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Inventiones mathematicae

Weighted cohomology

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Dedicated to Armand Borel

1. Introduction

(1.0) Weighted cohomology. In this paper, we introduce some invariants called *weighted cohomology groups* of an arithmetic group Γ. Suppose that $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is a torsion free (cf. §35.5) arithmetic subgroup of the group $\mathbf{G}(\mathbb{Q})$ of rational points of a semisimple linear algebraic group $\mathbf{G}$. (In the body of the paper we will address the slightly more general case of reductive groups.) Let E be a rational vectorspace on which is given an algebraic representation of $\mathbf{G}(\mathbb{Q})$. The weighted cohomology group $W^pH^i(\Gamma, E)$ of Γ with coefficients in E depends on an auxiliary parameter p called a *weight profile* (see §1.1). For certain values of p, $W^pH^i(\Gamma, E)$ is $H^i(\Gamma, E)$, the ordinary cohomology of the group Γ. For other values of p, it is more interesting:

Suppose D is the symmetric space associated to $\mathbf{G}$, i.e. the quotient of the group of real points of $\mathbf{G}$ by its maximal compact subgroup. Let X be the locally symmetric space $\Gamma\backslash D$. Then $H^i(\Gamma, E) = H^i(X, \mathbf{E})$, where $\mathbf{E}$ is the local system over X manufactured out of E. If D (and hence X) is Hermitian, a more interesting cohomology theory of X is it's $L_{(2)}$ cohomology $H^i_{(2)}(X, \mathbf{E} \otimes \mathbb{C})$. By the Zucker conjecture (i.e. the Looijenga [L], Saper-Stern [SS] theorem), there is a canonical isomorphism,

$$H^i_{(2)}(X, \mathbf{E} \otimes \mathbb{C}) \cong IH^i(\hat{X}, \mathbf{E}) \otimes \mathbb{C} \quad (1.0.1)$$

where $IH^i(\hat{X}, \mathbf{E})$ is the intersection homology of the Baily-Borel compactification $\hat{X}$ of X. A primary result of this paper is that, for a weight profile μ called the *middle* weight profile (see §1.1), we have a canonical isomorphism,

$$W^\mu H^i(\Gamma, E) \cong IH^i(\hat{X}, \mathbf{E}) \quad (1.0.2)$$

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Thus the weighted cohomology of Γ is a generalization of the intersection cohomology group $IH^i(\hat{X}, \mathbf{E})$. Unlike $IH^i(\hat{X}, \mathbf{E})$, weighted cohomology makes sense for all Γ, whether or not X is Hermitian.

The main advantage of weighted cohomology is its computability. This stems from its topological construction: The group $W^pH^i(\Gamma, E)$ is a cohomology theory on a compactification $\bar{X}$ of X which we call the *reductive Borel-Serre compactification*. (By this, we mean that $W^pH^i(\Gamma, E)$ is the cohomology of a complex of sheaves $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$ on $\bar{X}$. The complex $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$ may be thought of as the sheaf $\mathbf{E}$ on $X \subset \bar{X}$ extended to $\bar{X}$ with certain "support conditions" at $\bar{X} - X$, depending on p.) The compactification $\bar{X}$ is "almost nonsingular": it has singularities that are particularly amenable to computations.

In case X is Hermitian, there is a map $\pi : \bar{X} \to \hat{X}$ which should be thought of as a partial resolution of singularities of $\hat{X}$. The reason that the identity (1.0.2) above is surprising is the following: Whenever there is a map $\pi : \bar{X} \to \hat{X}$ of topological spaces, any cohomology theory on $\bar{X}$ can be computed as a cohomology theory on $\hat{X}$ by Grothendieck's $R\pi_*$ functor. However, it is rare that a cohomology theory on $\hat{X}$ can be computed as a cohomology theory on $\bar{X}$, as $IH^i(\hat{X}, \mathbf{E})$ can be computed as $W^\mu H^i(\Gamma, E)$.

Weighted cohomology is the main technical ingredient in the Topological Trace Formula [GM3],[GM6]. This is fixed point formula for the Lefschetz number of a Hecke correspondence operating on $IH^i(\hat{X}, \mathbf{E})$ (or, in fact, operating on any weighted cohomology group). Aside from the computability referred to above, the reason that weighted cohomology is useful in this regard is that Hecke operators extend to the reductive Borel-Serre compactification $\bar{X}$ (whereas they don't extend, for example, to the nonsingular toroidal compactifications).

Further salubrious uses for weighted cohomology will be discussed in §1.12 below. The other sections of the introduction are as follows: The first, 1.A (§1.1 - §1.3) gives the formal properties of weighted cohomology. The second, 1.B (§1.4 - §1.7) describes the geometry of the reductive Borel-Serre compactification $\bar{X}$. The third, 1.C (§1.8 - §1.11) gives the construction of weighted cohomology. These sections may be read more or less independently of each other.

1.A. Formal properties of weighted cohomology

(1.1) Weight profiles. The weighted cohomology groups $W^pH^i(\Gamma, E)$ depend on an auxiliary parameter p called a *weight profile*.

Let Δ be the set of $\mathbf{G}(\mathbb{Q})$ conjugacy classes of maximal rational parabolic subgroups of $\mathbf{G}$. In other words, Δ is the set of simple roots in a rational relative root system of $\mathbf{G}$. Let $\mathbb{Z} + \frac{1}{2}$ be the set $\{\cdots, -1\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \cdots\}$.

Definition. A *weight profile* is a function from Δ to $\mathbb{Z} + \frac{1}{2}$.

The set Δ is finite. Its cardinality is the $\mathbb{Q}$-rank of $\mathbf{G}$, which is 0 if and only if X is compact. In this case, $X = \hat{X}$ and by default, there is a unique weight profile ς. Then $W^\varsigma H^i(\Gamma, E) = H^i(\Gamma, E) = H^i(X, \mathbf{E}) = H^i(\hat{X}, \mathbf{E}) = IH^i(\hat{X}, \mathbf{E})$.

There are two particularly important weight profiles: the upper middle one and the lower middle one. For a maximal rational parabolic subgroup $\mathbf{Q} \subset \mathbf{G}$, let $2\rho_Q \in \mathbb{Z}$ be the sum of the positive roots of $\mathbf{G}$, restricted to a weight of S_Q, the split center of the Levi subgroup of Q, and considered as an integer by the usual identification of weights of S_Q with the integers. Then the *upper middle* weight profile μ is given by $\mu(Q) = -\rho_Q$ if $\rho_Q \in \mathbb{Z}+\frac{1}{2}$, and otherwise $\mu(Q) = -\rho_Q+\frac{1}{2}$. The *lower middle* weight profile ν is given by $\nu(Q) = -\rho_Q$ if $\rho_Q \in \mathbb{Z}+\frac{1}{2}$, and otherwise $\nu(Q) = -\rho_Q-\frac{1}{2}$. There is an involution $p \mapsto \bar{p}$ on the set of weight profiles given by $\bar{p}(Q) = -2\rho_Q - p(Q)$ which is then called the dual weight profile. This involution switches μ and ν.

Readers familiar with intersection cohomology will notice a resemblance between the notion of a weight profile and that of a perversity. There is an annoying sign difference between them, however, necessitated by the standard sign conventions for weights in algebraic group theory. Larger weight profiles behave like smaller perversities: the upper middle weighted cohomology group is the analogue of the lower middle intersection cohomology group and vice versa.

(1.2) Properties of the groups $W^pH^i(\Gamma, E)$. The collection of weighted cohomology groups of an arithmetic group should be seen as analogous to the collection of intersection cohomology groups of a variety. The weighted cohomology groups $W^pH^i(\Gamma, E)$ have the following properties, which are reminiscent of properties of intersection cohomology:

(0) The weighted cohomology group $W^pH^i(\Gamma, E)$ is a finite dimensional rational vector space, and $W^pH^i(\Gamma, E) = 0$ unless $0 \le i \le n$, where n is the dimension of X.

(1) If, for each $\alpha \in \Delta$, $p(\alpha)$ is small enough, then the weighted cohomology group is the ordinary cohomology group of Γ, i.e., $W^pH^i(\Gamma, E) \cong H^i(\Gamma, E) = H^i(X, \mathbf{E})$. If $p(\alpha)$ is large enough, then the weighted cohomology group is the ordinary homology group of Γ, i.e., $W^pH^i(\Gamma, E) \cong H_{n-i}(\Gamma, E) = H_{n-i}(X, \mathbf{E})$.

(2) If $p(\alpha) \ge q(\alpha)$ for all $\alpha \in \Delta$, then there is a canonical map

$$W^pH^i(\Gamma, E) \to W^qH^i(\Gamma, E)$$

"Most" of these maps are isomorphisms, so that there are only finitely many really distinct weighted cohomology groups. If X is Hermitian, then the map

$$W^\mu H^i(\Gamma, E) \xrightarrow{\cong} W^\nu H^i(\Gamma, E)$$

is an isomorphism, where μ and ν are the lower and upper middle weight profiles as in §1.1. So in case X is Hermitian, we may unambiguously speak of "the" middle weighted cohomology group $W^\mu H^i(\Gamma, E) = W^\nu H^i(\Gamma, E)$.

(3) *Poincaré duality.* There is a nonsingular pairing

$$W^pH^i(\Gamma, E) \times W^{\bar{p}}H^{n-i}(\Gamma, E^*) \to \mathbb{Q}$$

where E^* is the dual representation to E, and $p \mapsto \bar{p}$ is the involution of §1.1. As a consequence, if X is Hermitian, so $E \cong E^*$, then the middle weighted cohomology group $W^\mu H^i(\Gamma, E) = W^{\bar{\mu}}H^i(\Gamma, E)$ is dually paired to itself.

(4) If X is Hermitian, then the middle weighted cohomology group is isomorphic to the middle intersection cohomology group of the Baily-Borel compactification of X,

$$W^\mu H^i(\Gamma, E) \cong IH^i(\hat{X}, \mathbf{E}).$$

(5) A Hecke correspondence $(c_1, c_2) : \mathscr{H} \rightrightarrows X$ induces a map

$$\mathscr{H}^* : W^p H^i(\Gamma, E) \to W^p H^i(\Gamma, E)$$

from weighted cohomology to itself.

(1.3) Properties of the weighted cohomology sheaves $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$. The weighted cohomology group is defined geometrically as the hypercohomology group

$$W^p H^i(\Gamma, E) = H^i(\bar{X}; \mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})).$$

Here $\bar{X}$ is the reductive Borel-Serre compactification of X, and $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$ is a complex of sheaves on $\bar{X}$ called the *weighted cohomology sheaves.*

The weighted cohomology sheaves $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$ are analogous to the intersection cohomology sheaves $\mathbf{IC}^\bullet(\mathbf{E})$, as the following list of properties illustrates. In this list, each property is a refinement of the similarly numbered property in §1.2.

(0) The complexes $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$ are cohomologically constructible ([GM2]).

(1) If, for each $\alpha \in \Delta$, $p(\alpha)$ is small enough, then there is a quasi-isomorphism,

$$\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E}) \xrightarrow{\sim} Ri_*\mathbf{E}$$

where $i : X \to \bar{X}$ is the inclusion. If $p(\alpha)$ is large enough, then

$$\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E}) \xrightarrow{\sim} Ri_!\mathbf{E}.$$

(2) If $p(\alpha) \geq q(\alpha)$ for all $\alpha \in \Delta$, then there is a canonical sheaf map

$$\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E}) \to \mathbf{W}^q\mathbf{C}^\bullet(\mathbf{E}).$$

(3) If p and $\bar{p}$ are dual weight profiles then the sheaf $\mathbf{W}^{\bar{p}}\mathbf{C}^\bullet(\mathbf{E}^*)$ is Verdier-Borel-Moore dual to the sheaf $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$.

(4) If X is Hermitian, then there are quasi-isomorphisms,

$$R\pi_*\mathbf{W}^\mu\mathbf{C}^\bullet(\mathbf{E}) \xrightarrow{\sim} R\pi_*\mathbf{W}^\nu\mathbf{C}^\bullet(\mathbf{E}) \xrightarrow{\sim} \mathbf{IC}^\bullet(\mathbf{E})$$

where π is the projection (§1.0) from $\bar{X}$ to $\hat{X}$ and $\mathbf{IC}^\bullet(\mathbf{E})$ is the middle intersection complex on the Baily-Borel compactification $\hat{X}$.

(5) A Hecke correspondence $(c_1, c_2) : \mathscr{H} \rightrightarrows X$ on X extends to a correspondence $(\bar{c}_1, \bar{c}_2) : \bar{\mathscr{H}} \rightrightarrows \bar{X}$, and there is an induced map $\bar{c}_2^*\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E}) \to \bar{c}_1^!\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$.

Caveat. Even if X is Hermitian, it is not necessarily true that $\mathbf{W}^\mu\mathbf{C}^\bullet(\mathbf{E}) \to \mathbf{W}^\nu\mathbf{C}^\bullet(\mathbf{E})$ is a quasi-isomorphism. So $\mathbf{W}^\mu\mathbf{C}^\bullet(\mathbf{E})$ is not necessarily (Verdier-) self-dual on $\bar{X}$.

1.B. Geometry of the reductive Borel–Serre compactification

(1.4) The boundary strata. The reductive Borel–Serre compactification $\bar{X}$ of X is obtained from X by adding to it a union of nonsingular *boundary strata* X_P, one for every Γ- conjugacy class of rational parabolic subgroups P of $\mathbf{G}$.

Notation. For a rational parabolic subgroup P of $\mathbf{G}$, we will use the following notations: $P = M_P A_P \mathscr{U}_P$ is a rational Langlands decomposition for P (so $\mathscr{U}_P$ is the unipotent radical of P; $M_P A_P$ is a Levi subgroup; and A_P is the connected component of the (real points of the) $\mathbb{Q}$-split torus $\mathbf{S_P}$ in the center of $M_P A_P$).

Topology of the stratum X_P. The boundary stratum X_P corresponding to P is $\Gamma_M \backslash M_P / K_P$ where K_P is the maximal compact subgroup of M_P, and Γ_M is the projection of $\Gamma \cap P$ into M_P. This expression says that the reductive Borel-Serre boundary stratum X_P is the quotient of the M-homogeneous space M_P/K_P by the discrete group Γ_M. (The name "reductive" Borel-Serre comes from the fact that M is a reductive group.) If M_P is semi-simple, then X_P is itself a negatively curved locally symmetric. Otherwise it is a flat torus bundle over a locally symmetric space.

Closure relations among the strata. The boundary stratum $X_P \subset \bar{X}$ is in the closure of those boundary strata X_Q indexed by rational parabolic subgroups Q containing P. So, given P, there are $2^m - 1$ boundary strata having X_P in their closure (including X_P itself), where m is the dimension of A_P. Their closure relations among each other are the same as the closure relations among the faces and corners of a hyperquadrant $\{\mathbb{R}_{\geq 0}\}^m$, where $\mathbb{R}_{\geq 0}$ is the closed half-line $\{c \in \mathbb{R} | c \geq 0\}$.

Restriction to the $\mathbb{Q}$*-rank* 1 *case.* For purposes of the introduction, in the rest of §1.B and §1.C we will restrict our attention to the case that **G** has $\mathbb{Q}$-rank 1. This means that all rational parabolic subgroups Q are maximal, so that no boundary stratum of $\bar{X}$ has any other in its closure. This eliminates some complicated combinatorics (which is treated in the main body of the paper), and hopefully allows the main ideas to be clearer.

(1.5) The structure of a neighborhood T of a boundary stratum. A large part of this paper consists of the detailed study of certain neighborhoods T in $\bar{X}$ of a boundary stratum X_Q, called *geodesic neighborhoods*. Since Q is maximal (§1.4.4), a small neighborhood T of X_Q will intersect no strata other than X_Q and X. The following geometric structures in $T \cap X$ are very important in the construction of weighted cohomology.

The distinguished geodesics. For every point $x \in X \cap T$, there is a distinguished maximal geodesic line $l \subset X \cap T$ through x characterized by the property that it contains a single point $y \in X_Q$ in its closure. A point moving along l in one direction converges to y, and in the other direction it leaves T.

The geodesic retraction. The map $r_Q : X \cap T \to X_Q$ which sends the point $x \in X$ to the limit y of the distinguished geodesic through x is called the geodesic retraction. This map is a fibration.

The links. Each fiber F of the geodesic retraction $r : X \cap T \to X_Q$ has a product decomposition $F \cong \mathbb{R}_{>0} \times N_Q$, where $\mathbb{R}_{>0}$ is the open half-line. The half-lines $\mathbb{R}_{>0} \times n$ for $n \in N_Q$ are the distinguished geodesic lines. The submanifolds $c \times N_Q$ for $c \in \mathbb{R}$ are called the *links* of X_Q. They are determined geometrically by the property that their tangent spaces are perpendicular to the distinguished geodesic lines. In terms of groups, they are projections to $X = \Gamma \backslash D$ of the $\mathcal{U}_Q$ orbits on D. The manifold N_Q is topologically a nilmanifold. It is homeomorphic to $\Gamma_{\mathcal{U}} \backslash \mathcal{U}_Q$, where $\Gamma_{\mathcal{U}} = \Gamma \cap \mathcal{U}_Q$.

The geodesic action. The semigroup of nonnegative elements of the group $A_Q = \mathbf{S_Q}(\mathbb{R})^0$ acts on $X \cap T$ by moving a point $x \in X \cap T$ along the geodesic line l through it. This is called the *geodesic action*. The action $a : X \to X$ of $a \in A_Q$ is symbolized by $x \mapsto x \bullet a$. In terms of the product decomposition $F = \mathbb{R}_{>0} \times N_Q$, the action of $a \in A$ adds a constant to the first coordinate and fixes the second coordinate.

Caveat. The geodesic action by $a \in A_Q$ is not an isometry. The reason is that as c increases, the metric on the slice $c \times N_P \subset \mathbb{R}_{>0} \times N_P$ becomes smaller. The geodesic action is not the restriction of the action of G on D.

The connection on the link bundle. Let $\mathscr{B}$ be the quotient space of $X \cap T$ divided by the geodesic action of A_Q. The geodesic retraction induces a fibration $\theta : \mathscr{B} \to X_Q$ with fiber N_Q. This fibration is called the *link bundle* of X_Q. There is a canonical flat connection on the link bundle — i.e. there is a foliation of $\mathscr{B}$ by leaves of the same dimension of X_Q such that each leaf is transverse to each fiber $\theta^{-1}(y)$. The flat connection is constructed by choosing an appropriate lift of M_Q to Q (see §7.10).

(1.6) Looijenga Hecke correspondences. Let T be a special neighborhood of the boundary stratum X_Q, as in §1.5. Then there are some important continuous self-maps $L_\alpha : T \to T$ for certain $\alpha \in S_Q$, called *Looijenga Hecke correspondences.* The map L_α restricts to the identity on X_Q. For $x \in X \cap T$, $L_\alpha(x)$ may be defined as follows: Take any lift $\hat{x} \in D$ of $x \in X = \Gamma \backslash D$. Then $L_\alpha(x)$ is the projection of $\alpha\hat{x}$. This is not well defined for most $\alpha \in S_Q$, however for certain positive and "sufficiently divisible" values α it is well defined.

The geometry of Looijenga Hecke operators can be described as follows. The map L_α takes the each fiber F of r_Q into itself. It takes the manifold $c \times N_Q$ to a manifolds $c' \times N_Q$ by a covering projection which is a local isometry, where c' is obtained from c by adding a constant. The metric sizes of $c \times N_Q$ and $c' \times N_Q$ must satisfy a relation in order for there to exist a local isometry $c \times N_Q \to c' \times N_Q$. This is why only certain $\alpha \in S_Q$ can be used.

In this paper, we develop more general Looijenga Hecke correspondences that are not necessarily single valued, and that are defined for all $\alpha \in \mathbf{S_Q}(\mathbb{Q})$. Their composition has a compatibility with the group operation in $\mathbf{S_Q}(\mathbb{Q})$, so that if $\mathbf{S_Q}(\mathbb{Q})$ acts on a linear space (such as the stalk of a sheaf) through Looijenga Hecke correspondences, it acts as a representation of $\mathbf{S_Q}(\mathbb{Q})$.

Caveat. The action of $\mathbf{S_Q}(\mathbb{Q})$ on T by Looijenga Hecke correspondences is very different from the geodesic action of $A_Q \subset \mathbf{S_Q}(\mathbb{R})$ of §1.5 .4. In some sense it is the interplay of these two different actions that is the technical heart of this paper.

(1.7) The action of the torus S_Q on $Ri_*\mathbf{E}$. Let $\mathbf{E}$ be the local system over X obtained from a representation E of $\mathbf{G}$. Then $\mathbf{E}$ can be considered as a constructible (in fact, locally constant) sheaf on X. Let $i : X \to \bar{X}$ be the inclusion. There is a remarkable quasi–isomorphism class of complexes of sheaves $Ri_*\mathbf{E}$, due originally to Grothendieck. The salient feature of $Ri_*\mathbf{E}$ is:

(1.7.1) The stalk cohomology of $Ri_*\mathbf{E}$ at a point $x \in X_Q$ is (up to a dimension shift) the cohomology of the link N_Q of X_Q.

Any complex of sheaves $\mathbf{S}^\bullet$ on $\bar{X}$ in the quasi–isomorphism class of $Ri_*\mathbf{E}$ is called an *incarnation* of $Ri_*\mathbf{E}$. The hard part of verifying that a complex is an incarnation of $Ri_*\mathbf{E}$ comes down to verifying property 1.7.1.

In the derived category of the category of sheaves, there is a canonical "lift" $L_\alpha^*(Ri_*\mathbf{E})|T \to (Ri_*\mathbf{E})|T$ of L_α to the restriction $(Ri_*\mathbf{E})|T$ of $Ri_*\mathbf{E}$ to T. Since L_α is the identity on X_Q, this lift restricts to a map $\mathbf{L}_\alpha$ from $(Ri_*\mathbf{E})|X_Q$ to itself. Therefore, the Looijenga Hecke correspondences induce an action of $\mathbf{S_Q}(\mathbb{Q})$ on $(Ri_*\mathbf{E})|X_Q$.

1.C. Construction of weighted cohomology

(1.8) Truncation by weights. Let $\mathbf{T}^\bullet = \{\cdots \to \mathbf{T}^0 \to \mathbf{T}^1 \to \mathbf{T}^2 \to \cdots\}$ be a complex of sheaves (of vectorspaces) on $\bar{X}$. Suppose that we are given, for each rational (maximal) parabolic subgroup $\mathbf{Q}$, an action of $\mathbf{S_Q}$ by automorphisms on the restriction $\mathbf{T}^\bullet|X_Q$ of $\mathbf{T}^\bullet$ to the boundary stratum X_Q. Suppose that this action satisfies the following condition:

(1.8.1) The group $\mathbf{S_Q}$ acts semi–simply on $\mathbf{T}^i|X_Q$.

