to the largest stratum of X we conclude that

$$H^m(Y) \cong \bigoplus_j IH^{m-j}(X; H^j(F)) \bigoplus \text{ other terms}$$

where F is the fibre of $\pi : Y \to X$ over a point x in the largest stratum of X.

Example 3. If Y is a resolution of singularities of X, then the local system $H^j(F)$ of example (2) above is just the constant sheaf for $j = 0$ (and is 0 for $j \neq 0$). Thus we see that $\mathbf{IC}^\bullet(X)$ is a direct summand of $Rf_*(\mathbb{C}_Y)$ and so $IH^m(X)$ is a subgroup of the cohomology of any resolution of X.

Example 4. Suppose X^n has an isolated singularity $p \in X$ and that $\pi : Y \to X$ is a resolution of singularities, with a nonsingular exceptional divisor $E = \pi^{-1}(p)$. Let $U = X \cap B_\delta(p)$ denote the intersection of X with a small ball centered at p, and let $L = \partial U$ be the link of p in X. Then $\pi : L \to E$ is a bundle with fibre S^1 and the long exact cohomology sequence for the pair

$$H^*(U, L; Rf_*(\mathbb{C})) = H^*(\pi^{-1}(U), L; \mathbb{C})$$

is the Gysin sequence for the bundle $L \to E$:

$$\begin{array}{ccccccc}
H^i(\pi^{-1}(U), L) & \longrightarrow & H^i(\pi^{-1}(U)) & \longrightarrow & H^i(L) & \longrightarrow & H^{i+1}(\pi^{-1}(U); L) \\
|| & & || & & || & & || \\
H^{i-2}(E) & \xrightarrow{\cdot c^1} & H^i(E) & \xrightarrow{\pi^*} & H^i(L) & \xrightarrow{\pi_*} & H^{i-1}(E)
\end{array}$$

where $\cdot c^1$ denotes the cup product with the first Chern class of the normal bundle of E. On the other hand, the long exact cohomology sequence for $H^*(U, L; \mathbf{IC}^\bullet)$ breaks into short exact sequences since

$$IH^i(U) \cong \begin{cases} 0 & \text{for } i \geq n \\ H^i(L) & \text{for } i < n \end{cases}$$

and

$$IH^i(U,L) \cong \begin{cases} H^{i-1}(L) & \text{for } i > n \\ 0 & \text{for } i \le n \end{cases}$$

But the decomposition theorem states that $\mathbf{IC}^\bullet(X)$ is a direct summand of $Rf_*(\mathbb{C})$. If every term in this second exact sequence is a direct summand of the corresponding term in the first exact sequence, and if every third term in the two sequences are equal (to $H^*(L)$), we conclude that the homomorphism

$$\cdot c^1 : H^j(E) \to H^{j+2}(E)$$

is an injection for $j \le n-2$ and is a surjection for $j \ge n-2$. So in this case, the decomposition theorem amounts to the hard Lefschetz theorem on E. (In fact, c^1 is the hyperplane class for a particular projective embedding of E).

Example 5. Suppose X is an algebraic variety with an isolated singularity p and that $\pi : Y \to X$ is a resolution of singularities such that $\pi^{-1}(p)$ is a union of smooth divisors with normal crossings. The decomposition theorem can be used to give a formula for the intersection cohomology of X in terms of the cohomology groups of the divisors. This formula is completely described in [FR].

§9. Completion of the proof for $GU(n,1)$.

Let p be a singular point of $\overline{\Gamma\backslash X}$. As in § 6, the link L of p fibres over an abelian variety E (of complex dimension $d_1 = n-1$), with fibre S^1. The proof of the Zucker conjecture was there reduced to showing that the E_2 differential in the spectral sequence of this fibration,

$$d : H^{d_1-1}(E) \otimes H^1(S^1) \to H^{d_1+1}(E) \otimes H^0(S^1)$$

is injective. Let $\pi : Y \to \overline{\Gamma\backslash X}$ denote a toroidal resolution of singularities [AMRT]. This resolution may actually be chosen so that $\pi^{-1}(p) \cong E$ [La]. By example 4 of § 6, the decomposition theorem implies that $\cdot c^1 : H^{d_1-1}(E) \to H^{d_1+1}(E)$ is injective. But this is precisely the differential d.

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