In other words, each separate sheaf $\mathbf{T}^i|X_Q$ in the complex has a direct sum decomposition $\mathbf{T}^i|X_Q = \bigoplus_\alpha \mathbf{T}^i_\alpha$ such that $\mathbf{S_Q}$ acts on $\mathbf{T}^i_\alpha$ by a character α of $\mathbf{S_Q}$. Fix a weight profile p. Since we are restricting to the Q rank 1 case, and the set of simple roots Δ has only one element (§1.4), a weight profile (§1.1) is just a number in $\mathbb{Z}+\frac{1}{2}$. For each rational parabolic subgroup $\mathbf{Q} \subset \mathbf{G}$ we identify the weights of $\mathbf{S_Q}$ with the integers.

Definition. The *weight truncated* sheaf $\tau_{\geq p}\mathbf{T}^\bullet$ is the sub–complex of $\mathbf{T}^\bullet$ described as follows: Over $X \subset \bar{X}$, $\tau_{\geq p}\mathbf{T}^\bullet = \mathbf{T}^\bullet$, i.e. no truncation takes places except in the boundary strata. Over the boundary stratum $X_Q \subset X$, $\tau_{\geq p}\mathbf{T}^\bullet$ is defined as the subsheaf satisfying the condition that $\mathbf{S_Q}$ acts on $\tau_{\geq p}\mathbf{T}^i|X_Q$ by characters α of weight $\geq p$.

(1.9) The idea behind weighted cohomology. The idea behind the construction of weighted cohomology sheaves $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$ is the following: $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$ should be the sheaf $Ri_*\mathbf{E}$, equipped with the set of $\mathbf{S_Q}(\mathbb{Q})$ actions on $(Ri_*\mathbf{E})|\bar{X}_Q$ given by the Looijenga Hecke Correspondences (§1.7), truncated by the weight p (§1.8).

The problem with this "definition" is that $Ri_*\mathbf{E}$ is not a complex of sheaves with $\mathbf{S_Q}(\mathbb{Q})$ actions; it is rather an element of the derived category of sheaves, with $\mathbf{S_Q}(\mathbb{Q})$ actions in the derived category. We do not know how to perform a weight truncation on such an object. One difficulty is that the derived category is not an abelian category.

Instead, one must specify an actual incarnation $\mathbf{T}^\bullet$ of the quasi-isomorphism class $Ri_*\mathbf{E}$ with $\mathbf{S_Q}(\mathbb{Q})$ actions that agree with those of §1.7 and satisfy the condition (1.8.1). If the stalks of $\mathbf{T}^i$ were finite dimensional vector spaces, then the condition (1.8.1) would be automatic. However, this is rarely the case and our experience is that such incarnations $\mathbf{T}^\bullet$ are not easy to find.

We give two such complexes, and thereby two constructions of weighted cohomology. The first (§14) is a sheaf of complex vector spaces, which are differential forms of a special kind. The second (§28) is a sheaf of rational vector spaces. We prove (§29) that the second, tensored with the complex numbers, is quasi-isomorphic to the first (so the two constructions of weighted cohomology agree).

(1.10) The construction of weighted cohomology using differential forms. Let $\Omega^\bullet(\mathbf{E}\otimes\mathbb{C})$ be the complex of differential forms on X with coefficients in $\mathbf{E}\otimes\mathbb{C}$. The *special differential forms* $\Omega^\bullet_{\mathrm{sp}}(\mathbf{E}\otimes\mathbb{C}) \subset \Omega^\bullet(\mathbf{E}\otimes\mathbb{C})$ are those forms which, close to the boundary stratum X_Q, satisfy two properties: 1) they are 0 when evaluated on a set of vectors including the tangent to distinguished geodesic (§1.5), and 2) they are invariant under the $\mathcal{U}_Q$ action when pulled up to D. The special differential forms, considered as a sheaf on $\bar{X}$, is our candidate for the sheaf $\mathbf{T}^\bullet$ of §1.9. We define the sheaf $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E}\otimes\mathbb{C})$ to be the truncation by p of the sheaf of special differential forms.

To show that the special differential forms are an incarnation of $Ri_*(\mathbf{E}\otimes\mathbb{C})$, we verify condition (1.7.1) using the Nomizu–Van Est theorem, which says that the cohomology of the link N_Q is calculated by the forms which pull up to invariant forms on $\mathcal{U}_Q$.

The incarnation $\Omega^\bullet_{\mathrm{sp}}(\mathbf{E}\otimes\mathbb{C})$ of $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E}\otimes\mathbb{C})$ has the advantage that its cohomology is its global section cohomology. So $W^pH^i(\Gamma, E\otimes\mathbb{C})$ is just the quotient of the space of closed special i-forms on X by the space of exact special i-forms.

(1.11) The construction of rational weighted cohomology. Consider the sheaf $\mathbf{A}$ of functions on X which, when restricted to a geodesic neighborhood T of the boundary stratum X_Q, satisfy three properties: (1) they are constant on the distinguished geodesic lines (§1.5) so they pass to functions on the link bundle $\mathscr{B}$; (2) they are invariant under the connection on $\theta : \mathscr{B} \to X_Q$; and (3) they are polynomials with rational coefficients as functions on the fibers N_Q of the fibration $\theta : \mathscr{B} \to X_Q$. (This last condition makes sense because $N_Q = \Gamma_{\mathcal{U}}\backslash\mathcal{U}_Q$ and rational polynomials make sense on $\mathcal{U}_Q$.) Outside the geodesic neighborhood the sheaf $\mathbf{A}$ consists of locally constant functions.

Now, we resolve the sheaf $\mathbf{E}$ on X in the following standard way:

$$\mathbf{E} \xrightarrow{\epsilon_0} \mathbf{E}\otimes\mathbf{A} \xrightarrow{\epsilon_1} \mathrm{cokernel}(\epsilon_0)\otimes\mathbf{A} \xrightarrow{\epsilon_2} \mathrm{cokernel}(\epsilon_1)\otimes\mathbf{A} \xrightarrow{\epsilon_3} \cdots$$

Call this resolution $\mathbf{I}^\bullet(\mathbf{E})$. Our candidate for the complex $\mathbf{T}^\bullet$ of §1.9 is $i_*\mathbf{I}^\bullet(\mathbf{E})$. So we define $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$ to be the weight truncation by p of $i_*\mathbf{I}^\bullet(\mathbf{E})$.

To see that $i_*\mathbf{I}^\bullet(\mathbf{E})$ is an incarnation of $Ri_*\mathbf{E}$, it suffices to verify condition (1.7.1) which follows from the fact that the sheaf of polynomial functions on $\mathcal{U}_Q$ are $\Gamma_{\mathcal{U}}$-acyclic.

1.D. Applications of weighted cohomology

(1.12) The singularities of the reductive Borel–Serre compactification $\bar{X}$ are given away by the nilmanifolds N_Q which are the links. The cohomology of N_Q is equal to the cohomology of the Lie algebra of $\mathcal{U}_Q$ which is efficiently calculated by Kostant's theorem in terms of roots and weights (§11). This calculation extends to a calculation of the stalk cohomology of the weighted cohomology complex $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$.

As an application, the Euler characteristics of the weighted cohomology groups can all be calculated in terms of roots and weights. If X is Hermitian, then the weighted cohomology $W^\mu H^i(\Gamma, E)$ is the same as the intersection cohomology $IH^i(\hat{X}, \mathbf{E})$ of the Baily-Borel compactification, so the calculation gives a calculation of the intersection cohomology Euler characteristics.

Another application to the Baily-Borel compactification is this: The local intersection cohomology groups of $\hat{X}$ at a singular point p is a sum of particular weighted cohomology groups of an associated linear symmetric space. (This is similar to the fact that of the local cohomology of $Rj_*\mathbf{E}$, for $j : X \to \hat{X}$, is a sum of ordinary cohomology groups of the same linear symmetric space.) These applications will be the subject of a further paper.

In addition to these realized applications and the topological trace formula, there are possible (as yet, unrealized) applications described in §35.

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Contents

2. Subsheaves and truncation

(2.1) Suppose that Y is a locally compact Hausdorff space, with a decomposition into finitely many smooth (C^∞) manifolds (which we will call "strata"), $Y = \coprod_{i=1}^m Y_i$. (We assume each stratum is locally closed in Y.) Suppose that $\mathbf{S}$ is a sheaf on Y, and suppose that for each stratum $Y_i \subset Y$ a subsheaf $\mathbf{R_i} \subset \mathbf{S}|Y_i$ has been specified. Then there is *at most one* subsheaf $\mathbf{R} \subset \mathbf{S}$ such that, for each stratum Y_i, we have

$$\mathbf{R}|Y_i = \mathbf{R_i}. \tag{2.1.1}$$

(2.2) Definition. The collection of subsheaves $\{\mathbf{R_i}\}_{i \in I}$ is *compatible* if the following condition holds: For each stratum Y_i, for each point $y \in Y_i$, and for each element $v_y \in (R_i)_y$ in the stalk at y, there is a neighborhood U of y in Y and a section $v \in \Gamma(U, \mathbf{S})$ such that the section v restricts to the germ v_y at the point y, and such that for every stratum Y_j, we have

$$v|(U \cap Y_j) \in \Gamma(U \cap Y_j, \mathbf{R_j}). \tag{2.2.1}$$

(2.3) Lemma. *If the collection $\{\mathbf{R_i}\}$ of subsheaves is compatible, then there exists a unique subsheaf $\mathbf{R} \subset \mathbf{S}$ on Y such that (2.1.1) holds.*

(2.4) Proof. The unique possibility for $\mathbf{R}$ is the subsheaf of $\mathbf{S}$ which may be obtained as the sheafification of the presheaf $\mathbf{PR}$ whose sections over an open set $U \subset Y$ is given by:

$$\Gamma(U, \mathbf{PR}) = \{v \in \Gamma(U, \mathbf{S}) | \forall i \in I, v|(U \cap Y_i) \in \Gamma(U \cap Y_i, \mathbf{R_i})\}. \tag{2.4.1}$$

This is easily seen to satisfy (2.1.1) if the $\{\mathbf{R_i}\}$ are compatible. □

(2.5) Lemma. *Suppose that $\mathbf{S}$ is a sheaf on a topological space X, that $Y = \coprod_{i=1}^m Y_i \subset X$ is a closed and stratified subspace, and that $\mathbf{R_i} \hookrightarrow \mathbf{S}|Y_i$ is a collection of compatible sub-sheaves defined on Y. Then there is a unique sub-sheaf $\mathbf{S}' \subset \mathbf{S}$ on X such that*

1. $\mathbf{S}'|(X - Y) = \mathbf{S}|(X - Y)$
2. For each $i = 1, 2, \ldots, m$ we have, $\mathbf{S}'|Y_i = \mathbf{R_i}$

(2.6) Proof. In fact, $\mathbf{S}'$ is the sheafification of the presheaf $\mathbf{PS}'$ whose sections over an open set $U \subset X$ are given by

$$\Gamma(U, \mathbf{PS}') = \begin{cases} \Gamma(U, \mathbf{S}) & \text{if } \; U \subset X - Y \\ \Gamma(U \cap Y, \mathbf{R}) & \text{if } \; U \cap Y \neq \phi \end{cases} \tag{2.6.1}$$

where $\mathbf{R}$ is the subsheaf of $\mathbf{S}|Y$ which is guaranteed by Lemma 2.3. □

(2.7) Definition. The sheaf $\mathbf{S}'$ of (2.5.1) is said to be obtained by *truncating* the sheaf $\mathbf{S}$ to the subsheaves $\mathbf{R_i}$ along the subspaces Y_i.

3. Compact and parabolic subgroups

(3.1) Algebraic groups will be designated by bold face type ($\mathbf{G}, \mathbf{P}$, etc.). If an algebraic group is defined over the real numbers then its group of real points will be in Roman ($G = \mathbf{G}(\mathbb{R})$, $P = \mathbf{P}(\mathbb{R})$, etc.). The connected component of the identity is denoted with a superscript 0 ($\mathbf{G}^0$, $\mathbf{P}^0$, etc.) and the derived group is denoted with a superscript (1) ($\mathbf{G}^{(1)}$, $\mathbf{P}^{(1)}$, etc.).

(3.2) Throughout this paper we fix a connected reductive algebraic group $\mathbf{G}$ which is defined over the rational numbers $\mathbb{Q}$. Denote by $\mathbf{S_G}$ the maximal $\mathbb{Q}$−split torus in the center of $\mathbf{G}$ and let $A_G = \mathbf{S_G}(\mathbb{R})^0$ be the identity component of the group of real points of $\mathbf{S_G}$. Then the group of real points splits as a direct product ([BS] §1.2),

$$\mathbf{G}(\mathbb{R}) = A_G \times {}^0\mathbf{G}(\mathbb{R})$$

where

$${}^0\mathbf{G}(\mathbb{R}) = \bigcap_{\chi} \ker(\chi^2)$$

is the intersection of the kernels of the squares of all the algebraic rationally defined characters on $\mathbf{G}$. The group ${}^0\mathbf{G}(\mathbb{R})$ contains all compact and arithmetic subgroups of $\mathbf{G}(\mathbb{R})$.

(3.3) Let $K_Z \subset G$ be the maximal compact connected subgroup of the group of real points of the connected component of the center of $\mathbf{G}$. Choose a maximal compact subgroup $K^{(1)} \subset \mathbf{G}^{(1)}(\mathbb{R})$ of the derived group and choose $K \subset \mathbf{G}(\mathbb{R})$ to be a subgroup such that

$$K \supset K^{(1)}.K_Z.A_G.$$

and which is compact modulo A_G. Define

$$D = G/K. \tag{3.3.1}$$

Then D is a "generalized" symmetric space: it may fail to be connected and it may have trivial $\mathbb{R}^*$ factors. The group G acts transitively on D and the choice of K corresponds to a choice of basepoint $x_0 \in D$. We will often write $K = K(x_0)$. Choose a neat arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ and set

$$X = \Gamma\backslash D. \tag{3.3.2}$$

We will refer to X as a "locally symmetric space" although it may also fail to be connected. In this paper we will define and study the weighted cohomology groups for such a space.

(3.4) The compact subgroup $K^{(1)}.K_Z$ is contained in a *unique* maximal compact subgroup $K_1 \subset \mathbf{G}(\mathbb{R})$. Define $\tilde{K} = K_1.A_G$. Then $\tilde{D} = G/\tilde{K}$ is connected and there is a canonical projection $f : D \to \tilde{D}$ which is an isomorphism on each connected component. Thus we obtain a canonical identification between the various connected components of D. Furthermore the identity component of ${}^0\mathbf{G}(\mathbb{R})$ acts transitively on $\tilde{D}$ and the stabilizer of the point $f(x_0)$ is a maximal compact subgroup of $({}^0\mathbf{G}(\mathbb{R}))^0$. In this way we may canonically reduce to the following situation, which we will henceforth assume to be the case:

(3.5) Assumption. The affine algebraic group $\mathbf{G}$ is connected and reductive with ${}^0\mathbf{G} = \mathbf{G}$, and $A_G = \{1\}$; the group $K = K(x_0) \subset G$ is a maximal compact subgroup, and the ("generalized") symmetric space $D \cong G/K$ is connected and diffeomorphic to Euclidean space.

(3.6) For any rationally defined parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ we denote the unipotent radical by $\mathcal{U}_\mathbf{P}$, with real points $\mathcal{U}_P = \mathcal{U}_\mathbf{P}(\mathbb{R})$. The Levi quotient will be denoted $\mathbf{L_P} = \mathbf{P}/\mathcal{U}_\mathbf{P}$ and the projection will be denoted $\nu_P : \mathbf{P} \to \mathbf{L_P}$. Let $\mathbf{S_P}$ denote the maximal $\mathbb{Q}$-split torus in the center of $\mathbf{L_P}$. As in [BS] §0.3 we denote the $\mathbb{Q}$-split radical by $\mathbf{R_dP}$. There is an exact sequence of rational algebraic groups,

$$1 \to \mathcal{U}_\mathbf{P} \to \mathbf{R_dP} \to \mathbf{S_P} \to 1$$

which identifies $\mathbf{R_dP}$ as the semidirect product of $\mathcal{U}_\mathbf{P}$ with any lift of $\mathbf{S_P}$. As in [BS] §4.2, we define the connected real torus, $A_P = \mathbf{S_P}(\mathbb{R})^0$

(3.7) We identify the group $\chi(\mathbf{P})$ of algebraic rationally defined characters on $\mathbf{P}$ with the group $\chi(\mathbf{S_P})$ of algebraic characters of $\mathbf{S_P}$, and by abuse of notation we will sometimes denote this same set by $\chi(A_P)$. We also use the notation

$$\mathbf{M_P} = {}^0\mathbf{L_P}$$

for the intersection of the kernels of the squares of the rationally defined characters on $\mathbf{L_P}$, so that the real points of the Levi quotient split as a direct product $L_P = M_P \times A_P$.

(3.8) Associated to the maximal compact subgroup $K = K(x_0)$ there is a "Cartan involution" ([BS] §1.6, §1.9) θ of G whose fixed point set is K, and there is a unique lift $i_{x_o} : L_P \to P$ of the Levi factor, such that the image $L_P(x_0) = i_{x_0}(L_P)$ is θ-stable. We also obtain lifts $A_P(x_0), S_P(x_0), M_P(x_0)$ of the subgroups A_P, S_P, and M_P respectively, and hence a "rational Langlands decomposition",

$$P = M_P(x_0)A_P(x_0)\mathcal{U}_P. \tag{3.8.1}$$

with $(R_dP)^0 = A_P(x_0) \ltimes \mathcal{U}_P$. If $x_1 \in D$ is a different basepoint then the lifts $L_P(x_0)$ and $L_P(x_1)$ are conjugate by some element $u \in \mathcal{U}_P$. The group P acts transitively on D. The group $K_P(x_0) = K(x_0) \cap P$ is a maximal compact subgroup of P and it is the stabilizer of the point x_0. We denote by $K_P = \nu(K_P(x_0))$ the projection of $K_P(x_0)$ to the Levi quotient L_P and note ([BS] §1.4,1.5) that

1. K_P is a maximal compact subgroup of M_P
2. $K_P(x_0) = i_{x_0}(K_P) \subset M_P(x_0)$
3. The group $A_P(x_0)$ centralizes $K_P(x_0)$.

If $\mathbf{Q} \subset \mathbf{P}$ is another parabolic subgroup then $\mathcal{U}_P \subset \mathcal{U}_Q$, $R_dP \subset R_dQ$ and this inclusion determines a canonical embedding, $A_P \subset A_Q$ as the inclusion of identity components of the inclusion

$$S_P = R_dP/\mathcal{U}_P \hookrightarrow R_dQ/\mathcal{U}_Q = S_Q. \tag{3.8.2}$$

The choice of basepoint $x_0 \in D$ also determines inclusions, $K_Q(x_0) = Q \cap K_P(x_0) \subset K_P(x_0)$, $L_Q(x_0) \subset L_P(x_0)$, and $A_P(x_0) \subset A_Q(x_0)$. (This last inclusion agrees with the canonical embedding (3.8.2).)

(3.9) Rational Basepoints. Let $\mathbf{P} \subset \mathbf{G}$ be a rational parabolic subgroup. If the lift $\mathbf{L_P}(x_0)$ is a rational algebraic subgroup of $\mathbf{P}$, then we will say that the basepoint x_0 is *rational* for the parabolic subgroup $\mathbf{P}$. Every rational parabolic subgroup has rational basepoints, and in fact if x_0 is rational for $\mathbf{P}$ then it is also rational for every rational parabolic subgroup $\mathbf{Q} \supseteq \mathbf{P}$. If the basepoint x_0 can be chosen so that its stabilizer $K(x_0)$ is defined over $\mathbb{Q}$, then this basepoint is rational for all rational parabolic subgroups of $\mathbf{G}$. However such a "universally rational" basepoint does not exist in general.

4. Geodesic action

(4.1) The geodesic action of Borel and Serre [BS] §3 is a right action of the group A_P on D whose value $a \bullet y$ is defined by

$$a \bullet y = i_y(a).y$$

This geodesic action coincides with the action of the group $A_P(x_0)$ on $D = P/K_P(x_0)$ by multiplication from the right, i.e.

$$a \bullet gK_P(x_0) = gK_P(x_0)i_{x_0}(a) = gi_{x_0}(a)K_P(x_0) \tag{4.1.1}$$

for any $g \in P$. The resulting action of A_P on D is independent of the choice of basepoint. If $\mathbf{Q} \subset \mathbf{P}$ is another rational parabolic then the geodesic action of A_Q on D is compatible with the geodesic action of $A_P \supset A_Q$ on D.

5. Relative roots

(5.1) Fix once and for all a minimal rational parabolic subgroup $\mathbf{P}_0 \subset \mathbf{G}$. The rational parabolic subgroups $\mathbf{P}$ containing $\mathbf{P}_0$ are called *standard*. We also fix a maximal $\mathbb{Q}$−split torus $\mathbf{S_0}$ in the minimal parabolic subgroup $\mathbf{P}_0$. (Such an $\mathbf{S_0}$ is a rational lift of the maximal $\mathbb{Q}$-split torus $\mathbf{S_{P_0}}$ in the center of the Levi quotient.) Let ${}_{\mathbb{Q}}\Phi = {}_{\mathbb{Q}}\Phi(\mathbf{S_0}, \mathfrak{g})$ denote the (relative) roots of $\mathfrak{g}$ with respect to $\mathbf{S_0}$. The unipotent radical $\mathcal{U}_{P_0}$ of P_0 determines a linear order on the relative root system ${}_{\mathbb{Q}}\Phi$ such that the positive roots are those in $\mathfrak{N}(\mathbb{C}) = \mathrm{Lie}(\mathcal{U}_{P_0})(\mathbb{C})$,

$$\mathfrak{N}(\mathbb{C}) = \sum_{\alpha>0} \mathfrak{g}_\alpha(\mathbb{C}).$$

Let ${}_{\mathbb{Q}}\Delta \subset {}_{\mathbb{Q}}\Phi$ denote the corresponding basis of positive simple roots. The rational parabolic subgroups containing $\mathbf{P}_0$ are in one to one correspondence with subsets of ${}_{\mathbb{Q}}\Delta$ ([B1] §11.7). Each simple root $\alpha \in {}_{\mathbb{Q}}\Delta$ corresponds to a maximal parabolic subgroup $\mathbf{Q}_\alpha \supset \mathbf{P}_0$ and is the restriction to $\mathbf{S_0}$ of a generator χ_α of the character module of $\mathbf{Q}_\alpha$. If $\mathbf{Q}_{\alpha_1}, \mathbf{Q}_{\alpha_2}, \ldots, \mathbf{Q}_{\alpha_s}$ denote the maximal parabolic subgroups containing a standard rational parabolic subgroup $\mathbf{P}$, then $\mathbf{P} = \cap_{i=1}^{s}\mathbf{Q}_{\alpha_i}$ and the character module (of algebraic rationally defined characters) $\chi(\mathbf{P})$ is generated by $\chi_{\alpha_1}, \chi_{\alpha_2}, \ldots, \chi_{\alpha_s}$. In particular these roots determine a canonical isomorphism $A_P \cong (\mathbb{R}_{>0})^{({}_{\mathbb{Q}}\Delta)}$, by $a \mapsto (a^{\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_s})$.

6. The Borel-Serre compactification

(6.1) Definition. The Borel-Serre *boundary component* e_P corresponding to the rational parabolic subgroup $\mathbf{P}$ is the quotient of the symmetric space under the geodesic action,

$$e_P = D/A_P = P/K_P(x_0)A_P(x_0).$$

In [BS] §9.3, Borel and Serre construct a partial compactification $\bar{D}$ of D by attaching rational boundary components $e_P = D/A_P$ which are in one to one correspondence with rationally defined parabolic subgroups $\mathbf{P} \subset \mathbf{G}$ in such a way that the torus $A_P \cong (\mathbb{R}_{>0})^s$ is compactified by adding a point at infinity to each factor of $\mathbb{R}_{>0}$. The space $\bar{D}$ is a (noncompact) manifold with (infinitely many) corners. The closure $\bar{e}_P$ of e_P in $\bar{D}$ is the partial Borel-Serre compactification of e_P. The choice of basepoint $x_0 = \text{Fix}(K) \in D$ determines compatible basepoints $x_P = \text{Fix}(K \cap P) \in e_P$.

(6.2) The group $\mathbf{G}(\mathbb{Q})$ of rational points in $\mathbf{G}$ acts as a group of automorphisms of the partial compactification $\bar{D}$. The subgroup Γ acts freely on $\bar{D}$ and a fundamental result of Borel and Serre [BS] states that the quotient $\bar{X} = \Gamma\backslash\bar{D}$ is a compact manifold with corners. The corners (or Borel-Serre boundary *strata*) of $\bar{X}$ are in one to one correspondence with Γ conjugacy classes of rational parabolic proper subgroups of $\mathbf{G}$. If we fix a representative $\mathbf{P}$ of such a conjugacy class, and set $\Gamma_P = \Gamma \cap P$, then the corresponding boundary stratum is $Y_P = \Gamma_P\backslash e_P$. The closure $\bar{Y}_P$ of Y_P in $\bar{X}$ is the Borel-Serre compactification of Y_P.

(6.3) Geodesic retraction. For each pair of rational parabolic subgroups $\mathbf{P} \subseteq \mathbf{Q} \subseteq \mathbf{G}$, the group A_P acts from the right on the boundary component $e_Q = Q/K_Q A_Q(x_0) \cong P/K_P A_Q(x_0)$ with $A_Q \subset A_P$ acting trivially, and with quotient

$$e_Q/A_P(x_0) \cong P/K_P A_Q(x_0) A_P(x_0) = e_P.$$

(Here we include the case $D = e_G$.) Thus we obtain a canonical *geodesic retraction*

$$r_P : e_P^* = \coprod_{Q \supseteq P} e_Q \longrightarrow e_P \tag{6.3.1}$$

from the "open star" $e_P^* \subset \bar{D}$ of the corner e_P, which extends the geodesic retraction $D \to e_P$. The retraction r_P commutes with the action of P, i.e., for any $g \in P$ and for any $x \in e_P^*$, we have

$$r_P(gx) = g r_P(x). \tag{6.3.2}$$

If $\gamma \in \mathbf{G}(\mathbb{Q})$ and $P' = \gamma P \gamma^{-1}$ then left multiplication by γ maps e_P^* to $e_{P'}^*$ and the projections r_P and $r_{P'}$ are compatible with this action,

$$r_{P'}(\gamma x) = \gamma r_P(x) \tag{6.3.3}$$

Even more is true. Let

$$\bar{e}_P^* = \bigcup_{\mathbf{R} \subseteq \mathbf{P}} e_R^* = \bigcup \{ e_Q \mid \mathbf{Q} \cap \mathbf{P} \text{ is parabolic} \} \tag{6.3.4}$$

denote the open neighborhood of the closure $\bar{e}_P$ of the boundary component in $\bar{D}$. Then the geodesic retraction r_P extends canonically to a stratified retraction

$$r_P : \tilde{e}_P^* \longrightarrow \tilde{e}_P \tag{6.3.5}$$

with the property that for any rational parabolic subgroup $\mathbf{Q} \subset \mathbf{G}$, if $\mathbf{R} = \mathbf{Q} \cap \mathbf{P}$ is parabolic then the restriction $(r_P|e_Q) : e_Q \to e_R$ coincides with the geodesic retraction r_R.

(6.4) Lemma. *For every maximal rational parabolic subgroup $\mathbf{Q} \subset \mathbf{G}$ there is a neighborhood V_Q of the closure $\tilde{e}_Q \subset \tilde{D}$ and a smooth "distance function" $\rho_Q : V_Q \to [0, \epsilon)$ such that the mapping*

$$(r_Q, \rho_Q) : V_Q \xrightarrow{\cong} \tilde{e}_Q \times [0, \epsilon) \tag{6.4.1}$$

is a diffeomorphism of manifolds with corners, and such that

(a) $\rho_Q^{-1}(0) = \tilde{e}_Q$,
(b) V_Q is preserved by the groups $\Gamma_Q = \Gamma \cap Q$ and $\mathcal{U}_Q$,
(c) For all $\gamma \in \Gamma_Q$ and for all $x \in V_Q$, $\rho_Q(\gamma x) = \rho_Q(x)$,
(d) For all $u \in \mathcal{U}_Q$ and for all $x \in V_Q$, $\rho_Q(ux) = \rho_Q(x)$,
(e) If $\gamma \in \mathbf{G}(\mathbb{Q})$, and $\mathbf{Q}' = \gamma \mathbf{Q} \gamma^{-1}$ then $\rho_{Q'}(x) = \rho_Q(\gamma x)$,
(f) If $\mathbf{Q}'$ is a maximal rational parabolic subgroup and if $\mathbf{P} = \mathbf{Q} \cap \mathbf{Q}'$ is parabolic and if $x \in V_Q \cap V_{Q'}$ then $\rho_Q(r_{Q'}(x)) = \rho_Q(x)$, and $r_Q r_{Q'}(x) = r_{Q'} r_Q(x) = r_P(x)$.

(6.5) Outline of proof. Every manifold with corners admits a system of collared neighborhoods (6.4.1) of the closures of boundary strata which satisfy the commutativity conditions (f). By choosing such a system of collared neighborhoods for the boundary strata of the Borel-Serre compactification $\tilde{X}$ and then lifting these to $\tilde{D}$, we can guarantee that the resulting neighborhoods and projections have the appropriate Γ_Q and $\mathbf{G}(\mathbb{Q})$ invariance, properties (c) and (e). The $\mathcal{U}_Q$ invariance, property (d), follows from property (e) by continuity. □

It follows from the Γ_Q-invariance and the $\mathbf{G}(\mathbb{Q})$-invariance properties of r_Q and ρ_Q that these functions pass to mappings which are well defined on some neighborhood $\tilde{U}_Q$ of the closure $\tilde{Y}_Q$ of the boundary stratum Y_Q in the Borel Serre compactification $\tilde{X} = \Gamma \backslash D$. By rescaling the function ρ_Q we may replace the value ϵ by 2. We summarize these geometrical facts in the following statement:

(6.6) Proposition. *The Borel-Serre compactification $\tilde{X}$ of $X = \Gamma \backslash D$ is a manifold with corners. The closure $\tilde{Y}_Q$ of each maximal boundary stratum Y_Q admits an open "collared" neighborhood $\tilde{U}_Q \subset \tilde{X}$ and a diffeomorphism $(r_Q, \rho_Q) : \tilde{U}_Q \xrightarrow{\cong} \tilde{Y}_Q \times [0, 2)$ such that the resulting retraction $r_Q : \tilde{U}_Q \to \tilde{Y}_Q$ is given by the geodesic action of the one dimensional torus A_Q. It is possible to choose these neighborhoods in the following compatible way: If $\mathbf{Q_1}, \mathbf{Q_2}, \ldots, \mathbf{Q_s}$ are maximal rational parabolic subgroups and if $\mathbf{P} = \mathbf{Q_1} \cap \mathbf{Q_2} \cap \ldots \cap \mathbf{Q_s}$ is parabolic, then the s commuting geodesic actions determine a diffeomorphism of manifolds with corners,*

$$(r_P, \rho_P) : \tilde{U}_P = \tilde{U}_{Q_1} \cap \ldots \cap \tilde{U}_{Q_s} \xrightarrow{\cong} \tilde{Y}_P \times [0, 2)^s \tag{6.6.1}$$

where r_P is the geodesic retraction and

$$\rho_P(x) = (\rho_1(x), \rho_2(x), \ldots, \rho_s(x)). \tag{6.6.2}$$

Suppose $I \subset \{1, 2, \ldots, s\}$ is a proper subset, $\mathbf{Q_I} = \cap_{i \in I} \mathbf{Q_i} \supset \mathbf{P}$ is the corresponding rational parabolic subgroup, and $Y_I = Y_{Q_I}$ is the associated boundary stratum *with closure $\tilde{Y}_I$. Then the diffeomorphism (r_P, ρ_P) restricts to a diffeomorphism of manifolds with corners,*

$$(r_P, \rho_P) : \tilde{U}_P \cap \tilde{Y}_I \longrightarrow \tilde{Y}_P \times [0,2)^I$$

where

$$[0,2)^I = \{(\rho_1, \rho_2, \dots, \rho_s) \in [0,2]^s \mid \rho_j = 0 \text{ whenever } j \notin I\}$$

is the "half open face" of the rectangle $[0,2)^s$ which is determined by the index set I. □

Such a neighborhood $\tilde{U}_P$ is called a *geodesic neighborhood* of the closure $\tilde{Y}_P$ of the boundary stratum Y_P in the Borel-Serre compactification. We denote by $U_P = \tilde{U}_P \cap X$ the intersection of this neighborhood with the symmetric space X.

(6.7) Remark. The retractions r_P are canonically defined but the "distance functions" ρ_P are not. The "distance functions" cannot be chosen to be invariant with respect to the action of the full parabolic subgroup P. It is important that the "coordinates" $[0,2)^s$ (of (6.6.1)) on the geodesic neighborhood $\tilde{U}_P$ should not be confused with the partial compactification $\bar{A}_P$ of the torus, A_P (see [Z1] §3.19).

7. The nilmanifold fibration

(7.1) As above, let $\mathbf{P}$ denote a rational parabolic subgroup with unipotent radical $\mathcal{U}_\mathbf{P}$ and let $\Gamma_P = \Gamma \cap P$ and $\Gamma_{\mathcal{U}} = \Gamma \cap \mathcal{U}_P$. Then $N_P = \Gamma_{\mathcal{U}} \backslash \mathcal{U}_P$ is a nilmanifold with universal covering space $pr : \mathcal{U}_P \to N_P$.

(7.2) We now describe a fibration of the Borel-Serre boundary stratum Y_P by nilmanifolds isomorphic to N_P. The Borel-Serre boundary component $e_P = P/K_P(x_0)A_P(x_0)$ is acted upon (from the left) by Γ_P and by $\mathcal{U}_P$. The quotient by Γ_P is the Borel-Serre stratum Y_P,

$$Y_P = \Gamma_P \backslash e_P = \Gamma_P \backslash P/K_P(x_0)A_P(x_0). \tag{7.2.1}$$

The quotient of e_P by $\mathcal{U}_P$ is denoted D_P and we denote the quotient mapping by $\mu : e_P \to D_P$. Thus,

$$D_P = \mathcal{U}_P \backslash P/K_P(x_0)A_P(x_0) = L_P/K_P A_P \cong M_P/K_P \tag{7.2.2}$$

may be canonically identified with the (generalized) symmetric space M_P/K_P corresponding to the Levi quotient L_P. We remark that D_P is canonically diffeomorphic to the quotient $R_d P \backslash D$.

(7.3) Definition. The *reductive Borel-Serre stratum* X_P is the quotient space

$$X_P = \Gamma_P \backslash D_P.$$

(7.4) The projection $P \to X_P$ factors through Y_P and gives a map $\pi : Y_P \to X_P$. If $y \in P(\mathbb{R})$ then in terms of double cosets,

$$\pi(\Gamma_P y K_P(x_0)A_P(x_0)) = \Gamma_P \mathcal{U}_P y K_P(x_0)A_P(x_0)$$
$$\in X_P = \Gamma_P \mathcal{U}_P \backslash P/K_P(x_0)A_P(x_0).$$

The projection $\nu : P \to L_P$ induces an identification

$$X_P = \Gamma_P\backslash D_P \cong \Gamma_L\backslash L_P/K_PA_P \cong \Gamma_L\backslash M_P/K_P$$

(where $\Gamma_L = \nu(\Gamma_P) \cong \Gamma_P/\Gamma_{\mathcal{U}}$) of X_P with a locally symmetric space for the Levi quotient, L_P. The spaces e_P, Y_P, D_P, and X_P form a commutative diagram of fiber bundles,

$$\begin{array}{ccc} e_P & \xrightarrow{\tilde{\tau}} & Y_P \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\pi} \\ D_P & \xrightarrow[\tau]{} & X_P \end{array} \tag{7.4.1}$$

(7.5) Remark. There does not exist a global action of $\mathcal{U}_P$ on Y_P whose orbits are the fibers of π.

(7.6) Definition. We define a mapping (which depends on the choice of basepoint $x_0 \in X$),

$$F : \mathcal{U}_P \times D_P \to e_P$$

by

$$F(u, zK_PA_P) = u.i_{x_0}(z)K_P(x_0)A_P(x_0) \in e_P = P/K_P(x_0)A_P(x_0)$$

(here, $u \in \mathcal{U}_P$, $zK_PA_P \in D_P = L_P/K_PA_P$, and $i_{x_0} : L_P \to P$ is the θ-stable lift which is determined by the basepoint x_0.)

Proposition. ([Z1] §3.4) *The map F is a diffeomorphism.* □

(7.7) In this paragraph we describe the action of the parabolic group P on the Borel-Serre boundary component e_P in terms of the coordinates $\mathcal{U}_P \times D_P$ which are determined by the diffeomorphism F above. Let us denote by $\nu : P \to L_P = P/\mathcal{U}_P$ the canonical projection to the Levi quotient, and by $i = i_{x_0} : L_P \to P$ the lift (which is determined by the choice of basepoint x_0). For any $x \in P$ the element $\kappa(x) = x.i\nu(x^{-1})$ is in $\mathcal{U}_P = \ker(\nu)$. Then, $\forall x \in P, \forall u \in \mathcal{U}_P$, we have $xui\nu(x^{-1}) = xux^{-1}\kappa(x) \in \mathcal{U}_P$.

(7.8) Lemma. *The action of P on $\mathcal{U}_P \times D_P$ is given by*

$$x.(u, zK_PA_P) = (xui\nu(x^{-1}), \nu(x)zK_PA_P) \tag{7.8.1}$$

Proof. The proof is a computation:

$$\begin{aligned} F(xui\nu(x^{-1}), \nu(x)zK_PA_P) &= x.ui(z)K_P(x_0)A_P(x_0) \\ &= x.F(u, zK_PA_P). \end{aligned}$$ □

(7.9) Consequences.

1. If $x \in \mathcal{U}_P$ then $x.(u, zK_PA_P) = (xu, zK_PA_P)$ which shows that $\mathcal{U}_P$ acts only on the first factor, and it does so by translation.
2. If $x \in A_P(x_0)$ then $\nu(x)$ is in the center of L_P, so $x.(u, zK_PA_P) = (u^x, zK_PA_P)$ which shows that $A_P(x_0)$ acts only on the first factor, and it does so by conjugation.

3. For any $y = zK_PA_P \in D_P$, the partial map $F_y : \mathcal{U}_P \to e_P$ induces a diffeomorphism $f_y : N_P \cong \pi^{-1}(x)$ of the nilmanifold N_P with the fiber over the point $x = \Gamma_L zK_PA_P \in X_P$, which in terms of double cosets, is given by

$$f_y(\Gamma_U v) = \Gamma_P v i_{x_0}(z) K_P(x_0) A_P(x_0).$$

(7.10) Proposition. *The product structure $F : \mathcal{U}_P \times D_P \to e_P$ induces an integrable connection on the nilmanifold fibration $\pi : Y_P \to X_P$, in other words, a decomposition of each tangent space,*

$$T_yY_P \cong \ker(d\pi(y)) \oplus H_y$$

into vertical and horizontal components such that the plane fields $\{H_y\}$ are tangent to smooth "horizontal" submanifolds.

(7.11) Proof. The product structure determined by F on the Borel-Serre boundary component e_P certainly determines an integrable connection on the (trivial) fibration $e_P \to D_P$. In order for this to pass to an integrable connection on the bundle $\pi : Y_P \to X_P$ it is necessary for the horizontal subspaces H_y to be preserved under the discrete group Γ_P. However, the action of Γ_P on the product space $\mathcal{U}_P \times D_P$ is given in §7.8,

$$\gamma.(u, zK_PA_P) = (\gamma u i\nu(\gamma^{-1}), \nu(\gamma)zK_PA_P)$$

and it does not mix the two factors. Therefore the differential of this action preserves the decomposition into horizontal and vertical subspaces. □

(7.12) Proposition. *The integrable connection on the nilmanifold fibration $\pi : Y_P \to X_P$ is independent of the choice of basepoint which was used in its definition.*

(7.13) Proof. It suffices to show that the integrable connection on e_P which is determined by the product structure $F : \mathcal{U}_P \times D_P \to e_P$ is independent of the choice of basepoint. Choose two basepoints, $x_0, x_1 \in D$ and choose $g \in P$ so that $gx_0 = x_1$. Then $i_{x_1}(z) = g i_{x_0}(z) g^{-1}$ for any $z \in L_P$. Let F_0, F_1 be the corresponding diffeomorphism $\mathcal{U}_P \times D_P \to e_P$. Then

$$\begin{aligned} F_1(u, zK_PA_P) &= u i_{x_1}(z) K_P(x_1) A_P(x_1) \in P/K_P(x_1)A_P(x_1) \\ &= u(g i_{x_0}(z) g^{-1}) g K_P(x_0) A_P(x_0) \in P/K_P(x_0)A_P(x_0) \end{aligned}$$

Now write $g = v i_{x_0}(l)$ for some $v \in \mathcal{U}_P, l \in L_P$. Then

$$F_1(u, zK_PA_P) = uv i_{x_0}(lz) K_P(x_0) A_P(x_0) = F_0(uv, lzK_PA_P)$$

Thus the change of basepoint gives a diffeomorphism

$$F_0^{-1}F_1 : \mathcal{U}_P \times D_P \to \mathcal{U}_P \times D_P$$

by $F_0^{-1}F_1(u, zK_PA_P) = (uv, lzK_PA_P)$ which does not mix the factors. Therefore the differential of this map preserves the horizontal subspaces. □

(7.14) Remark. If both x_0 and x_1 are chosen so that the lifts $L_P(x_0)$ and $L_P(x_1)$ are rationally defined subgroups of P, then the resulting element $v \in \mathcal{U}_P$ is a rational point. Consequently, the rational structures on each fiber $\mathcal{U}_P \times \{zK_PA_P\}$ are preserved under the diffeomorphism $F_0^{-1}F_1$.

8. The reductive Borel-Serre compactification

(8.1) We will be concerned with the reductive Borel-Serre compactification $\bar{X}$ of X, which, so far as we know, was first described in [Z1] §4.2, p. 190. It is a stratified space whose strata are the reductive Borel-Serre strata described above, and it is obtained as a quotient of the Borel-Serre compactification as follows.

(8.2) Definition. The reductive Borel-Serre compactification $\bar{X}$ of X is the topological space which is obtained from the Borel-Serre compactification $\tilde{X}$ by collapsing each nilmanifold fiber of the map $\pi : Y_P \to X_P$ to a point.

The rest of §8 contains a description of the local structure of the singularities of $\bar{X}$. First we recall the notion of a "link".

(8.3) Definition. A stratified submersion $\theta : \mathscr{B} \to Y$ is a stratified space $\mathscr{B}$ and a proper surjective mapping to a smooth manifold Y such that (1) the map θ is a locally trivial fiber bundle and (2) the restriction of θ to each stratum S of $\mathscr{B}$ is a smooth locally trivial submersion $S \to Y$.

For such a map, the mapping cone

$$M(\theta) = (\mathscr{B} \times [0,1) \coprod Y)/\sim$$

(where, $\forall b \in \mathscr{B}$, the point $(b, 0)$ is identified with $\theta(b) \in Y$), has a canonical stratification with an $s+1$ dimensional stratum $S \times (0,1)$ for each s-dimensional stratum $S \subset \mathscr{B}$, and with an additional stratum Y.

Now suppose that X is a stratified space and that $Y \subset X$ is a stratum.

(8.4) Definition. A *link bundle* for Y is a stratified submersion $\theta : \mathscr{B} \to Y$ together with a stratum preserving homeomorphism,

$$\phi : M(\theta) \to T_Y \subset X$$

to some neighborhood T_Y of Y, which is smooth on each stratum of $M(\theta)$, and which takes the stratum $Y \subset M(\theta)$ identically to $Y \subset T_Y$.

If a link bundle for Y exists, the stratification is said to be *locally trivial* along Y. The neighborhood T_Y is called a *tubular neighborhood* of Y, and the fiber $\mathscr{L}_y = \theta^{-1}(y) \subset \mathscr{B}$ is called the *link* of the stratum Y (at the point y). By local triviality, the link completely determines the local topology of X near the point $y \in Y$. In the rest of this section we will find a link bundle for the stratum X_P in the reductive Borel-Serre compactification.

(8.5) If $\mathbf{P} \subset \mathbf{Q}$ are rational parabolic subgroups with unipotent radicals $\mathscr{U}_\mathbf{P} \supset \mathscr{U}_\mathbf{Q}$ and associated nilmanifolds $N_P = \Gamma_{\mathscr{U}_P} \backslash \mathscr{U}_P$ and $N_Q = \Gamma_{\mathscr{U}_Q} \backslash \mathscr{U}_Q$ then $\mathbf{P}$ determines a parabolic subgroup $\mathbf{P^Q} = \mathbf{P}/\mathscr{U}_\mathbf{Q} \subset \mathbf{L_Q} = \mathbf{Q}/\mathscr{U}_\mathbf{Q}$ with unipotent radical $\mathscr{U}_\mathbf{P}^\mathbf{Q} = \mathscr{U}_\mathbf{P}/\mathscr{U}_\mathbf{Q}$ and discrete subgroup $\Gamma_P^Q = \Gamma_{\mathscr{U}_P} \backslash \Gamma_{\mathscr{U}_Q}$. We denote by $N_P^Q = \Gamma_P^Q \backslash \mathscr{U}_P^Q$ the associated nilmanifold. The group $\mathscr{U}_Q$ acts (from the right) on N_P with quotient

$$N_P/\mathscr{U}_Q = \Gamma_{\mathscr{U}_P} \backslash \mathscr{U}_P/\mathscr{U}_Q = \Gamma_P^Q \backslash \mathscr{U}_P^Q = N_P^Q. \qquad (8.5.1)$$

Thus we have a fiber bundle,

$$\tau_{PQ} : N_P \to N_P^Q \qquad (8.5.2)$$

with fiber $\tau_{PQ}^{-1}(1) = N_Q$. If $\mathbf{P} \subset \mathbf{R} \subset \mathbf{Q}$ are rational parabolic subgroups then we obtain a further fiber bundle

$$\tau_{PQ}^R : N_P^Q = N_P/\mathscr{U}_Q \to N_P/\mathscr{U}_R = N_P^R \tag{8.5.3}$$

with fiber N_R^Q.

(8.6) We now refer to the notation of §6.6. Suppose that $\mathbf{P}$ is the intersection of maximal rational parabolic subgroups, $\mathbf{P} = \mathbf{Q_1} \cap \mathbf{Q_2} \cap \ldots \cap \mathbf{Q_s}$ and that $I \subset \{1, 2, \ldots, s\}$ is a proper subset, with $\mathbf{Q} = \cap_{i \in I} \mathbf{Q_i} \supset \mathbf{P}$ the corresponding parabolic subgroup. Define the simplex

$$\Delta_P = \Delta_P^G = \{(\rho_1, \rho_2, \ldots, \rho_s) \in [0,1]^s \mid \textstyle\sum_{i=1}^s \rho_i = 1\}$$
$$\Delta_P^Q = \{(\rho_1, \rho_2, \ldots, \rho_s) \in \Delta_P | \rho_j = 0 \text{ if } j \notin I\} = [0,1]^I \cap \Delta_P.$$

If $\mathbf{P} \subset \mathbf{R} \subset \mathbf{Q}$ are rational parabolic subgroups then we obtain an inclusion as a face,

$$\Delta_P^R \hookrightarrow \Delta_P^Q \subset \Delta_P.$$

(8.7) Define the stratified space,

$$\mathscr{S}_P = N_P \times \Delta_P^G / \sim \tag{8.7.1}$$

with the following identifications: For each $\mathbf{Q} \supset \mathbf{P}$ divide by the action of $\mathscr{U}_Q$ on the subset $N_P \times \Delta_P^Q \subset N_P \times \Delta_P$ (where $\mathscr{U}_Q$ acts on the first factor only).

Consider the (topological) category $\mathscr{C}_1$ with one object N_P^Q for each rational parabolic subgroup $\mathbf{Q} \supsetneqq \mathbf{P}$ (including the case $N_P^G = N_P$) and a unique morphism $\mathrm{Hom}(N_P^Q, N_P^R) = \{\tau_{PQ}^R\}$ (see (8.5.3)) whenever $\mathbf{P} \subsetneqq \mathbf{R} \subsetneqq \mathbf{Q}$. Then $\mathscr{S}_P$ is the classifying space of the category $\mathscr{C}_1$.

(8.8) Definition. The map

$$\delta : \mathscr{S}_P \to \Delta_P$$

to the $(s-1)$-dimensional simplex Δ_P is the map induced from the projection to the second factor in (8.7.1).

Consider the (topological) category $\mathscr{C}_2$ with one object, $\{Q\}$, (a point) for each rational parabolic subgroup $\mathbf{Q} \supsetneqq \mathbf{P}$ (including the case $\mathbf{Q} = \mathbf{G}$) and with a unique morphism $\{Q\} \to \{R\}$ whenever $\mathbf{P} \subsetneqq \mathbf{R} \subsetneqq \mathbf{Q}$. Then the classifying space of the category $\mathscr{C}_2$ may be canonically identified with the simplex Δ_P. The projection δ is the map which is induced on classifying spaces by the functor $F : \mathscr{C}_1 \to \mathscr{C}_2$ which collapses each space $N_P^Q \in \mathscr{C}_1$ to the point $\{Q\} \in \mathscr{C}_2$.

(8.9) Proposition. *The map δ is a well defined proper stratified mapping. For each rational parabolic subgroup $\mathbf{Q} \supset \mathbf{P}$ the pre-image of the interior of the face Δ_P^Q is diffeomorphic to a product,*

$$\delta^{-1}(\Delta_P^Q)^o \cong N_P^Q \times (\Delta_P^Q)^o$$

with δ given by projection to the second factor. □

(8.10) Theorem. *The boundary strata X_P form a locally trivial stratification of the reductive Borel-Serre compactification $\bar{X}$. The closure $\bar{X}_P$ of the stratum X_P in $\bar{X}$ is the reductive Borel-Serre compactification of X_P. The link of each point $x \in X_P$ is the stratified space $\mathscr{L}_P$ (of §8.7) which is stratified by its intersection with the strata of X, so that $\mathscr{L}_P \cap X_Q = \mathscr{L}_P^Q$. Every point $x \in X_P$ has a neighborhood basis consisting of "distinguished" neighborhoods* ([GM2] §1.1), *each of which is homeomorphic (by a stratum preserving homeomorphism which is smooth on each stratum) to the space*

$$\mathrm{cone}(\mathscr{L}_P) \times D^r$$

where D^r denotes the r-dimensional disk ($r = \dim_{\mathbb{R}}(X_P)$).

(8.11) Preliminaries to the proof. The proof involves a study of the "corner" $(\mathbb{R}_{\geq 0})^s$, which is canonically stratified by the coordinate subspaces. Although the distance functions ρ_i give us a canonical "cubical" (closed) neighborhood $[0,1]^s$ of the origin, we will find it convenient to replace this with a "simplicial" neighborhood,

$$N_s = \{(\rho_1, \rho_2, \dots, \rho_s) \in [0,1]^s \mid \textstyle\sum_{i=1}^s \rho_i \leq 1\}$$

because it is easily seen to coincide with the cone over the standard simplex

$$\Delta^s = \partial N_s = \{(\rho_1, \rho_2, \dots, \rho_s) \in [0,1]^s \mid \textstyle\sum_{i=1}^s \rho_i = 1\}$$

and also because the "faces" of ∂N_s are in one to one correspondence with the strata of the corner.

(8.12) Proof of Theorem 8.10. Proposition 6.6 gives a diffeomorphism of manifolds with corners,

$$(r_P, \rho_P) : \tilde{U}_P \to \tilde{Y}_P \times [0,2)^s.$$

Since r_P is equivariant with respect to the action of $\mathscr{U}_P$, and since ρ_P is invariant with respect to the action of $\mathscr{U}_P$ (§6.4), the functions r_P and ρ_P pass to mappings

$$(r_P, \rho_P) : \bar{U}_P \to \bar{X}_P \times [0,2)^s$$

although this map is no longer a diffeomorphism. (Here, $\bar{U}_P$ is a geodesic neighborhood of the closure $\bar{X}_P$ of the stratum X_P in the reductive Borel-Serre compactification $\bar{X}$), and we have a commutative diagram

$$\begin{array}{ccc} \tilde{U}_P & \xrightarrow{(r_P,\rho_P)} & \tilde{Y}_P \times [0,2)^s \\ \pi \downarrow & & \downarrow \pi \times I \\ \bar{U}_P & \xrightarrow[(r_P,\rho_P)]{} & \bar{X}_P \times [0,2)^s \end{array}$$

Define subsets of $\tilde{U}_P$ by $\tilde{T}_P = (r_P, \rho_P)^{-1}(Y_P \times N_s)$, $\partial\tilde{T}_P = (r_P, \rho_P)^{-1}(Y_P \times \partial N_s)$ and set $\bar{T}_P = \pi(\tilde{T}_P)$ and $\partial\bar{T}_P = \pi(\partial\tilde{T}_P)$. The composition

$$\tilde{\Psi} : \partial\tilde{T}_P \xrightarrow{\cong} Y_P \times \partial N_s \to Y_P$$

is a link bundle for $Y_P \subset \tilde{X}$ and in fact the mapping cone of $\tilde{\Psi}$ is canonically identified with $Y_P \times N_s$. It follows that the composition

$$\bar{\Psi} : \partial\bar{T}_P \to X_P \times \partial N_s \to X_P$$

is a link bundle for $X_P \subset \bar{X}$ whose mapping cone may be canonically identified with $\bar{T}_P$. Restricting the commutative diagram

$$\begin{array}{ccccc} \partial\tilde{T}_P & \xrightarrow{\cong} & Y_P \times \partial N_s & & \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} & & \\ \partial\bar{T}_P & \longrightarrow & X_P \times \partial N_s & \longrightarrow & X_P \end{array}$$

to the pre-image of a point $x \in X_P$ and identifying the fiber $\pi^{-1}(x)$ with the nilmanifold N_P gives a diagram

$$\begin{array}{ccc} \pi^{-1}\bar{\Psi}^{-1}(x) & \xrightarrow{\cong} & N_P \times \Delta_P \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \bar{\Psi}^{-1}(x) & \longrightarrow & \Delta_P \end{array}$$

which shows that the link $\mathscr{L} = \bar{\Psi}^{-1}(x)$ is obtained from the product $N_P \times \Delta_P$ by making the identifications (8.4) on the faces of the simplex. □

9. Weight profiles

(9.1) Let $\mathbf{Q}$ be a *maximal* rational parabolic subgroup of $\mathbf{G}$. The unipotent radical $\mathscr{U}_\mathbf{Q}$ determines a linear ordering on the set of algebraic characters $\chi(\mathbf{S_Q})$ of the split torus $\mathbf{S_Q}$ such that the weights occurring in $\mathfrak{N}_Q = \mathrm{Lie}(\mathscr{U}_Q)$ are positive. This gives a canonical isomorphism,

$$\phi_Q : \chi(S_Q) \cong \mathbb{Z}.$$

(9.2) Definition. A *weight cutoff* for Q is a choice of half integer weight, $t_Q = \phi_Q^{-1}(n + \frac{1}{2}) \in \chi(S_Q) \otimes_\mathbb{Z} \mathbb{Q}$.

The resulting subsets,

$$\chi(S_Q)_+ = \{\alpha \in \chi(S_Q) \mid \alpha \geq t_Q\}$$
$$\chi(S_Q)_- = \{\alpha \in \chi(S_Q) \mid \alpha \leq t_Q\}$$

are disjoint and decompose the character module into "high" and "low" weights: If $\alpha \in \chi(S_Q)_+$ and $\beta \in \chi(S_Q)_-$ then $\alpha \geq \beta$. Conversely, a decomposition of the character module into disjoint subsets $\chi(S_Q)_\pm$ of high and low weights determines a unique weight cutoff.

(9.3) Let $\mathbf{Q_1}, \mathbf{Q_2}, \ldots \mathbf{Q_r}$ be the collection of standard maximal rational parabolic subgroups of $\mathbf{G}$. Choose weight cutoffs, $t_1, t_2, \ldots, t_r$ for each of these maximal standard parabolic subgroups. They determine weight cutoffs t_Q for every maximal rational parabolic subgroup $\mathbf{Q}$, by conjugation. Let $\mathbf{P}$ be an arbitrary rational parabolic subgroup of $\mathbf{G}$. These weight cutoffs determine disjoint subsets, $\chi(S_P)_+ \subset \chi(S_P)$ and $\chi(S_P)_- \subset \chi(S_P)$ (which do not necessarily account for all the characters of S_P) by

$$\begin{aligned} \chi(S_P)_+ &= \{\alpha \in \chi(S_P) \mid \forall \text{ maximal } Q \supset P,\ (\alpha|S_Q) \geq t_Q\} \\ \chi(S_P)_- &= \{\alpha \in \chi(S_P) \mid \forall \text{ maximal } Q \supset P,\ (\alpha|S_Q) \leq t_Q\}. \end{aligned} \tag{9.3.1}$$

The collection of these induced subsets $\chi(S_P)_\pm$ will be called the *weight profile* which is determined by the choice of r standard maximal weight cutoffs t_i $(i = 1, 2, \ldots r)$. If $M = \oplus M_\alpha$ is an S_P-module, we denote by

$$M_+ = \bigoplus_{\alpha \in \chi(S_P)_+} M_\alpha$$

the submodule consisting of all weight spaces with weights in the "high" region as determined by the weight profile, and similarly for M_-. When several weight profiles are being considered, we will use $\chi(S_P)^p_\pm$ (or $M^p_\pm$) to refer to the subset corresponding to the profile p.

(9.4) Special weight profiles. Recall from §5 the standard minimal parabolic subgroup $\mathbf{P_0} = \bigcap_{i=1}^r \mathbf{Q_i}$. Choose a Borel subgroup $\mathbf{B}(\mathbb{C}) \subset \mathbf{G}(\mathbb{C})$ such that $\mathbf{B}(\mathbb{C}) \subset \mathbf{P_0}(\mathbb{C})$ and choose a Cartan subgroup $\mathbf{H}(\mathbb{C}) \subset \mathbf{B}(\mathbb{C})$ such that $\mathbf{S_P}(x_0)(\mathbb{C}) \subset \mathbf{H}(\mathbb{C}) \subset \mathbf{L_{P_0}}(x_0)(\mathbb{C})$. Let $\Phi = \Phi(\mathbf{H}, \mathfrak{g}(\mathbb{C}))$ be the resulting system of roots. The Borel subgroup $\mathbf{B}(\mathbb{C})$ determines a system of positive roots Φ^+ and a basis $\Delta \subset \Phi^+$. Define

$$\rho = \rho_G = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \qquad (9.4.1)$$

and for each standard maximal parabolic subgroup $\mathbf{Q}$ set $\rho_Q = \rho | S_Q(x_0)$. Then ρ_Q coincides with the restriction to $S_Q(x_0)$ of the weight

$$\rho(\mathcal{U}_Q) = \frac{1}{2} \sum_{\alpha \in \Phi(\mathbf{H}, \mathfrak{N}_Q)} \alpha$$

although it is not equal to

$$\frac{1}{2} \sum_{\beta \in \Phi(\mathbf{S_Q}, \mathfrak{N}_Q)} \beta$$

since the corresponding root spaces may fail to be one-dimensional.

(9.5) Definition. We define six special weight profiles, $\mu, \nu, 0, d, \infty, -\infty$, (called the upper middle, lower middle, zero, dualizing, infinity and minus infinity weight profiles respectively) by their weight cutoffs on each standard maximal rational parabolic subgroup $\mathbf{Q}$ as follows.

$$\begin{aligned}
\chi(S_Q)^\mu_+ &= \{\alpha \in \chi(S_Q) | \alpha > -\rho_Q\} \\
\chi(S_Q)^\nu_+ &= \{\alpha \in \chi(S_Q) | \alpha \geq -\rho_Q\} \\
\chi(S_Q)^0_+ &= \{\alpha \in \chi(S_Q) | \alpha \geq 0\} \\
\chi(S_Q)^d_+ &= \{\alpha \in \chi(S_Q) | \alpha > -2\rho_Q\} \\
\chi(S_Q)^\infty_+ &= \phi \\
\chi(S_Q)^{-\infty}_+ &= \chi(S_Q)
\end{aligned}$$

These weight cutoffs determine weight profiles as described above.

(9.6) Dual of a weight profile. Suppose p is a weight profile which is determined by weight cutoffs $(t_1, t_2, \ldots, t_r)$ with respect to the standard maximal parabolic subgroups $\mathbf{Q_1}, \mathbf{Q_2}, \ldots, \mathbf{Q_r}$ respectively. Define the dual weight profile to be the weight profile which is determined by the weight cutoffs $(-t_1-2\rho_1, -t_2-2\rho_2, \dots, -t_r-2\rho_r)$ where $\rho_i = \rho_{Q_i} = \rho|S_{Q_i}(x_0)$. Then μ and ν are dual, 0 and d are dual, and ∞ and $-\infty$ are dual.

(9.7) Sum of weight profiles. Suppose p is a weight profile corresponding to weight cutoffs $\{s_1, s_2, \dots, s_r\}$ and suppose q is a weight profile corresponding to weight cutoffs $\{t_1, t_2, \dots, t_r\}$ with respect to the standard maximal rational parabolic subgroups. We will define the weight profile $p+q$ so that, for every maximal rational parabolic subgroup **Q**, if $\alpha \in \chi(S_Q)^p_+$ and if $\beta \in \chi(S_Q)^q_+$ then the tensor product character $\alpha \otimes \beta \in \chi(S_Q)^{p+q}_+$. Since we have made the universal choice that weight cutoffs correspond to half integers, we are forced into the following:

(9.8) Definition. The weight profile $p+q$ is given by weight cutoffs $\{s_1+t_1+\frac{1}{2}, s_2+t_2+\frac{1}{2}, \dots, s_r+t_r+\frac{1}{2}\}$ with respect to the standard maximal rational parabolic subgroups.

10. Weights of the cohomology of $\mathfrak{N}$

(10.1) Fix a rational parabolic subgroup **P** of **G** and let $\mathbf{G} \to \mathbf{Gl}(E)$ be an irreducible algebraic representation of $\mathbf{G}(\mathbb{R})$ on some complex vectorspace E. The unipotent radical $\mathcal{U}_P$ is normal in P so we obtain an action (by conjugation) of P on the exterior algebra $\wedge^\bullet \mathfrak{N}_P$. Since P acts on E we obtain an action of P on the Koszul complex ([Kz], [V] §3.1, [BW], [CE] XIII §3),

$$C^\bullet(\mathfrak{N}_P, E) = \mathrm{Hom}_{\mathbb{R}}(\wedge^\bullet(\mathfrak{N}_P), E) = \mathrm{Hom}_{\mathbb{C}}(\wedge^\bullet(\mathfrak{N}_P(\mathbb{C})), E) \qquad (10.1.1)$$

which computes the cohomology of $\mathfrak{N}_P = \mathfrak{N}_P(\mathbb{R}) = \mathrm{Lie}(\mathcal{U}_P)$ with coefficients in E. The lift $S_P(x_0) \subset P$ of S_P acts algebraically on this complex and this action commutes with the Lie algebra differential, inducing a weight-decomposition

$$C^\bullet(\mathfrak{N}_P, E) = \bigoplus_{\alpha \in \chi(S_P)} C^\bullet(\mathfrak{N}_P, E)_\alpha$$

into eigenspaces for $S_P(x_0)$.

(10.2) Remark. This weight decomposition is not canonical: it changes with the choice of lift $S_P(x_0)$ of S_P. However, there are two canonical structures which may be associated to these decompositions and which are independent of choices: (1) the induced decomposition on the cohomology $H^*(\mathfrak{N}_P, E)$, and (2) the filtration of the differential forms by weights.

(10.3) Fix a weight profile (§9.3), $\chi(S_P)_\pm$ which is determined by a choice of weight cutoffs for each of the maximal standard rational parabolic subgroups of **G**. Fix a rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$.

Definition. The complex

$$C^\bullet(\mathfrak{N}_P, E)_+ := \bigoplus_{\alpha \in \chi(S_P)_+} C^\bullet(\mathfrak{N}_P, E)_\alpha$$

is the subcomplex of $C^\bullet(\mathfrak{N}_P, E)$, consisting of those eigen-complexes with weights $\alpha \in \chi(S_P)_+$.

(10.4) Proposition.

1. The subcomplex $C^\bullet(\mathfrak{N}_P, E)_+$ does not depend on the choice of lift $S_P(x_0)$.
2. The subcomplex $C^\bullet(\mathfrak{N}_P, E)_+$ is preserved by the action of P on $C^\bullet$.

(10.5) Proof. The first statement follows from the second, since the various lifts of S_P are conjugate by elements of P. The subgroup $L_P(x_0)$ actually preserves each of the weight spaces since it commutes with $S_P(x_0)$, so it suffices to consider the action of each element $u \in \mathcal{U}_P$ on each weight space $C^\bullet(\mathfrak{N}_P, E)_\alpha$. Let $S'_P = uS_P(x_0)u^{-1}$ be the new lift of S_P. Then S'_P acts semisimply on $C^\bullet(\mathfrak{N}_P, E)$ and the corresponding weight space $C^\bullet(\mathfrak{N}_P, E)'_\alpha$ is the image of $C^\bullet(\mathfrak{N}_P, E)_\alpha$ under the (coadjoint) action of u. This action is given infinitesimally by the Lie algebra representation of $\mathfrak{N}_P = \oplus_{\beta\in\Delta}\mathfrak{N}_\beta$ on $C^\bullet(\mathfrak{N}_P, E)$. If $n \in \mathfrak{N}_\beta$ then $n.C^\bullet(\mathfrak{N}_P, E)_\alpha \subset C^\bullet(\mathfrak{N}_P, E)_{\beta+\alpha}$, in other words, the action of $\mathfrak{N}_P$ does not lower weights. Therefore the same is true for the action of $\mathcal{U}_P$. But the weight cutoffs are defined with the following property: if $\alpha \in \chi(S_P)_+$ and if $\beta \geq \alpha$ then $\beta \in \chi(S_P)_+$. Thus, $u.C^\bullet(\mathfrak{N}_P, E)_\alpha \subset \oplus_{\beta\in\chi(S_P)_+}C^\bullet(\mathfrak{N}_P, E)_\beta$, which gives

$$\bigoplus_{\alpha\in\chi(S_P)_+} C^\bullet(\mathfrak{N}_P, E)'_\alpha \subset \bigoplus_{\beta\in\chi(S_P)_+} C^\bullet(\mathfrak{N}_P, E)_\beta$$

and the reverse inclusion is the same. □

(10.6) Definition. The *weighted Lie algebra cohomology* $H^k(\mathfrak{N}_P, E)_+$ is the cohomology of the complex $C^\bullet(\mathfrak{N}_P, E)_+$.

(10.7) The inclusion $C^\bullet(\mathfrak{N}_P, E)_+ \hookrightarrow C^\bullet(\mathfrak{N}_P, E)$ induces an injection

$$H^k(\mathfrak{N}_P, E)_+ \hookrightarrow H^k(\mathfrak{N}_P, E)$$

which identifies the weighted Lie algebra cohomology with the subgroup of the full Lie algebra cohomology which is spanned by those classes with weights $\alpha \in \chi(S_P)_+$. In particular, there is no contradiction with §9.3 in our use of the symbol $H^k(\mathfrak{N}_P, E)_+$ to denote the weighted Lie algebra cohomology.

(10.8) The following lemma will be key in the proof that weighted cohomology satisfies Poincaré duality. Fix a weight profile p and its dual weight profile q. Suppose that $\Phi : E_1 \times E_2 \to \mathbb{C}$ is a nondegenerate pairing of algebraic finite dimensional G−modules. Let **P** be a rational parabolic subgroup of **G** and let $m = \dim_{\mathbb{R}}(N_P) = \dim(\mathfrak{N}_P)$ be the dimension of the nilmanifold. Then wedge product of differential forms induces a pairing

$$\tilde{\Phi} : H^i(\mathfrak{N}_P; E_1) \times H^{m-i}(\mathfrak{N}_P; E_2) \to H^m(\mathfrak{N}_P; \mathbb{C}) = \mathbb{C}$$

(10.9) Lemma. *The pairing $\tilde{\Phi}$ is nondegenerate and it restricts to a nondegenerate pairing*

$$H^i(\mathfrak{N}_P; E_1)^p_+ \times H^{m-i}(\mathfrak{N}_P, E_2)^q_- \to \mathbb{C}.$$

(10.10) Proof. The pairing $\tilde{\Phi}$ is nondegenerate because N_P is a compact orientable manifold which satisfies Poincaré duality. This pairing is S_P−equivariant and the top cohomology $H^m(\mathfrak{N}_P;\mathbb{C})$ has S_P−weight equal to $-2\rho_P = -2\rho|S_P$ (see §11.6). Hence, for each weight $\alpha \in \chi(S_P)$, the subspace $H^i(N_P;\mathbf{E}_1)_\alpha$ is dually paired under $\tilde{\Phi}$ with $H^{m-i}(N_P,E_2)_\beta$ where $\beta = -\alpha - 2\rho_P$. On the other hand, by the definition of dual profiles, $\alpha \in \chi(S_P)_+^p$ if and only if $-\alpha - 2\rho_P \in \chi(S_P)_-^q$. □

11. Theorems of Nomizu, van Est and Kostant

(11.1) Let $\mathbf{P} \subset \mathbf{G}$ denote a rational parabolic subgroup with unipotent radical $\mathcal{U}_\mathbf{P}$ and let $\Gamma_P = \Gamma \cap P$ and $\Gamma_\mathcal{U} = \Gamma \cap \mathcal{U}$. Then $N_P = \Gamma_\mathcal{U} \backslash \mathcal{U}_P$ is a nilmanifold with universal covering space $pr : \mathcal{U}_P \to N_P$, and the tangent space to the basepoint $\Gamma_\mathcal{U} I$ may be canonically identified with the Lie algebra $\mathfrak{N}_P = \mathrm{Lie}(\mathcal{U}_P)$. Suppose $\Psi : G \to GL(E)$ is an algebraic irreducible representation on some complex vectorspace E. The restriction of Ψ to $\Gamma_\mathcal{U}$ determines a local system $\mathbf{E} = E \times_{\Gamma_\mathcal{U}} \mathcal{U}_P$ over the nilmanifold N_P. If $\omega \in \Omega^r(N_P, \mathbf{E})$ is an $\mathbf{E}$-valued differential form on N_P, then its pullback $\mathrm{pr}^*(\omega) \in \Omega^r(\mathcal{U}_P, \mathbf{E})$ is a Γ_P-invariant $\mathbf{E}$-valued differential form on $\mathcal{U}_P$.

(11.2) Definition. A smooth differential i-form ω on N_P with values in $\mathbf{E}$ is called *invariant* if $pr^*(\omega)$ is invariant under the (left) action of $\mathcal{U}_P$.

(11.3) We denote the complex of invariant differential forms by $\Omega^\bullet_{inv}(N_P, \mathbf{E})$. The projection induces an isomorphism,

$$\begin{aligned} pr^* : \Omega^\bullet_{inv}(N_P, \mathbf{E}) &\cong C^\bullet(\mathfrak{N}_P, E) \\ &= \mathrm{Hom}_\mathbb{R}(\wedge^\bullet(\mathfrak{N}_P), E) \\ &= \mathrm{Hom}_\mathbb{C}(\wedge^\bullet(\mathfrak{N}_P(\mathbb{C})), E) \end{aligned}$$

between the complex of invariant differential forms on N_P and the Koszul complex which computes the Lie algebra cohomology of $\mathfrak{N}_P$. We shall use the following basic result of Nomizu and van Est:

(11.4) Theorem. ([N], [E]) *The inclusion $\Omega^\bullet_{inv}(N_P, \mathbf{E}) \hookrightarrow \Omega^\bullet(N_P, \mathbf{E})$ of the invariant differential forms into the complex of all smooth differential forms on N_P induces an isomorphism*

$$H^k(\mathfrak{N}_P, E) \cong H^k(N_P, \mathbf{E})$$

between the Lie algebra cohomology of $\mathfrak{N}_P$ and the de Rham cohomology of N_P.

(11.5) The group $\mathcal{U}_P$ acts trivially on the cohomology $H^k(\mathfrak{N}, E)$ leaving an action of the quotient $L_P = P/\mathcal{U}_P$ on the $\mathfrak{N}$-cohomology. As in §9.4 we fix Cartan and Borel subgroups $\mathbf{H}(\mathbb{C}) \subset \mathbf{B}(\mathbb{C})$ with $\mathbf{H}(\mathbb{C}) \subset \mathbf{L}_\mathbf{P}(x_0)(\mathbb{C})$. This gives rise to a system of positive roots Φ^+ with

$$\rho = \rho_G = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha. \tag{11.5.1}$$

Let

$$W = W(\mathfrak{g}(\mathbb{C}), \mathfrak{h}(\mathbb{C}))$$

denote the Weyl group for G, and let $W_P = W(\mathfrak{l}_P(\mathbb{C}), \mathfrak{h}(\mathbb{C}))$ denote the Weyl group for $L_P(x_0)$, with $W_P \subset W$. For each $w \in W$ set

$$\Phi^+(w) = \{\alpha \in \Phi^+ | w^{-1}\alpha \in \Phi^-\}. \tag{11.5.2}$$

Then $|\Phi^+(w)| = \ell(w)$ is the length of w and the set

$$W_P^1 = \{w \in W | \Phi^+(w) \subset \Phi(\mathbf{H}, \mathfrak{N}_P(\mathbb{C}))\} \tag{11.5.3}$$

consists of the unique element of minimal length from each of the cosets $W_P x \in W_P \backslash W$ ([Sp] §10.2, [V] §3.2.1). Suppose the representation $\Psi : G \to Gl(E)$ is algebraic and is irreducible with highest weight λ. For any dominant weight μ, let V_μ denote the irreducible $\mathbf{L_P}(\mathbb{C})$-module with highest weight μ.

(11.6) Theorem. ([K], Theorem 5.14; [V], Corollary 3.2.16) *The $L_P(\mathbb{C})$ representation $H^*(\mathfrak{N}_P, E)$ is algebraic and is given by*

$$H^*(\mathfrak{N}_P, E) = \bigoplus_{w \in W_P^1} V_{w(\lambda+\rho)-\rho}[-\ell(w)]$$

where $\ell(w) = |\Delta^+(w)|$ is the length of w, and $[-\ell(w)]$ indicates that the module appears in degree $\ell(w)$. If E is defined over $\mathbb{Q}$ then so is $H^*(\mathfrak{N}_P, E)$. □

(11.7) Suppose $\mathbf{P} \subset \mathbf{Q}$ are rational parabolic subgroups of $\mathbf{G}$. The image $\bar{\mathbf{P}}$ of $\mathbf{P}$ in $\mathbf{L_Q} = \mathbf{Q}/\mathcal{U}_\mathbf{Q}$ is a parabolic subgroup of $\mathbf{L_Q}$ with unipotent radical $\mathcal{U}_{\bar{\mathbf{P}}} = \mathcal{U}_\mathbf{P}/\mathcal{U}_\mathbf{Q}$. Let $\Gamma_{\mathcal{U}_{\bar{P}}} = (\Gamma \cap \mathcal{U}_P)/(\Gamma \cap \mathcal{U}_Q)$. Then corresponding to the exact sequence of unipotent groups,

$$1 \to \mathcal{U}_\mathbf{Q} \to \mathcal{U}_\mathbf{P} \to \mathcal{U}_{\bar{\mathbf{P}}} \to 1$$

we have a fibration of nilmanifolds $N_P \to N_{\bar{P}}$ with fiber N_Q, where $N_{\bar{P}} = \Gamma_{\mathcal{U}_{\bar{P}}} \backslash \mathcal{U}_{\bar{P}}$. The group $\mathcal{U}_P$ acts by conjugation on $\mathcal{U}_Q$, hence on $\mathfrak{N}_Q$, hence on the Lie algebra cohomology $H^*(\mathfrak{N}_Q, E)$. The subgroup $\mathcal{U}_Q$ acts trivially on this Lie algebra cohomology, so we obtain an action of the quotient group $\mathcal{U}_{\bar{P}}$ on $H^*(\mathfrak{N}_Q, E)$.

(11.8) Proposition. *The complex of $\mathcal{U}_P^Q$-invariant differential forms induces a canonical identification of the E_2 term of the Leray spectral sequence for the fibration $N_P \to N_{\bar{P}}$ with the spectral sequence for Lie algebra cohomology,*

$$E_2^{ij} = H^i(N_{\bar{P}}; \mathbf{H}^j(N_Q; \mathbf{E})) \cong H^i(\mathfrak{N}_{\bar{P}}; H^j(\mathfrak{N}_Q; E)). \tag{11.8.1}$$

(Here, $\mathbf{H}^j(N_Q; \mathbf{E})$ denotes the local system $R^j\pi_*(\mathbf{E})$.) This spectral sequence degenerates at E_2 and determines an isomorphism

$$H^a(N_P; \mathbf{E}) \cong H^a(\mathfrak{N}_P, E) \cong \bigoplus_{i+j=a} E_2^{ij} \tag{11.8.2}$$

If $\alpha \in \chi(S_Q)$ is a weight of the smaller torus $S_Q \subset S_P$ then the isotypical component $H^*(\mathfrak{N}_P, E)_\alpha$ of weight α is given by

$$H^a(\mathfrak{N}_P, E)_\alpha = \bigoplus_{i+j=a} H^i(\mathfrak{N}_{\bar{P}}; H^j(\mathfrak{N}_Q, E)_\alpha) \tag{11.8.3}$$

(11.9) Proof. This follows from Kostant's theorem. Alternatively, it may be proven by the same method as in Prop. 12.14. □

12. Differential forms on the nilmanifold fibration

(12.1) The result of §9 and §10 is a decomposition of the cohomology and of the cochain complex of the nilmanifold N_P into weight subcomplexes. We want to do the same for the cochain complex of the nilmanifold fibration $\pi : Y_P \to X_P$ by filtering the cochain complex of the fibers of π.

(12.2) We refer to the notation of §7, in other words, **P** is a rational parabolic subgroup of **G**, $\boldsymbol{\mathcal{U}}_{\mathbf{P}}$ is the unipotent radical, $\nu : \mathbf{P} \to \mathbf{L_P} = \mathbf{P}/\boldsymbol{\mathcal{U}}_{\mathbf{P}}$ is the projection to the Levi quotient, $\Gamma_P = \Gamma \cap P$, $\Gamma_L = \nu(\Gamma_P)$, $\pi : Y_P \to X_P$ is the nilmanifold fibration with fibers diffeomorphic to $N_P = \Gamma_{\mathcal{U}} \backslash \mathcal{U}_P$, and $\tilde{\tau} : e_P \to Y_P$ is the quotient of the Borel-Serre boundary component by the action of the discrete group Γ_P. Let $G \to Gl(E)$ be an algebraic, finite dimensional, representation of G on some complex vectorspace E. As in §11.1 we obtain a local system

$$\mathbf{E} = E \times_{\Gamma_P} e_P$$

over Y_P whose restriction to N_P is also denoted by **E**. For any differential form $\omega \in \Omega^r(Y_P, \mathbf{E})$, the pullback $\tilde{\tau}^*(\omega) \in \Omega^r(e_p, E)$ is a Γ_P-invariant form.

(12.3) Definition. A differential form $\omega \in \Omega^r(Y_P, \mathbf{E})$ is called ($\mathcal{U}_P$-) *invariant* if its pullback $\tilde{\tau}^*(\omega) \in \Omega^r(e_P, E)$ is invariant under the action (from the left) of $\mathcal{U}_P$.

It is possible to check locally whether a differential form ω is $\mathcal{U}_P$-invariant: its Lie derivative with respect to all the "left invariant" vectorfields defined by the Lie algebra $\mathfrak{N}_P$ must vanish. Thus we may make the following definition:

(12.4) Definition. The complex $\boldsymbol{\Omega}^\bullet_{\text{inv}}(Y_P, \mathbf{E})$ is the complex of sheaves on Y_P whose sections over an open set $U \subset Y_P$,

$$\Gamma(U, \boldsymbol{\Omega}^\bullet_{\text{inv}}(Y_P, \mathbf{E})) = \Omega^\bullet_{\text{inv}}(U, \mathbf{E})$$

consist of all $\mathcal{U}_P$-invariant differential forms (on U) with values in **E**.

(12.5) Let us consider the complex of sheaves $\mathbf{T}^\bullet_P = \pi_* \boldsymbol{\Omega}^\bullet_{\text{inv}}(Y_P, \mathbf{E})$ on X_P whose sections over an open set $U \subset X_P$ are the ($\mathcal{U}_P$-) invariant **E**−valued differential forms on $\pi^{-1}(U)$. For each point $x \in X_P$ let us also consider the vectorspace of invariant differential forms $\Omega^q_{\text{inv}}(N_x, \mathbf{E})$ along the fiber, $N_x = \pi^{-1}(x)$. These vectorspaces form a finite dimensional vectorbundle over X_P, which we denote by $\mathbf{C}^q(N_P, \mathbf{E})$. The exterior differential $d : \mathbf{C}^q(N_P, \mathbf{E}) \to \mathbf{C}^{q+1}(N_P, \mathbf{E})$ is a vectorbundle mapping.

(12.6) Proposition. *The vectorbundle $\mathbf{C}^q(N_P, \mathbf{E})$ over X_P admits a canonical flat connection. The complex of sheaves $\mathbf{T}^\bullet_P$ is canonically isomorphic to the direct sum,*

$$\mathbf{T}^j_P \cong \bigoplus_{p+q=j} \boldsymbol{\Omega}^p(X_P; \mathbf{C}^q(N_P, \mathbf{E})).$$

In other words, it is the single complex associated to the double complex of sheaves of smooth differential forms on X_P with coefficients in the flat vectorbundle of invariant differential forms tangent to the fiber.

(12.7) Proof. The integrable connection for the fiber bundle $\pi : Y_P \to X_P$ was constructed in §7.10. Thus, for sufficiently small $U \subset X_P$, there is a canonical product structure $\pi^{-1}(U) \cong U \times N_x$ (where $N_x = \pi^{-1}(x)$ is the fiber over some point $x \in U$). The invariant differential forms are therefore preserved by parallel transport. Hence there is a canonical decomposition of the space of sections

$$\Gamma(\pi^{-1}(u), \boldsymbol{\Omega}^{\bullet}_{\mathrm{inv}}) = \Omega^{\bullet}_{\mathrm{inv}}(\pi^{-1}(U)) \cong \Omega^{\bullet}(U) \otimes \Omega^{\bullet}_{\mathrm{inv}}(N_x, \mathbf{E})$$

for which the differential also splits into horizontal and vertical components. But this is precisely the sheaf (or rather, the single complex associated to the double complex of sheaves) $\boldsymbol{\Omega}^{\bullet}(X_P, \mathbf{C}^{\bullet}(N_P, \mathbf{E}))$. □

(12.8) The group $S_P(x_0)$ acts algebraically and semisimply on $\mathbf{C}^q(N_P, \mathbf{E})$ and commutes with the differential $d : \mathbf{C}^q(N_P, \mathbf{E}) \to \mathbf{C}^{q+1}(N_P, \mathbf{E})$. In fact, left multiplication of $S_P(x_0)$ on $e_P = P/K_P(x_0)A_P(x_0)$ preserves the fibers of the map to D_P, takes $\mathcal{U}_P$-invariant forms to $\mathcal{U}_P$-invariant forms, and preserves the flat connection, as may be seen from §7.8. We obtain a weight decomposition of flat vectorbundles,

$$\mathbf{C}^{\bullet}(N_P, \mathbf{E}) \cong \bigoplus_{\alpha \in \chi(S_P)} \mathbf{C}^{\bullet}(N_P, \mathbf{E})_{\alpha} \tag{12.8.1}$$

and a corresponding decomposition of the sheaf of differential forms,

$$\mathbf{T}^{\bullet}_P \cong \boldsymbol{\Omega}^{\bullet}(X_P; \mathbf{C}^{\bullet}(N_P, \mathbf{E})) \cong \bigoplus_{\alpha \in \chi(S_P)} \boldsymbol{\Omega}^{\bullet}(X_P; \mathbf{C}^{\bullet}(N_P, \mathbf{E})_{\alpha})) \tag{12.8.2}$$

Fix a weight profile $\{\chi(S_P)_{\pm}\}$.

(12.9) Definition. The *weight subbundle*, $\mathbf{C}^{\bullet}(N_P, \mathbf{E})_+ \subset \mathbf{C}^{\bullet}(N_P, \mathbf{E})$ is the flat vectorbundle on X_P,

$$\mathbf{C}^{\bullet}(N_P, \mathbf{E})_+ = \bigoplus_{\alpha \in \chi(S_P)_+} \mathbf{C}^{\bullet}(N_P, \mathbf{E})_{\alpha}.$$

The *weight subcomplex* $\mathbf{T}^{\bullet}_{P+} \subset \mathbf{T}^{\bullet}_P$ is the complex of sheaves on X_P which is given by

$$\mathbf{T}^{j}_{P+} = \bigoplus_{p+q=j} \boldsymbol{\Omega}^{p}(X_P; \mathbf{C}^{q}(N_P, \mathbf{E})_+)$$

(12.10) Proposition. *The weight subbundle $\mathbf{C}^{\bullet}(N_P, \mathbf{E})_+ \subset \mathbf{C}^{\bullet}(N_P, \mathbf{E})$ and the weight subcomplex $\mathbf{T}^{\bullet}_{P+} \subset \mathbf{T}^{\bullet}_P$ do not depend on the choice of basepoint $x_0 \in D$.*

(12.11) Proof. The proof follows from Prop. 10.4. □

(12.12) In the next two sections we use the van Est theorem to identify the flat vectorbundle $\mathbf{C}^{\bullet}(N_P, \mathbf{E})$ with a local system which is obtained from the Koszul complex.

The choice of basepoint $x_0 \in D$ determines a lift $i = i_{x_0} : L_P \to P$ of the Levi quotient and hence an action of L_P on the unipotent radical $\mathcal{U}_P$ by $\bar{g}.u = i(\bar{g})ui(\bar{g})^{-1}$ for $\bar{g} \in L_P$. This action restricts to a representation of $\Gamma_L = \nu(\Gamma_P)$ on the Lie algebra $\mathfrak{N}_P$ and hence also on the Koszul complex $C^\bullet(\mathfrak{N}_P, E)$. Define the flat vectorbundle

$$C^\bullet(\mathfrak{N}_P, E) \times_{\Gamma_L} D_P$$

to be the local system on $X_P = \Gamma_L \backslash D_P$ which corresponds to the above representation of Γ_L on the Koszul complex.

(12.13) Proposition. *The choice of basepoint x_0 gives rise to an $S_P(x_0)$-equivariant isomorphism*

$$h : \mathbf{C}^\bullet(N_P, \mathbf{E}) \to C^\bullet(\mathfrak{N}_P, E) \times_{\Gamma_L} D_P \tag{12.13.1}$$

of flat vectorbundles over X_P.

Proof. Let $\tilde{\tau} : e_P \to Y_P$ and $\tau : D_P \to X_P$ denote the projections. We first define a map,

$$\tilde{h} : \tau^*\mathbf{C}^\bullet(N_P, \mathbf{E}) \to C^\bullet(\mathfrak{N}_P, E) \times D_P \tag{12.13.2}$$

of (trivial) flat vectorbundles on D_P, and then we will check that $\tilde{h}$ is equivariant with respect to the Γ_P actions on both sides. Let $F : \mathcal{U}_P \times D_P \to e_P$ be the diffeomorphism of §7.6 which is determined by the choice of basepoint x_0. A section $\tilde{\omega}$ of the flat vectorbundle $\tau^*\mathbf{C}^\bullet(N_P, \mathbf{E})$ is a differential form on e_P along the fibers of $\tilde{\pi} : e_P \to D_P$ which is left invariant under the group $\Gamma_P \mathcal{U}_P$. Evaluating $F^*(\tilde{\omega})$ at the identity $I \in \mathcal{U}_P$ gives a section of the trivial bundle $C^\bullet(\mathfrak{N}_P, E) \times D_P$. This defines the mapping $\tilde{h}$, in other words,

$$\tilde{h}(\tilde{\omega}) = F^*(\tilde{\omega})|_{I \times D_P}. \tag{12.13.3}$$

It is clearly an isomorphism of flat vectorbundles. It remains to show that, for all $\gamma \in \Gamma_P$ we have

$$\tilde{h}\gamma^*(\tilde{\omega}) = \bar{\gamma}^*(\tilde{h}\tilde{\omega}) \tag{12.13.4}$$

where $\bar{\gamma} = \nu(\gamma) \in \Gamma_L$ acts on $C^\bullet(\mathfrak{N}_P, E)$ by the lifted action of §12.12. By (7.8.1) an element $\gamma \in \Gamma_P$ acts on $\mathcal{U}_P \times D_P$ by

$$\gamma.(u, x) = (\gamma u i \nu(\gamma^{-1}), \nu(\gamma)x) \tag{12.13.5}$$

There is a unique $\gamma_1 \in \mathcal{U}_P$ such that $\gamma = \gamma_1 i(\bar{\gamma})$, hence

$$\gamma.(u, x) = (\gamma_1 i(\bar{\gamma})ui(\bar{\gamma})^{-1}, \bar{\gamma}x) \tag{12.13.6}$$

Left multiplication by γ_1 acts as the identity on $\mathcal{U}_P$-invariant differential forms, so the action of $\gamma \in \Gamma_P$ on the differential form $\tilde{\omega}$ is the same as that induced by the following action:

$$\gamma \bullet (u, x) = (i(\bar{\gamma})ui(\bar{\gamma})^{-1}, \bar{\gamma}x) \tag{12.13.7}$$

which is precisely the lifted action of Γ_L on $C^\bullet(\mathfrak{N}_P, E) \times D_P$. We conclude that $\tilde{h}$ induces an isomorphism between the flat vectorbundles $\mathbf{C}^\bullet(N_P, \mathbf{E})$ and $C^\bullet(\mathfrak{N}_P, E) \times_{\Gamma_L} D_P$ as desired. The $S_P(x_0)$ equivariance is a similar calculation. □

(12.14) Corollary. *For any $\alpha \in \chi(S_P)$, the map h restricts to an isomorphism*

$$h_\alpha : \mathbf{C}^\bullet(N_P, \mathbf{E})_\alpha \to C^\bullet(\mathfrak{N}_P, E)_\alpha \times_{\Gamma_L} D_P$$

of flat vectorbundles on X_P.

(12.15) Proposition. [H2] *The complex of local systems $\mathbf{C}^\bullet(N_P, \mathbf{E})$ on X_P is quasi-isomorphic to the sum of its cohomology sheaves,*

$$\mathbf{C}^\bullet(N_P, \mathbf{E}) \sim R\pi_*(\mathbf{E}) \sim \bigoplus_q \mathbf{H}^q(N_P, \mathbf{E})[-q]. \tag{12.15.1}$$

The same is true for the complex of local systems $\mathbf{C}^\bullet(N_P, \mathbf{E}_\alpha)$, for any $\alpha \in \chi(S_P)$. In particular, the spectral sequence for the cohomology of the fiber bundle $\pi : Y_P \to X_P$ collapses at E_2 and determines an isomorphism,

$$H^i(Y_P, \mathbf{E}) \cong \bigoplus_{p+q=i} H^p(X_Q; \mathbf{H}^q(N_P, \mathbf{E})) \tag{12.15.2}$$

(12.16) Proof. By Prop. 12.13, the complex of local systems $\mathbf{C}^\bullet(N_P, \mathbf{E})$ is given by a complex of algebraic representations,

$$C^0(\mathfrak{N}_P, E) \to C^1(\mathfrak{N}_P, E) \to \dots$$

of the reductive algebraic group L_P. The category of algebraic representations of L_P is a semi-simple abelian category, so every complex in this category is quasi-isomorphic to its homology. This proves the first two statements. The spectral sequence for the cohomology of the fiber bundle $\pi : Y_P \to X_P$ is the spectral sequence for the complex of sheaves $R\pi_*(\mathbf{E})$. The theorem of van Est and Nomizu implies that the complex of local systems $\mathbf{C}^\bullet(N_P, \mathbf{E})$ on X_P is quasi-isomorphic to the complex of sheaves $R\pi_*(\mathbf{E})$, which, together with the first statement proves that the spectral sequence degenerates. □

13. Differential forms on the reductive Borel-Serre compactification

(13.1) Let $\Psi : G \to \mathrm{Gl}(E)$ be a finite dimensional algebraic representation of G on some complex vectorspace E. Then Ψ determines a local coefficient system $\mathbf{E}$ on $X = \Gamma \backslash G/K$ by restricting this representation to Γ,

$$\mathbf{E} = D \times_\Gamma E$$

For each rational parabolic subgroup P, we similarly obtain local coefficient systems (also denoted $\mathbf{E}$) on the nilmanifold $N_P = \Gamma_{\mathcal{U}} \backslash \mathcal{U}_P$ and on the Borel-Serre boundary stratum $Y_P = \Gamma_P \backslash e_P$. Let us denote the inclusion of the locally symmetric space into the reductive Borel-Serre compactification by $i : X \to \bar{X}$.

(13.2) Definition. A differential i-form ω on $X = \Gamma \backslash D$ with values in $\mathbf{E}$ is called *special* if for each stratum Y_P of the Borel-Serre compactification, there exists a neighborhood of Y_P in $\tilde{X}$ (which depends on the differential form ω), such that in this neighborhood, the following two conditions hold:

1. the differential form ω is the pull-up of a differential form $\omega_P \in \Omega^\bullet(Y_P, \mathbf{E})$ from the boundary stratum, via the geodesic retraction (§6.3), and
2. the form ω_P is $\mathcal{U}_P$-invariant, i.e. $\omega_P \in \Omega^i_{\mathrm{inv}}(Y_P, \mathbf{E})$.

We denote by $\boldsymbol{\Omega}^{\bullet}_{\text{sp}}$ the complex of pre-sheaves of special differential forms on X, whose sections over an open set $U \subset X$ are

$$\Gamma(U;\boldsymbol{\Omega}^{\bullet}_{\text{sp}}) = \{\omega \in \Omega^{\bullet}(U,\mathbf{E}) \,|\, \begin{matrix}\omega \text{ is the restriction to } U \\ \text{of a special differential form}\end{matrix}\}$$

Let Sh denote the sheafification functor which assigns to any morphism of pre-sheaves $f : \mathbf{A} \longrightarrow \mathbf{B}$ the corresponding morphism of sheaves, $Sh(f) : Sh(\mathbf{A}) \longrightarrow Sh(\mathbf{B})$. There is a canonical morphism $g : \mathbf{A} \longrightarrow Sh(\mathbf{A})$. If the presheaf B were already a sheaf, then the map f factors through g.

(13.3) Definition. We define the following complex of sheaves,

$$\bar{\boldsymbol{\Omega}}^{\bullet}_{\text{sp}} = \bar{\boldsymbol{\Omega}}^{\bullet}_{\text{sp}}(\bar{X},\mathbf{E}) = Sh(i_*\boldsymbol{\Omega}^{\bullet}_{\text{sp}})$$

(13.4) Proposition. *The sheaf* $\bar{\boldsymbol{\Omega}}^{\bullet}_{\text{sp}}$ *satisfies the following properties:*

1. The sheaf $\bar{\boldsymbol{\Omega}}^{\bullet}_{\text{sp}}$ *is fine.*
2. For any boundary stratum $X_P \subset \bar{X}$, *the restriction* $\bar{\boldsymbol{\Omega}}^{\bullet}_{\text{sp}}|X_P$ *is canonically isomorphic to the complex of sheaves,* $\mathbf{T}^{\bullet}_P$ *of* $\mathcal{U}_P$*-invariant* $\mathbf{E}$*-valued differential forms (§12.5).*

(13.5) Proof. The sheaves $\bar{\boldsymbol{\Omega}}^{i}_{\text{sp}}$ are modules over the sheaf of "special" functions, $\bar{\boldsymbol{\Omega}}^{0}_{\text{sp}}$, which is easily seen to be a fine sheaf since each stratum admits partitions of unity and the neighborhood in which the function is assumed to be geodesic-invariant is arbitrarily small. Thus, $\bar{\boldsymbol{\Omega}}^{\bullet}_{\text{sp}}$ is also fine. The second statement is clear: we associate to any germ of a differential form ω the corresponding invariant form, ω_P. □

(13.6) Lemma. *The quasi-isomorphism* $\boldsymbol{\Omega}^{\bullet}(X,\mathbf{E}) \xrightarrow{\sim} \mathbf{E}$ *of complexes of sheaves on* X *induces a quasi-isomorphism,* $\bar{\boldsymbol{\Omega}}^{\bullet}_{\text{sp}}(\bar{X}_P,\mathbf{E}) \xrightarrow{\sim} Ri_*(\mathbf{E})$.

The rest of §13 consists of a proof of this lemma.

(13.7) Definition. Define the complex of sheaves of special differential forms on the Borel-Serre compactification,

$$\tilde{\boldsymbol{\Omega}}^{\bullet}_{\text{sp}}(\tilde{X},\mathbf{E}) = Sh(j_*\boldsymbol{\Omega}^{\bullet}_{\text{sp}})$$

(13.8) Lemma. *The map* $\pi : \tilde{X} \longrightarrow \bar{X}$ *takes the complex of special differential forms to the complex of special differential forms,*

$$\pi_*\tilde{\boldsymbol{\Omega}}^{\bullet}_{\text{sp}}(\tilde{X},\mathbf{E}) = \bar{\boldsymbol{\Omega}}^{\bullet}_{\text{sp}}(\bar{X},\mathbf{E})$$

(13.9) Proof. The proof is straightforward. □

(13.10) Proof of Lemma 13.6. The inclusion $i : X \longrightarrow \bar{X}$ factors through the Borel-Serre compactification,

$$X \underset{j}{\longrightarrow} \tilde{X} \underset{\pi}{\longrightarrow} \bar{X}.$$

Since the sheafification $Sh(\boldsymbol{\Omega}^\bullet_{\mathrm{sp}})$ is precisely the complex of sheaves $\boldsymbol{\Omega}^\bullet(X, \mathbf{E})$ of all smooth differential forms on X, we obtain natural sheaf maps,

$$\bar{\boldsymbol{\Omega}}^\bullet_{\mathrm{sp}} = \pi_*(\tilde{\boldsymbol{\Omega}}^\bullet_{\mathrm{sp}}) \to \pi_* j_* \boldsymbol{\Omega}^\bullet(X, \mathbf{E}) \to R\pi_* Rj_*(\mathbf{E})$$

We need to check that the above morphisms induce isomorphisms on the stalk cohomology at each point $x \in X_P$. By §13.4 (part 2), the Poincaré lemma, and the van Est theorem, the stalk cohomology of this complex is $H^*(\pi^{-1}(x), \mathbf{E})$ which maps isomorphically to the stalk cohomology of $R\pi_* Rj_*(\mathbf{E})$. □

(13.11) Remark. The complex of sheaves $\tilde{\boldsymbol{\Omega}}^\bullet_{sp}$ (on the Borel-Serre compactification) is not fine, nor is it quasi-isomorphic to the complex $Rj_*(\mathbf{E})$, because the sheaf of $\mathscr{U}_P$-invariant differential forms on N_P is not fine and is not quasi-isomorphic to the constant sheaf.

14. Weighted cohomology over $\mathbb{C}$

(14.1) As above, fix an algebraic finite dimensional representation of G on some complex vectorspace E, and let $\mathbf{E}$ denote the corresponding local coefficient systems on X, on Y_P, and on N_P. Fix a weight profile (§9.3), $\{\chi(S_P)_\pm\}$. By Prop. 13.4, part (2), for each boundary stratum X_P the weight sub-sheaves (§12.9) may be identified as sub-sheaves of the sheaf of special differential forms,

$$\mathbf{T}^\bullet_{P+} \cong \boldsymbol{\Omega}^\bullet(X_P, \mathbf{C}^\bullet(N_P, \mathbf{E})_+) \hookrightarrow \bar{\boldsymbol{\Omega}}^\bullet_{\mathrm{sp}}(\bar{X}; \mathbf{E}))|X_P. \qquad (14.1.1)$$

It is easy to see that the collection of these subsheaves $\{\mathbf{T}^\bullet_{P+}\}$ satisfy the compatibility condition of §2.2. By Lemma 2.5, there is a unique sub-sheaf, $\mathbf{WC}^\bullet(\bar{X}, \mathbf{E}) \subset \bar{\boldsymbol{\Omega}}^\bullet_{\mathrm{sp}}(\bar{X}; \mathbf{E}))$ with the property that, for each stratum X_P we have,

$$\mathbf{WC}^\bullet(\bar{X}, \mathbf{E}) \mid X_P = \mathbf{T}^\bullet_{P+} \qquad (14.1.2)$$

(14.2) Definition. The *weighted cohomology sheaf* $\mathbf{WC}^\bullet(\bar{X}, \mathbf{E})$ is the sheaf obtained by truncating the sheaf $\bar{\boldsymbol{\Omega}}^\bullet_{\mathrm{sp}}(\bar{X}; \mathbf{E}))$ of special differential forms to the subsheaves $\mathbf{T}^\bullet_{P+}$ along the strata X_P.

(14.3) Remark. If the representation E is defined over $\mathbb{R}$ then the weighted cohomology may also be defined over $\mathbb{R}$ by taking differential forms with coefficients in $E(\mathbb{R})$. The weighted cohomology sheaf is fine. Since the construction of weighted cohomology involves unravelling a string of definitions, we will now describe the vectorspace of sections $\Gamma(U; \mathbf{WC}^\bullet(\mathbf{E}))$ of the weighted cohomology complex over an open set $U \subset \bar{X}$. This space of sections consists of *certain* differential forms $\omega \in \Omega^\bullet(U \cap X; \mathbf{E})$ on the big open stratum $X \subset \bar{X}$. Note that the open set $U \cap X \subset \bar{X}$ may be canonically identified with the open subset $\pi^{-1}(U) \cap X \subset \tilde{X}$ of the Borel-Serre compactification $\tilde{X}$, and so ω may be viewed as a differential form on the interior stratum of the Borel-Serre compactification. Now suppose that X_P is a stratum of $\bar{X}$ and suppose that $U \cap X_P \neq \phi$. Then the differential form ω must satisfy the following restrictions:

1. Near $\pi^{-1}(U \cap X_P) \subset Y_P$, the form ω is the pullback of a form $\omega_P \in \Omega^\bullet(\pi^{-1}(U \cap X_P); \mathbf{E})$, via the geodesic retraction.
2. The form ω_P is $\mathscr{U}_P$-invariant, i.e.

$$\omega_P \in \Omega^{\bullet}_{\mathrm{inv}}(\pi^{-1}(U \cap X_P);\mathbf{E}) \cong \bigoplus_{p+q=\bullet} \Omega^p(U;\mathbf{C}^q(N_P,\mathbf{E}))$$

3. The form ω_P lies in the subcomplex

$$\Omega^{\bullet}_{\mathrm{inv}}(\pi^{-1}(U \cap X_P);\mathbf{E})_+ = \bigoplus_{p+q=\bullet} \bigoplus_{\alpha\in\chi(S_P)_+} \Omega^p(U;\mathbf{C}^q(N_P,\mathbf{E})_\alpha).$$

If the open set $U \subset \bar{X}$ does not intersect the boundary stratum X_P, then no restrictions (relative to this stratum) are placed on the section ω.

(14.4) The six special weight profiles $\mu, \nu, 0, t, \infty, -\infty$ (§9.5) give rise to particular weighted cohomology complexes which we denote by $\mathbf{W}^\mu\mathbf{C}^\bullet(\mathbf{E})$, $\mathbf{W}^\nu\mathbf{C}^\bullet(\mathbf{E})$, $\mathbf{W}^0\mathbf{C}^\bullet(\mathbf{E})$, $\mathbf{W}^d\mathbf{C}^\bullet(\mathbf{E})$, $\mathbf{W}^\infty\mathbf{C}^\bullet(\mathbf{E})$, $\mathbf{W}^{-\infty}\mathbf{C}^\bullet(\mathbf{E})$, respectively. Since

$$\mathbf{W}^{-\infty}\mathbf{C}^\bullet(\mathbf{E}) = \bar{\Omega}^{\bullet}_{\mathrm{sp}}$$

(i.e. there is no weight cutoff) we have a canonical quasi-isomorphism,

$$\mathbf{W}^{-\infty}\mathbf{C}^\bullet(\mathbf{E}) \xrightarrow{\sim} Ri_*(\mathbf{E}),$$

with the direct image sheaf (§13.6). For the weight profile $p = \infty$ we have $C^*(\mathfrak{N}_P,\mathbf{E})_+ = 0$. Therefore sections of $\mathbf{W}^\infty\mathbf{C}^\bullet(\mathbf{E})$ vanish on $\bar{X} - X$ so we have a canonical quasi-isomorphism $\mathbf{W}^\infty\mathbf{C}^\bullet(\mathbf{E}) \xrightarrow{\sim} Ri_!(\mathbf{E})$ with the extension by zero sheaf. We will show in §19 that

$$\mathbf{W}^d\mathbf{C}^\bullet(\mathbb{C}) \cong \mathbb{D}_{\bar{X}}[-n]$$

is the dualizing sheaf, shifted by $n = \dim \bar{X}$, that $\mathbf{W}^0\mathbf{C}^\bullet(\mathbb{C}) \cong \mathbb{C}$ is the constant sheaf, and in the Hermitian case that $\mathbf{W}^\mu\mathbf{C}^\bullet(\mathbb{C})$ and $\mathbf{W}^\nu\mathbf{C}^\bullet(\mathbb{C})$ both become isomorphic to the (middle) intersection homology complex of sheaves, when pushed down to the Baily Borel compactification.

(14.5) There are canonical maps between various weighted cohomology complexes with a fixed local coefficient system $\mathbf{E}$: If p is a weight profile corresponding to cutoff values $(t_1, t_2, \ldots, t_r)$ and if q is a weight profile corresponding to cutoff values $(s_1, s_2, \ldots, s_r)$ and if $t_i \le s_i$ for each $i = 1, 2, \ldots, r$ then we will say $p \le q$ and in this case $\mathbf{W}^q\mathbf{C}^\bullet(\mathbf{E})$ is a subcomplex of $\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E})$. In particular we obtain canonical morphisms,

$$\mathbf{W}^\infty\mathbf{C}^\bullet \to \mathbf{W}^0\mathbf{C}^\bullet \to \mathbf{W}^\mu\mathbf{C}^\bullet \to \mathbf{W}^\nu\mathbf{C}^\bullet \to \mathbf{W}^d\mathbf{C}^\bullet \to \mathbf{W}^{-\infty}\mathbf{C}^\bullet \qquad (14.5.1)$$

15. Looijenga's local Hecke correspondences

(15.1) In this section we fix a rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$. One of the key ingredients in weighted cohomology is the action of S_P on the invariant differential forms tangent to the fiber of $\pi : Y_P \to X_P$. Unfortunately this action does not arise from a geometric action of S_P on Y_P. Even the action of S_P on $\mathfrak{N}_P = \mathrm{Lie}(\mathcal{U}_P)$ does not arise from a geometric action of S_P on the nilmanifold fibers $N_P = \Gamma_{\mathcal{U}}\backslash\mathcal{U}_P$. However these actions do come from a geometric "action" of $\mathbf{S_P}(\mathbb{Q})$ on a neighborhood

of X_P by *correspondences* which play a key role in Looijenga's proof of the Zucker conjecture ([L] §3.7), and which we now describe. In fact the rational points of the whole split radical $\mathbf{R_d P} \subset \mathbf{P}$ acts by correspondences on a neighborhood of X_P.

The results in this section are not needed in order to define weighted cohomology nor to determine its basic properties. However they give a geometric and natural way to think about the locally defined action of the torus $A_P = \mathbf{S_P}(\mathbb{R})^0$.

(15.2) Throughout this section we fix $\alpha \in (\mathbf{R_d P})(\mathbb{Q})$. Define the discrete subgroup $\Gamma'_P = \Gamma_P \cap \alpha^{-1}\Gamma_P\alpha$ with finite indices,

$$\begin{aligned} r &= [\Gamma_P : \Gamma'_P] \\ s &= [\alpha^{-1}\Gamma_P\alpha : \Gamma'_P] \end{aligned} \tag{15.2.1}$$

It gives rise to a correspondence on Y_P in the usual way,

$$(c_1, c_2) : Y'_P = \Gamma'_P \backslash P / K_P(x_0) A_P(x_0) \rightrightarrows Y_P \tag{15.2.2}$$

by

$$\begin{aligned} c_1(\Gamma'_P x K_P(x_0) A_P(x_0)) &= \Gamma_P x K_P(x_0) A_P(x_0) \\ c_2(\Gamma'_P x K_P(x_0) A_P(x_0)) &= \Gamma_P \alpha x K_P(x_0) A_P(x_0) \end{aligned} \tag{15.2.3}$$

which is easily seen to be well defined and independent of the choice of basepoint x_0. This correspondence preserves the fibers of the map $\pi : Y_P \to X_P$ because $\alpha \in R_d P$ acts trivially on X_P. The correspondence is also defined on a neighborhood of Y_P by omitting the factor $A_P(x_0)$ in (15.2.3).

(15.3) Let $\mathbf{C}^\bullet(N_P, \mathbf{E})$ denote the complex (§12.5) of flat vectorbundles on X_P, consisting of the $\mathscr{U}_P$-invariant, $\mathbf{E}$-valued smooth differential forms along the fibers of $\pi : Y_P \to X_P$, and let $\mathbf{C}^\bullet(N'_P, \mathbf{E})$ denote the flat vectorbundle on X_P of invariant forms along the fibers of $\pi' : Y'_P \to X_P$. Define $c_2^* : \mathbf{C}^\bullet(N_P, \mathbf{E}) \to \mathbf{C}^\bullet(N'_P, \mathbf{E})$ to be the vectorbundle map which is obtained by pulling back differential forms from Y_P to Y'_P, and let $(c_1)_* : \mathbf{C}^\bullet(N'_P, \mathbf{E}) \to \mathbf{C}^\bullet(N_P, \mathbf{E})$ denote the vectorbundle map which is obtained by summing the values of the differential form over the finitely many points in the fiber of c_1.

(15.4) Definition. The Looijenga Hecke operator associated to the element $\alpha \in \mathbf{R_d P}(\mathbb{Q})$ is the vectorbundle mapping,

$$c^* = \frac{1}{r}(c_1)_*(c_2)^* : \mathbf{C}^\bullet(N_P, \mathbf{E}) \to \mathbf{C}^\bullet(N_P, \mathbf{E}).$$

Now let $h : \mathbf{C}^\bullet(N_P, \mathbf{E}) \to C^\bullet(\mathfrak{N}_P, E) \times_{\Gamma_L} D_P$ be the isomorphism of flat vectorbundles which was constructed in §12.13.

(15.5) Proposition. *Suppose the basepoint x_0 is chosen to be rational for* **P**, *so that the torus $\mathbf{S_P}(x_0) \subset \mathbf{R_d P}$ is a rationally defined algebraic subgroup of* **P**. *Suppose that $\alpha \in S_P(x_0) \cap \mathbf{G}(\mathbb{Q})$. Then the action of the Looijenga Hecke operator for α agrees with the adjoint action of α on the Koszul complex of flat vectorbundles, in the sense that the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{C}^\bullet(N_P, \mathbf{E}) & \xrightarrow{h} & C^\bullet(\mathfrak{N}_P, E) \times_{\Gamma_L} D_P \\ {\scriptstyle c^*}\downarrow & & \downarrow{\scriptstyle Ad(\alpha)} \\ \mathbf{C}^\bullet(N_P, \mathbf{E}) & \xrightarrow{h} & C^\bullet(\mathfrak{N}_P, E) \times_{\Gamma_L} D_P \end{array} \tag{15.5.1}$$

(15.6) Proof. Fix a point $x \in X_P$, identify the fiber $\pi^{-1}(x) \cong N_P$, (using §7.9 (part 3)), and consider the resulting homomorphism $h \circ (c_1)_* \circ h^{-1} : C^\bullet(\mathfrak{N}_P, E) \to C^\bullet(\mathfrak{N}_P, E)$ on the invariant differential forms along $\pi^{-1}(x)$. This map is given by multiplication by r since it consists of adding the values of an invariant differential form over r points. The mapping $c_2 : Y'_P \to Y_P$ lifts to a map $\tilde{c}_2 : e_P \to e_P$ which is given by left multiplication with α. If $F : \mathcal{U}_P \times D_P \to e_P$ is the diffeomorphism (§7.6) determined by the basepoint, then in these coordinates the mapping $\tilde{c}_2$ is given by

$$\begin{aligned}\tilde{c}_2(u, zK_PA_P) &= \alpha.(u, zK_PA_P) = (\alpha u i\nu(\alpha^{-1}), \nu(\alpha)zK_PA_P) \\ &= (\alpha u\alpha^{-1}, zK_PA_P)\end{aligned}$$

since α is in the center of $i(L_P)$. Thus the action on the van Est complex of invariant differential forms on $\mathcal{U}_P$ is given by conjugation with α. □

16. Splitting the weighted cohomology sheaf

(16.1) The results in this section are somewhat technical but they will be used in the proof (§23) that the middle weighted cohomology pushes down to the intersection cohomology on the Baily-Borel Satake compactification.

(16.2) Let $\mathbf{P_0}$ be a minimal rational parabolic subgroup of $\mathbf{G}$: we may take it to be the standard one. Let $\mathbf{Q_1}, \mathbf{Q_2}, \ldots \mathbf{Q_r}$ be the maximal rational parabolic (proper) subgroups containing $\mathbf{P_0}$, which have been ordered in some way, say $\mathbf{Q_1} \prec \mathbf{Q_2} \prec \ldots \prec \mathbf{Q_r}$. Let $S_i = S_{Q_i}(x_0)$ be the corresponding 1-dimensional tori. For each subset $J \subseteq \{1, 2, \ldots, r\}$ let $\mathbf{Q_J} = \cap_{i \in J}\mathbf{Q_i}$ be the corresponding standard rational parabolic subgroup. Let us say that an arbitrary rational parabolic subgroup has *depth* $\leq d$ if it is $\mathbf{G}(\mathbb{Q})$-conjugate to a standard rational parabolic subgroup $\mathbf{Q_J}$ such that $J \subseteq \{1, 2, \ldots, d\}$. Consider the following filtration (which depends on the choice of ordering of the standard maximal parabolic subgroups) of the reductive Borel-Serre compactification $\bar{X}$ by subsets $X = U_0 \subset U_1 \subset \ldots \subset U_r = \bar{X}$ where

$$U_d = X \cup \bigcup \{X_P | \text{ depth}(P) \leq d\}.$$

Then each U_d is open. Fix d with $0 \leq d \leq r-1$. Let us denote by $i : U_d \to U_{d+1}$ the inclusion. The complement $U_{d+1} - U_d$ contains the maximal boundary stratum $X_{Q_{d+1}}$ (and its $\mathbf{G}(\mathbb{Q})$-conjugates) as a dense open subset. For convenience, we will focus attention on the single "standard" stratum which we will denote by X_{d+1} and we will denote its closure in U_{d+1} by $\bar{X}_{d+1} = U_{d+1} - U_d$. Fix a weight profile which is given by standard weight cutoffs $t_i \in \chi(S_i) \otimes \mathbb{Q}$, and choose an algebraic complex representation E of G. Let $\mathbf{WC^\bullet(E)}$ denote the weighted cohomology complex of sheaves (with respect to the local system $\mathbf{E}$) on the reductive Borel-Serre compactification $\bar{X}$.

(16.3) Consider the two complexes of sheaves

$$\begin{aligned}\mathbf{A^\bullet} &= \mathbf{WC^\bullet(E)}|U_{d+1} \\ \mathbf{B^\bullet} &= i_*i^*\mathbf{WC^\bullet(E)} = i_*(\mathbf{WC^\bullet}|U_d)\end{aligned}$$

Then $\mathbf{A^\bullet}$ is a subsheaf of $\mathbf{B^\bullet}$.

(16.4) Proposition. *There is a complex of sheaves* $\mathbf{F}^\bullet$ *on the closure (in* U_{d+1}*)* $\bar{X}_{d+1}$ *of the (maximal) boundary stratum* X_{d+1} *such that the inclusion* $\mathbf{A}^\bullet|\bar{X}_{d+1} \subset \mathbf{B}^\bullet|\bar{X}_{d+1}$ *extends to a quasi-isomorphism,*

$$\mathbf{A}^\bullet|\bar{X}_{d+1} \oplus \mathbf{F}^\bullet \xrightarrow{\sim} \mathbf{B}^\bullet|\bar{X}_{d+1}.$$

Furthermore, the sheaf $\mathbf{F}^\bullet$ *may be chosen so that, for each* $x \in \bar{X}_{d+1}$*, the induced decomposition of the stalk cohomology,*

$$H_x^*(\mathbf{WC}^\bullet) \oplus H_x^*(\mathbf{F}^\bullet) \cong H_x^*(\mathbf{B}^\bullet)$$

is precisely the decomposition of the stalk cohomology $H_x^*(\mathbf{B}^\bullet)$ *into the subspaces with* S_{d+1}*-weights* $\leq t_{d+1}$ *and* $\geq t_{d+1}$ *respectively.*

(16.5) Corollary. *For any subset* $T \subset \bar{X}_{d+1}$*, the group* S_{d+1} *acts on the hypercohomology*

$$H^*(T;\mathbf{B}^\bullet|T) \cong H^*(T;\mathbf{F}^\bullet|T) \oplus H^*(T;\mathbf{WC}^\bullet|T)$$

with weights $\leq t_{d+1}$ *on the first factor and with weights* $\geq t_{d+1}$ *on the second factor.*

(16.6) Proof. The E_2 term of the spectral sequence for hypercohomology is

$$E_2^{pq} = H^p(\bar{X}_{d+1};\mathbf{H}^q\mathbf{B}^\bullet). \tag{16.13.1}$$

The S_{d+1}-weights are given by their values on the coefficients, i.e., on the stalk cohomology groups of $\mathbf{B}^\bullet$. By Prop. 16.4, the decomposition of stalk cohomology,

$$H_x^q(\mathbf{B}^\bullet) \cong H_x^q(\mathbf{F}^\bullet) \oplus H_x^q(\mathbf{WC}^\bullet)$$

is precisely the decomposition into S_{p+1}-weights $\leq t_{d+1}$ and weights $\geq t_{d+1}$ respectively. Since the differentials in this spectral sequence preserve weights, the same weights appear in the hypercohomology. □

(16.7) The rest of §16 is devoted to the proof of Prop. 16.4. The idea of the proof is the following: on the interior of the stratum X_{d+1} the sheaf $\mathbf{B}^\bullet$ is quasi-isomorphic to the sheaf $\mathbf{B}^\bullet_{\mathrm{sp}}$ of special differential forms on X_{d+1} with coefficients in a certain complex of flat vectorbundles. The coefficient complex decomposes under the action of the torus S_{d+1}. One summand gives rise to weighted cohomology, and we take $\mathbf{F}^\bullet$ to be the complementary summand. The same torus S_{d+1} acts at the boundary of the stratum X_{d+1}, in fact, even a bigger torus acts. So the same procedure may be used to split off a complement of the weighted cohomology sheaf over the whole closure $\bar{X}_{d+1}$. (The complex $\mathbf{F}^\bullet$ depends on the choice of basepoint $x_0 \in D$ while the sheaves $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ do not.)

(16.8) Let us define the "special" direct image sheaf, $\mathbf{B}^\bullet_{\mathrm{sp}}$, to be the sheaf whose sections over an open set $V \subset U_{d+1}$ consist of differential forms ω on $V \cap X$ which satisfy the following restrictions whenever X_P is a stratum in U_{d+1} and $V \cap X_P \neq \phi$:

1. Near $\pi^{-1}(V \cap X_P) \subset Y_P$ the form ω is the pullback (under the A_P-geodesic action) of a differential form ω_P on $\pi^{-1}(V \cap X_P)$.
2. The form ω_P is $\mathcal{U}_P$-invariant.
3. If $X_P \subset U_d$ then $\omega_P \in \Omega^\bullet_{\mathrm{inv}}(\pi^{-1}(V \cap X_P), \mathbf{E})_+$.

In other words, the form ω is "special" near every stratum, and it also satisfies weight restrictions near strata which are contained in the smaller open set U_d. Then $\mathbf{B}^\bullet_{\text{sp}}$ is canonically a sub-sheaf of $\mathbf{B}^\bullet = i_*(\mathbf{WC}^\bullet|U_d)$.

(16.9) Lemma. *The inclusion* $\mathbf{B}^\bullet_{\text{sp}} \to \mathbf{B}^\bullet$ *is a quasi-isomorphism.*

(16.10) Proof. The proof is very similar to the proof of Lemma 13.6. □

(16.11) For any open set $V \subset U_{d+1}$, the group of sections of the weighted cohomology complex, $\Gamma(V, \mathbf{WC}^\bullet(\mathbf{E}))$ is the subgroup of $\Gamma(V, \mathbf{B}^\bullet_{\text{sp}})$ (see §16.6) which also satisfy a weight restriction with respect to S_{d+1} (if $V \cap \bar{X}_{d+1} \neq \phi$). In particular it is a subgroup and so we have an inclusion of sub-sheaves,

$$\mathbf{WC}^\bullet(\mathbf{E}) \subset \mathbf{B}^\bullet_{\text{sp}} \subset \mathbf{B}^\bullet$$

which makes explicit the map $\mathbf{A}^\bullet \to \mathbf{B}^\bullet$. To complete the proof of the proposition, we will find a complement $\mathbf{F}^\bullet$ to $\mathbf{WC}^\bullet(\mathbf{E})|\bar{X}_P$ in the complex of sheaves $\mathbf{B}^\bullet_{\text{sp}}|\bar{X}_P$.

(16.12) *Notation.* For any subset $J \subseteq \{1, 2, \ldots r\}$, let X_J denote the stratum X_{Q_J} and let N_J denote the nilmanifold N_{Q_J}. Denote by S_J the torus $S_{Q_J}(x_0) \subset Q_J$. If $I \subseteq J \subseteq \{1, 2, \ldots, r\}$, we define the subset

$$\chi(S_J)_{+(I)} = \{\alpha \in \chi(S_J) |\ \forall i \in I,\ \alpha|S_i \in \chi(S_i)_+\}$$

If M is a finite dimensional module over S_J then we may consider the *I-partial truncation*,

$$M_{+(I)} = \bigoplus \{M_\alpha |\ \alpha \in \chi(S_J)_{+(I)}\}$$

Then $M_{+(I)} = \cap_{i \in I} M_{+(i)}$ where

$$M_{+(i)} = \bigoplus \{M_\alpha |\ \alpha \in \chi(S_J) \text{ and } \alpha|S_i \in \chi(S_i)_+\}.$$

(16.13) Fix a stratum $X_J \subset \bar{X}_{d+1}$ and let $I = J - \{d+1\}$. As in §12.8, the coefficient vectorbundle $\mathbf{C}^\bullet(N_J, \mathbf{E})$ breaks into a direct sum of modules $\mathbf{C}^\bullet(N_J, \mathbf{E})_\alpha$ for $\alpha \in \chi(S_J)$. We break the collection of characters $\chi(S_J)_{+(I)}$ into a disjoint union,

$$\chi(S_J)_{+(I)} = \chi(S_J)_{+(J)} \cup \chi(S_J)' \tag{16.13.1}$$

where $\chi(S_J)_+ = \chi(S_J)_{+(J)}$ is the set of characters α such that $\alpha|S_i \in \chi(S_i)_+$ for $i \in J$, and where $\chi(S_J)'$ is the remaining characters α with

$$\begin{cases} \alpha|S_i \in \chi(S_i)_+ & \text{for} \quad i \in I \\ \alpha|S_i \in \chi(S_i)_- & \text{for} \quad i = d+1 \end{cases} \tag{16.13.2}$$

Thus the coefficient bundle breaks up accordingly as

$$\mathbf{C}^\bullet(N_J, \mathbf{E})_{+(I)} \cong \mathbf{C}^\bullet(N_J, \mathbf{E})_{+(J)} \oplus \mathbf{C}^\bullet(N_J, \mathbf{E})' \tag{16.13.3}$$

with

$$\mathbf{C}^\bullet(N_J, \mathbf{E})' = \bigoplus_\alpha \{\mathbf{C}^\bullet(N_J, \mathbf{E})_\alpha |\ \alpha \in \chi(S_J)'\}$$

Therefore the sheaves of differential forms break up accordingly,

$$\boldsymbol{\Omega}^\bullet(X_J;\mathbf{C}^\bullet(N_J,\mathbf{E})_{+(I)}) \cong \boldsymbol{\Omega}^\bullet(X_J;\mathbf{C}^\bullet(N_J,\mathbf{E})_{+(J)}) \oplus \boldsymbol{\Omega}^\bullet(X_J;\mathbf{C}^\bullet(N_J,\mathbf{E})') \quad (16.13.4)$$

(16.14) Definition. Let $\mathbf{F}^\bullet$ be the complex of sheaves on $\bar{X}_{d+1}$ whose sections over an open set $V \subset \bar{X}_{d+1}$ consist of smooth differential forms

$$\omega \in \Omega^\bullet(V \cap X_{d+1};\mathbf{C}^\bullet(N_{d+1},\mathbf{E})) \cong \Omega^\bullet_{\text{inv}}(\pi^{-1}(V \cap X_{d+1}),\mathbf{E})$$

which satisfy the following conditions, whenever X_J is a stratum in $\bar{X}_{d+1}$ and $V \cap X_J \neq \phi$:

1. Near $\pi^{-1}(V \cap X_J) \subset Y_J$ the form ω is the pullback (under the A_J geodesic action on $\tilde{X}$) of a differential form ω_J on $\pi^{-1}(V \cap X_J)$.
2. The form ω_J is $\mathcal{U}_J$-invariant.
3. $\omega_J \in \Omega^\bullet(V \cap X_J;\mathbf{C}^\bullet(N_J,\mathbf{E})')$.

(16.15) Proof of Prop. 16.4. The complex of sheaves $\mathbf{F}^\bullet$ is a subcomplex of $\mathbf{B}^\bullet_{\text{sp}}$. For every stratum $X_J \subset \bar{X}_{d+1}$, the restriction $\mathbf{F}^\bullet|X_J$ is canonically isomorphic to the complex of sheaves $\boldsymbol{\Omega}^\bullet(X_J;\mathbf{C}^\bullet(N_J,\mathbf{E})')$, so by (16.13.4) it is a complement to $\mathbf{A}^\bullet$ in $\mathbf{B}^\bullet_{\text{sp}}$. □

17. Stalk cohomology

(17.1) In this chapter we give an explicit formula for the restriction of the cohomology sheaf of the weighted cohomology to a boundary stratum. In a later paper we will analyze the restriction of the cohomology sheaf to the closure of a boundary stratum.

Choose a weight profile $\{\chi(S_P)_\pm\}$. Fix a rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ and let $\pi : Y_P \to X_P$ denote the nilmanifold fibration associated to the Borel-Serre boundary stratum Y_P and the reductive Borel-Serre stratum X_P. Fix $x \in X_P$. A choice of $y \in D_P$ which projects to x determines (§7.9 (part 3)) a diffeomorphism $f_y : N_P \to \pi^{-1}(x)$, which is well defined up to conjugation by Γ_P on N_P. By Prop. 10.4, the subgroup

$$H^i(\pi^{-1}(x),\mathbf{E})_+ = (f_y)_*(H^i(N_P,\mathbf{E})_+)$$

is independent of the choice of lift $y \in D_P$ so we obtain a local system,

$$\mathbf{H}^i(\pi^{-1}(x),\mathbf{E})_+ \subset R^i\pi_*(\mathbf{E}) \quad (17.1.1)$$

on the stratum X_P. Since $\Gamma_L \subset M_P = P/\mathcal{U}_P A_P(x_0)$ acts on $H^i(\mathfrak{N}_P, E)$ we obtain a second local system,

$$\mathbf{H}^i(\mathfrak{N}_P,E)_+ = H^i(\mathfrak{N}_P,E)_+ \times_{\Gamma_L} D_P \quad (17.1.2)$$

on $X_P = \Gamma_L \backslash D_P$. (Here, Γ_L denotes the projection of $\Gamma_P = \Gamma \cap P$ to the Levi quotient L.) Finally, the stalk cohomology of the weighted cohomology complex $\mathbf{WC}^\bullet(\mathbf{E})$ gives a local system

$$\mathbf{H}^i(\mathbf{WC}^\bullet(\mathbf{E})) \quad (17.1.3)$$

on X_P.

(17.2) Proposition. *There are canonical isomorphisms of local systems*

$$\mathbf{H}^i(\mathbf{WC}^\bullet(\mathbf{E}))\ |X_P \cong \mathbf{H}^i(\pi^{-1}(x), \mathbf{E})_+$$

and a choice of basepoint determines an isomorphism of local systems,

$$\mathbf{H}^i(\mathbf{WC}^\bullet(\mathbf{E}))\ |X_P \cong \mathbf{H}^i(\mathfrak{N}_P, E)_+$$

on the stratum X_P.

(17.3) Proof. The proof follows immediately from Eq. (14.1.2), Prop. 12.6, and Prop. 12.13. □

(17.4) In the next two sections we use Kostant's theorem to make this formula more explicit. Let $p = \{\chi(S_P)\}_\pm$ denote a fixed weight profile. As in §9.4 we fix Cartan and Borel subgroups $\mathbf{H}(\mathbb{C}) \subset \mathbf{B}(\mathbb{C})$ with $S_P(x_0) \subset H \subset L_P(x_0)$. Suppose the representation E of $\mathbf{G}$ is irreducible with highest weight λ. Let V_α denote the $\mathbf{L_P}$-representation with highest weight α. Recall (11.5.3) that the subset

$$W_P^1 = \{w \in W | \Phi^+(w) \subset \Phi(\mathbf{H}, \mathfrak{N}_P(\mathbb{C}))\}$$

consists of the unique element of minimal length from each of the cosets $W_P x \in W_P \backslash W$.

(17.5) Definition. Let $W_P^p(E)$ denote the subset,

$$W_P^p(E) = \{w \in W_P^1 |\ (w(\lambda + \rho) - \rho)|S_P \in \chi(S_P)_+\}.$$

(17.6) Theorem. *The restriction* $\mathbf{H}^i(\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E}))|X_P$ *is isomorphic to the local system* $V \times_{\Gamma_L} D_P$, *where* V *is the sum of irreducible* $\mathbf{L_P}$*-representations,*

$$V = \bigoplus_{\substack{w \in W_P^p(E) \\ \ell(w) = i}} V_{w(\lambda+\rho)-\rho}.$$

(17.7) Proof. By Prop. 17.2, the cohomology local system is given by the local system on X_P corresponding to the $\mathbf{L_P}$ representation on $H^i(\mathfrak{N}_P, E)_+$. By Kostant's theorem this is a direct sum of certain irreducible representations $V_{w(\lambda+\rho)-\rho}$ where $w \in W_P^1$ and $\ell(w) = i$. The S_P-weight of each module V_α is constant and is equal to $\alpha|S_P$, so the truncation procedure either retains the full module $V_{w(\lambda+\rho)-\rho}$ or eliminates it. Those which are retained are specified by $w \in W_P^p(E)$ and $\ell(w) = i$. □

(17.8) Euler characteristic. Fix a weight profile p and an algebraic irreducible representation $G \to Gl(E)$ with highest weight λ. Let $\mathbf{P_1}, \mathbf{P_2}, \ldots, \mathbf{P}_\ell$ be a collection of representatives, one from each Γ-conjugacy class of proper rational parabolic subgroups of $\mathbf{G}$. For each representative $\mathbf{P_j}$ let $\Gamma_{L_j} = \nu(\Gamma \cap P_J)$ denote the projection of $\Gamma \cap P_j$ to the Levi quotient, and denote by $\chi(\Gamma_{L_j}) = \chi(X_{P_j})$ the Euler characteristic of the group Γ_{L_j}. Let $V_\alpha^{(j)}$ be the irreducible $\mathbf{L_j}$ representation with highest weight α.

(17.9) Theorem. *The Euler characteristic of the weighted cohomology,*

$$W^p\chi(\Gamma, \mathbf{E}) = \sum_{i=0}^{\dim(X)} (-1)^i \dim W^p H^i(\bar{X}, \mathbf{E})$$

may be expressed as the sum

$$W^p\chi(\Gamma, \mathbf{E}) = \chi(\Gamma). \dim(E) + \sum_{j=1}^{\ell} \chi(\Gamma_{L_j}) \sum_{w \in W^p_{P_j}} (-1)^{\ell(w)} \dim(V^{(j)}_{w(\lambda+\rho)-\rho})$$

(17.10) Proof. For any complex of sheaves $\mathbf{S}^\bullet$, which is cohomologically constructible with respect to a stratification of a space space $Y = \coprod_{i=1}^{\ell} Y_i$, we have [GM4],

$$\chi_c(Y, \mathbf{S}^\bullet) = \sum_{i=1}^{\ell} \chi_c(Y_i) \sum_j (-1)^j \dim(H^j_y(\mathbf{S}^\bullet))$$

where χ_c denotes the Euler characteristic with compact supports, and H^j_y denotes the stalk cohomology group at any point $y \in Y_i$. In the case that Y_i is a locally symmetric space and that the discrete group $\Gamma_{Y_i} \cong \pi_1(Y_i)$acts freely on its universal covering space, we have ([H1]) $\chi_c(Y_i) = \chi(\Gamma_{Y_i})$ since in general these two Euler characteristics differ by the Euler characteristic of the boundary of the Borel-Serre compactification, which is 0 since it is stratified by nilmanifold fibrations. □

(17.11) Remark. If the derived group of the Levi factor L_j does not have a compact Cartan subgroup then $\chi(\Gamma_{L_j}) = 0$.

18. Weighted cohomology of the link

(18.1) Fix a weight profile, an algebraic, finite dimensional, complex representation E of $\mathbf{G}$, and a rational parabolic subgroup $\mathbf{P}$ of rational rank s. Let $\mathscr{L}_P \subset \bar{X}$ be the link of a point $x_P \in X_P$ in the reductive Borel-Serre stratum corresponding to $\mathbf{P}$. By §8.9, there is a natural map $\delta : \mathscr{L}_P \to \Delta_P$ to the closed $s-1$-dimensional simplex, whose fiber N_x over any interior point $x \in \mathscr{L}_P \cap X$ is diffeomorphic to the nilmanifold N_P. In fact, a choice of $y \in D_P$ determines a diffeomorphism $N_x \cong N_P$. If $WH^j(\mathscr{L}_P, \mathbf{E})$ denotes the cohomology of the restriction of the complex of sheaves $\mathbf{WC}^\bullet$ to the link $\mathscr{L}_P$, then the inclusion $i : N_x \hookrightarrow \mathscr{L}_P$ induces a homomorphism $i^* : WH^j(\mathscr{L}_P, \mathbf{E}) \to H^j(N_P, \mathbf{E})$ because the restriction $\mathbf{WC}^\bullet|N_x$ is quasi-isomorphic to the restriction of the local system $\mathbf{E}$ to $N_x \cong N_P$. There is also a Gysin homomorphism $i_* : H^{j-(r-1)}(N_P, \mathbf{E}) \to H^j(\mathscr{L}_P, \mathbf{E})$ which is given by wedge product with the differential form $\delta^*(\epsilon.d\mathrm{vol})$ where $\epsilon : \Delta_P \to [0,1]$ is a bump function which vanishes near the edges of the simplex.

(18.2) Proposition. *The following sequence is split exact:*

$$0 \longrightarrow H^{j-s+1}(N_P, \mathbf{E})_- \xrightarrow{i_*} WH^j(\mathscr{L}_P, \mathbf{E}) \xrightarrow{i^*} H^j(N_P, \mathbf{E})_+ \longrightarrow 0 \qquad (18.2.1)$$

(18.3) Remark. This proposition says that the weighted cohomology of the link breaks into a direct sum of the stalk cohomology, $H^*(N_P, \mathbf{E})_+$ at the point $x_P \in X_P$,

and the "opposite" cohomology, $H^*(N_P, \mathbf{E})_-$ with a dimension shift. The proof of this proposition will use the Mayer-Vietoris spectral sequence corresponding to the standard open covering of the $s-1$-dimensional simplex Δ_P which consists of the open stars of the vertices. The multi-intersections of these open sets are the open stars of the various (open) faces of Δ_P.

(18.4) Let $\mathbf{Q}_1, \mathbf{Q}_2, \dots \mathbf{Q}_s$ be the collection of maximal rational parabolic subgroups containing $\mathbf{P}$. Each subset $I \subset \{1, 2, \dots, s\}$ corresponds to a parabolic subgroup $\mathbf{Q}_I = \cap_{i \in I} \mathbf{Q}_i \supset \mathbf{P}$ (with $\mathbf{Q}_\phi = \mathbf{G}$ and $\mathbf{Q}_{\{1,2,\dots,s\}} = \mathbf{P}$) and to a stratum $X_{Q_I} \subset \bar{X}$ and hence to an open face $(\Delta_P^{Q_I})^o$ of the simplex Δ_P (but $\Delta_P^P = \phi$). In particular, the *vertices* of the simplex Δ_P corresponds to subsets of cardinality $s-1$, which is unfortunate. For any subset $I \subset \{1, 2, \dots, s\}$ of cardinality $\leq s-1$, let us denote by $V_I \subset \mathscr{S}_P$ the open subset

$$V_I = \delta^{-1}(St(\Delta_P^{Q_I})^o)$$

which is obtained by pulling back to $\mathscr{S}_P$ the open star of the open face $(\Delta_P^{Q_I})^o$. The cohomology $H^j(\mathfrak{N}_P, E)$ is a finite dimensional S_P-module, then it is also a finite dimensional $S_{Q_I}-$module and the decomposition into S_P-weight spaces refines the decomposition into $S_{Q_I}-$weight spaces. As in (16.12), for each $i \in I$ let

$$H^j(\mathfrak{N}_P, E)_{+(i)} = \bigoplus \{H^j(\mathfrak{N}_P, E)_\alpha | \ \alpha | S_{Q_i} \in \chi(S_{Q_i})_+\}$$

We will denote by

$$\begin{aligned} H^j(\mathfrak{N}_P, E)_{+(I)} &= \oplus\{H^j(\mathfrak{N}_P, E)_\alpha | \quad \forall i \in I, \quad \alpha | S_{Q_i} \in \chi(S_{Q_i})_+\} \\ &= \oplus\{H^j(\mathfrak{N}_P, E)_\alpha | \quad \alpha | S_{Q_I} \in \chi(S_{Q_I})_+\} \\ &= \bigcap_{i \in I} H^j(\mathfrak{N}_P, E)_{+(i)} \end{aligned}$$

the submodule of $H^j(\mathfrak{N}_P, E)$ consisting of those isotypical components whose $S_{Q_I}-$weights are in the "high" region as defined by the weight profile.

(18.5) Lemma. *If $I \subset \{1, 2, \dots, s\}$ is a proper subset then the weighted cohomology of the open set V_I is given by*

$$WH^j(V_I) = H^j(\mathfrak{N}_P, E)_{+(I)}$$

If $I = \phi$ then the weighted cohomology of the open set $V_I = \delta^{-1}((\Delta_P^G)^o)$ is the full Lie algebra cohomology,

$$WH^j(V_I) = H^j(\mathfrak{N}_P, \mathbf{E}).$$

(18.6) Proof. Fix any point $t \in \Delta_P^{Q_I}$ in the face which corresponds to Q_I. Then V_I is a distinguished neighborhood (§8.10) of the stratum

$$\mathscr{S}_P \cap X_{Q_I} = \delta^{-1}(\Delta_P^{Q_I}) \cong \delta^{-1}(t) \times \Delta_P^{Q_I}$$

in $\mathscr{S}_P$, so by the Künneth formula,

$$WH^j(V_I, \mathbf{E}) = H^j(\delta^{-1}(t); \mathbf{WC}^\bullet | \delta^{-1}(t))$$

The fiber $\delta^{-1}(t)$ is diffeomorphic to $N_P^{Q_I}$ (see §8.9) and by Prop. 17.2 the stalk cohomology of the complex of sheaves $\mathbf{WC}^\bullet$ when restricted to the stratum X_{Q_I} is given by $\mathbf{H}^*(\mathfrak{N}_{Q_I}, E)_{+(I)}$. Therefore the spectral sequence for this cohomology group has an E_2 term which may be identified with

$$E_2^{a,b} = H^a(\mathfrak{N}_P^{Q_I}; H^b(\mathfrak{N}_{Q_I}, E)_{+(I)})$$

This spectral sequence is a direct sum of S_{Q_I}-isotypical components of the spectral sequence from §11.8, hence it also degenerates at E_2. These groups were identified in §11.8.3 as

$$\bigoplus_{a+b=j} E_2^{a,b} = H^j(\mathfrak{N}_P, E)_{+(I)} \quad \square$$

(18.7) Proof of Proposition 18.2. Let us denote by $\bar{J}$ the complement of any subset $J \subset \{1, 2, \ldots, s\}$. Each subset of cardinality $s-1$, say, $I = \overline{\{i\}}$, corresponds to an open set $V_I \subset \mathscr{Y}_P$ which is the pre-image under δ of the open star of a *vertex* of Δ_P^G, and these $s-1$ subsets form an open cover of $\mathscr{Y}_P$. We wish to consider the Mayer-Vietoris spectral sequence ([Go] ch. 2 §5.3 or [BT] p.96, prop. 8.8) for this cover. The groups $E_1^{a,b}$ are cohomology groups of multi-intersections of open sets in this cover,

$$E_1^{a,b} = \bigoplus_{|J|=a+1} WH^b(\bigcap_{j\in J} V_{\{\bar{j}\}}; \mathbf{E})$$

where $J \subset \{1, 2, \ldots, s\}$. Note that

$$\bigcap_{j\in J} V_{\{\bar{j}\}} = V_{\bar{J}}.$$

It follows that

$$E_1^{a,b} = \bigoplus_{|I|=s-1-a} WH^b(V_I; \mathbf{E}) = \bigoplus_{|I|=s-1-a} H^b(\mathfrak{N}_P, E)_{+(I)}$$

by the preceding lemma. Thus each group $E_1^{a,b}$ is a subgroup of the cohomology group $H^b(\mathfrak{N}_P, E)$, and the E_1 differential is given (up to sign) by inclusions of subgroups.

Claim. We claim that $E_2^{a,b} = 0$ unless $a = 0$ or $a = s-1$, and that $E_2^{0,b} = H^b(\mathfrak{N}_P, \mathbf{E})_+$ and $E_2^{s-1,b} = H^b(\mathfrak{N}_P, E)_-$ (where now the subscripts $\pm$ refer to the decomposition into S_P weight subspaces).

In other words, the E_2 term is concentrated in two columns. It follows from this claim that the spectral sequence degenerates at E_2 because the whole spectral sequence decomposes with respect to S_P weights, the differentials are induced by inclusion of subgroups, and they preserve the decomposition into S_P-isotypical components. However, $\chi(S_P)_+ \cap \chi(S_P)_- = \phi$ so for each differential the domain and target groups have distinct weights. Hence all the differentials vanish. In summary, the weighted cohomology of the link is the direct sum of the two groups $E_2^{0,*}$ and $E_2^{s-1,*}$.

It is not too hard to see that the image of the restriction map $i^* : WH^b(\mathscr{Y}_P; \mathbf{E}) \to H^b(N_P; \mathbf{E})$ is the first column of the E_2 term of the spectral sequence. In fact, the nilmanifold is contained in the intersection, $N_P \subset V_I$, for each (proper subset) $I \subset \{1, 2, \dots, s\}$. So restriction determines a homomorphism $\Phi : E_1^{0,*} \to H^*(\mathfrak{N}_P, \mathbf{E})$ whose image is the sum $\sum_{|I|=s-1} H^*(\mathfrak{N}_P; \mathbf{E})_{+(I)}$. The desired map i^* is the restriction of Φ to $E_2^{0,*} = \ker(d^0) \cong H^*(\mathfrak{N}_P; \mathbf{E})_+$. A similar argument may be used to see that the inclusion of $H^b(\mathfrak{N}_P, E)_- \subset H^{b+s-1}(\mathscr{S}_P, \mathbf{E})$ may be identified with the Gysin homomorphism. □

(18.8) Lemma. *Let $W_1, W_2, \dots, W_s$ be subspaces of a vectorspace W. For any subset $I \subset \{1, 2, \dots, s\}$ put $W_I = \bigcap_{i \in I} W_i$ and let $W_\phi = W_1 + W_2 + \dots + W_s \subset W$. These determine a semi-simplicial vectorspace and the associated complex*

$$0 \to W_{\{1,2,\dots,s\}} \to \dots \to \bigoplus_{|I|=q} W_I \to \dots \to \bigoplus_i W_i \to W_\phi \to 0 \tag{18.8.1}$$

is exact. □

(18.9) Proof of claim. To verify the claim, consider the first two columns of E_1,

$$E_1^{0,b} = \bigoplus_{|I|=s-1} H^b(\mathfrak{N}_P, E)_{+(I)} \xrightarrow{d_0} \bigoplus_{|I|=s-2} H^b(\mathfrak{N}_P, E)_{+(I)}$$

The kernel of this differential contains the common intersection $H^b(\mathfrak{N}_P, E)_+$. Similarly, consider the last two columns of E_1,

$$E_1^{s-2,b} = \bigoplus_{|I|=1} H^b(\mathfrak{N}_P, E)_{+(I)} \xrightarrow{d_{s-2}} E_1^{s-1,b} = H^b(\mathfrak{N}_P, E)$$

The image of this differential is the sum

$$\sum_{|I|=1} H^b(\mathfrak{N}_P, E)_{+(I)} \subset H^b(\mathfrak{N}_P, E)$$

We form an augmented term $\tilde{E}_1$ which is obtained by introducing a new column

$$\tilde{E}_1^{-1,b} = \ker(d_0) = H^b(\mathfrak{N}_P, E)_+$$

and by replacing the last column $E_1^{s-1,b} = H^b(\mathfrak{N}_P, E)$ by the image of the (last) differential d_{s-2},

$$\tilde{E}_1^{s-1,b} = \sum_{|I|=1} H^b(\mathfrak{N}_P, E)_{+(I)}.$$

(Note that this is not a direct sum.) Then the new sequence

$$\dots \to \tilde{E}_1^{a,b} \to \tilde{E}_1^{a+1,b} \to \dots$$

is exact, as may be seen by applying Lemma 18.8 to the collection of subspaces

$$W_i = H^b(\mathfrak{N}_P, E)_{+(i)} \subset W = H^b(\mathfrak{N}_P, E).$$

It follows that

$$E_2^{0,b} = \tilde{E}_1^{-1,b} = H^b(\mathfrak{N}_P, E)_+$$

and

$$E_2^{s-1,b} = H^b(\mathfrak{N}_P, E)/\tilde{E}_1^{s-1,b} = H^b(\mathfrak{N}_P, E)_-$$

as claimed. □

(18.10) The splitting. Let $d = \dim_{\mathbb{R}}(X_P)$ denote the real dimension of the stratum X_P of $\bar{X}$ which contains the point x_P. Choose a distinguished neighborhood U of x_P (§8.10). Thus there is a stratum preserving homeomorphism,

$$U \cong B^d \times c(\mathscr{L}_P) \tag{18.10.1}$$

where B^d is an open ball in $\mathbb{R}^d$ and $c(\mathscr{L}_P)$ denotes the cone over the link. Restriction of differential forms gives a homomorphism,

$$H^i(\mathfrak{N}_P, E)_+ \cong WH^i(U; \mathbf{E}) \to WH^i(\mathscr{L}_P, \mathbf{E})$$

which is independent of the choice of neighborhood U or of the homeomorphism (18.10.1), and which defines a canonical splitting of the sequence (18.2.1).

(18.11) Proposition. *The long exact sequence for the pair $WH^*(U, \partial U; \mathbf{E})$ splits into short exact sequences which may be identified as follows:*

$$\begin{array}{ccccccccc} 0 & \to & WH^i(U, \mathbf{E}) & \to & WH^i(\partial U, \mathbf{E}) & \to & WH^{i+1}(U, \partial U; \mathbf{E}) & \to & 0 \\ & & \cong\downarrow & & & & \cong\downarrow & & \\ & & H^i(\mathfrak{N}_P, E)_+ & & & & H^{i-s-d+1}(\mathfrak{N}_P, E)_- & & \end{array}$$

(18.12) Proof. Let $V = c(\mathscr{L}_P)$ denote the "normal slice", with boundary $\partial V = \mathscr{L}_P$. As pairs of spaces, we have $(U, \partial U) \cong (B^d, \partial B^d) \times (V, \partial V)$. By the Künneth formula, it suffices to verify the proposition for the long exact sequence of the pair $WH^*(V, \partial V; \mathbf{E})$. However the terms in this sequence may be identified as follows: $WH^i(V; \mathbf{E}) = WH^i_{x_P}(\mathbf{E}) \cong H^i(\mathfrak{N}_P, \mathbf{E})_+$ is the stalk cohomology, and $WH^i(\partial V; \mathbf{E}) = WH^i(\mathscr{L}_P, \mathbf{E}) \cong H^i(\mathfrak{N}_P, E)_+ \oplus H^{i-s+1}(\mathfrak{N}_P, E)_-$ by Prop. 18.2. Thus the restriction map

$$WH^i(V, \mathbf{E}) \to WH^i(\partial V, \mathbf{E})$$

is injective so $j^* : WH^i(V, \partial V; \mathbf{E}) \to WH^i(\partial V, \mathbf{E})$ vanishes. □

(18.13) Corollary. *The compactly supported stalk cohomology of the weighted cohomology complex is given by*

$$WH^i_{c,x_P} = WH^i(U, \partial U; \mathbf{E}) \cong H^{i-d-s}(\mathfrak{N}_P, \mathbf{E})_- \quad \square$$

19. The dualizing complex

(19.1) In this section we consider the weighted cohomology complex

$$\mathbf{WC}^\bullet(\mathbb{C})$$

corresponding to the "dualizing" weight profile (§9.5) and the constant coefficient system $\mathbb{C}$ on X. Let $n = \dim(X)$. Let $\mathbf{P}$ be a rational parabolic subgroup of $\mathbf{G}$ and let $N_P = \Gamma_P\backslash\mathcal{U}_P$ be the nilmanifold of §7. If s denotes the rational rank of $\mathbf{P}$, and if c denotes the $\mathbb{R}$- codimension of the stratum X_P in $\bar{X}$, then $c = s + \dim_{\mathbb{R}}(N_P)$.

(19.2) Lemma. *With respect to the dualizing weight profile d of §9.5 and the trivial coefficient system $\mathbb{C}$, the "low" weight subgroup of the cohomology of N_P is given by*

$$H^i(N_P, \mathbb{C})_- = \begin{cases} \mathbb{C} & \text{if } i = c - s \\ 0 & \text{otherwise} \end{cases} \tag{19.2.1}$$

(19.3) Proof. As in §9.4 choose Borel and Cartan subgroups with $\mathbf{B}(\mathbb{C}) \subset \mathbf{P}(\mathbb{C})$ and

$$\mathbf{S_P}(x_0)(\mathbb{C}) \subset \mathbf{H}(\mathbb{C}) \subset \mathbf{L_P}(x_0)(\mathbb{C})$$

and let $\Phi^+ \subset \Phi(\mathbf{H}, \mathfrak{g})$ denote the corresponding root system and collection of positive roots. The cohomology group $H^i(N_P; \mathbb{C})_-$ is a sum of $L_P(\mathbb{C})$−modules with S_P-weights $\alpha \leq -2(\rho|S_P)$. By Kostant's theorem, the highest weight of such a module is of the form $w\rho - \rho$ with $w \in W_P^1$. However there is a unique $w \in W_P^1$ of greatest length, namely $w = w_0^P w_0^G$ where $w_0^P \in W_P$ is the unique longest element, and $w_0^G \in W$ is the unique longest element. But $w_0^G \Phi^+ = -\Phi^+$ so $w_0^G \rho|S_P = -\rho|S_P$. Also, w_0^P acts trivially on S_P. Therefore $w\rho|S_P = -\rho|S_P$ which shows that the S_P−weight of the top cohomology group is $\alpha = -2\rho|S_P$. (In fact this module is the top exterior power of $\mathfrak{N}_P(\mathbb{C})$.) In summary there is a unique subgroup of the cohomology appearing in $H^*(N_P, \mathbb{C})_-$ and it is precisely the top cohomology group $H^{c-s}(N_P, \mathbb{C})$, which is one dimensional. □

(19.4) Proposition. *The isomorphism $\mathbf{W}^d\mathbf{C}^\bullet(\mathbb{C})|X \xrightarrow{\cong} \Omega^\bullet(X, \mathbb{C})$ extends to a unique quasi-isomorphism $\mathbf{W}^d\mathbf{C}^\bullet(\mathbb{C}) \xrightarrow{\sim} \mathbb{D}^\bullet_{\bar{X}}(\mathbb{C})[-n]$ between the "dualizing" weighted cohomology sheaf on $\bar{X}$ and the dualizing complex on $\bar{X}$ (shifted by the dimension of $\bar{X}$).*

(19.5) Remark. The dualizing sheaf of a smooth orientable manifold of real dimension n is quasi-isomorphic to $\mathbb{C}[n]$, the constant sheaf appearing in degree $-n$. This shift of n is used in order to guarantee that the Verdier duality theorem (see, for example, [Ve], [GM2] §1.12, or [B-] V §7.8) holds with no shifts.

(19.6) Proof. Since $\bar{X}$ is a normal pseudomanifold ([GM1]) (i.e. the link of each stratum is connected), the dualizing complex $\mathbb{D}_{\bar{X}}$ is the intersection complex $\mathbf{I}^t\mathbf{C}^\bullet_{\bar{X}}$ with top perversity t. By [GM2] (axioms [AX1]) the quasi-isomorphism

$$\mathbf{W}^d\mathbf{C}^\bullet|X \xrightarrow{\sim} \mathbb{C}_X = \mathbb{D}^\bullet_X[-n]$$

(on the interior) extends to a unique quasi-isomorphism $\mathbf{W}^d\mathbf{C}^\bullet(\mathbb{C}) \xrightarrow{\sim} \mathbf{I}^t\mathbf{C}^\bullet_{\bar{X}}$ provided the "support" and "attaching" axioms can be verified: Fix $x_P \in X_P$ and let U be a distinguished neighborhood of x_P in $\bar{X}$. Let c denote the real codimension of the stratum X_P in $\bar{X}$. For the dualizing sheaf, (after unshifting by n), these axioms read:

1. $H^i(U; \mathbf{W}^d\mathbf{C}^\bullet) = 0 \quad \forall i \geq c - 1$,
2. $H^i(U, \mathbf{W}^d\mathbf{C}^\bullet) \to H^i(\mathcal{L}_P; \mathbf{W}^d\mathbf{C}^\bullet)$ is an isomorphism $\forall i \leq c - 2$.

By Prop. 17.2, the stalk cohomology is

$$W^d H^i(U;\mathbb{C}) \cong H^i(\mathfrak{N}_P;\mathbb{C})_+.$$

We must show this vanishes for $i \geq c-1$. If s denotes the rational rank of **P** then $\dim_{\mathbb{R}}(N_P) = c - s$ so $H^i(N_P;\mathbb{C}) = 0$ for $i \geq c - s + 1$. Thus the support condition holds automatically unless $s = 1$, in which case **P** is a maximal rational parabolic subgroup. In this case, $H^{c-1}(N_P;\mathbb{C})$ is the top degree cohomology and by the preceding lemma, it coincides with $H^*(\mathfrak{N}_P;\mathbb{C})_-$. Thus $H^*(\mathfrak{N}_P;\mathbb{C})_+$ has no components in the top degree, $c-1$, i.e. $H^{c-1}(\mathfrak{N}_P;\mathbb{C})_+ = 0$. This verifies axiom (1).

By Prop. 18.2, the weighted cohomology of the link is

$$\begin{aligned} WH^i(\mathscr{L}_P;\mathbb{C}) &= H^i(\mathfrak{N}_P;\mathbb{C})_+ \oplus H^{i-s+1}(\mathfrak{N}_P;\mathbb{C})_- \\ &= W^d H^i(U,\mathbb{C}) \oplus H^{i-s+1}(\mathfrak{N}_P;\mathbb{C})_- \end{aligned}$$

By the preceding Lemma 19.2, for the dualizing weight profile, the "low" cohomology group occurs in only one degree of $\mathfrak{N}_P$-cohomology, namely $c-s$. Thus

$$W^d H^i(\mathscr{L}_P;\mathbb{C}) = \begin{cases} W^d H^i(U;\mathbb{C}) & \text{for } i \leq c-2 \\ H^{c-s}(\mathfrak{N}_P;\mathbb{C}) & \text{for } i = c-1 \\ 0 & \text{otherwise} \end{cases}$$

In particular the restriction homomorphism is an isomorphism for $i \leq c-2$ which verifies axiom (2). □

20. Remarks on Borel-Moore-Verdier duality

(20.1) Let k be a field and suppose that $\mathbf{S_1}^\bullet$ and $\mathbf{S_2}^\bullet$ are cohomologically constructible complexes of sheaves of k-vectorspaces on a stratified pseudomanifold $\bar{X}$ of $\mathbb{R}$-dimension n. Let $\mathbb{D}_{\bar{X}}$ denote the dualizing complex. A morphism Φ : $\mathbf{S_1}^\bullet \otimes \mathbf{S_2}^\bullet \to \mathbb{D}_{\bar{X}}[-n]$ is called a *Verdier dual pairing* if the induced map $\mathbf{S_1}^\bullet \to \mathbf{RHom}(\mathbf{S_2}^\bullet, \mathbb{D}_{\bar{X}})[-n]$ is a quasi isomorphism, in other words, if it induces isomorphisms on the stalk cohomology at each point $x \in \bar{X}$. In this case, for each open set $U \subset \bar{X}$, the induced pairing

$$H^j_c(U, \mathbf{S_1}^\bullet) \otimes H^{n-j}(U, \mathbf{S_2}^\bullet) \to H^n_c(U, \mathbb{D}^\bullet) \to k$$

is nondegenerate for all j ([BM], [Ve], [GM2]).

(20.2) Lemma. *The pairing Φ is a Verdier dual pairing iff for each point $x \in \bar{X}$ the induced pairing*

$$H^j_c(U_x, \mathbf{S_1}^\bullet) \times H^{n-j}(U_x, \mathbf{S_2}^\bullet) \to k$$

is nondegenerate for all j, where U_x denotes a distinguished neighborhood (§8.10) of $x \in \bar{X}$.

(20.3) Proof. We must check that, for each $x \in \bar{X}$ the resulting morphism

$$i_x^*(\mathbf{S_1}^\bullet) \to i_x^*\mathbf{RHom}(\mathbf{S_2}^\bullet, \mathbb{D}_{\bar{X}})[-n]$$

induces an isomorphism on cohomology, where $i_x : x \to \bar{X}$ denotes the inclusion of the point x. By double duality, this is equivalent to the statement that

$$i_x^!(\mathbf{S_1}^\bullet) \to i_x^! \mathbf{RHom}(\mathbf{S_2}^\bullet, \mathbb{D}_{\bar X})[-n]$$

induces isomorphism on cohomology, where

$$i_x^! = \text{dual} \circ i_x^* \circ \text{dual}$$

is the Grothendieck-Verdier functor discussed in [Ve], [GM2] §12, [B-] V §7.14. However the cohomology of these groups may be identified as follows:

$$H^j(i_x^!(\mathbf{S_1}^\bullet)) \cong H_c^j(U_x, \mathbf{S_1}^\bullet)$$

and

$$\begin{aligned} H^j(i_x^!(\mathbf{RHom}(\mathbf{S_2}^\bullet, \mathbb{D}_{\bar X}^\bullet))[-n] &= H^j(\mathbf{RHom}(i_x^*(\mathbf{S_2}^\bullet), i_x^! \mathbb{D}_{\bar X}^\bullet)[-n] \\ &= \mathrm{Hom}^j(H^*(U_x; \mathbf{S_2}^\bullet), H_c^*(U_x; \mathbb{D}_{\bar X}^\bullet)[-n]) \\ &= \mathrm{Hom}(H^{n-j}(U_x, \mathbf{S_2}^\bullet), k) \end{aligned}$$

because

$$H_c^i(U_x, \mathbb{D}_{\bar X}^\bullet[-n]) = \begin{cases} k & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

which verifies the lemma. □

21. Duality for weighted cohomology

(21.1) Let $\mathbf{P}_0$ denote the standard minimal rational parabolic subgroup of $\mathbf{G}$. Let $\mathbf{Q}_1, \mathbf{Q}_2, \ldots \mathbf{Q}_r$ denote the standard maximal parabolic subgroups containing $\mathbf{P}_0$. Suppose p and q are weight profiles corresponding to weight cutoffs $\{s_1, s_2, \ldots, s_r\}$ and $\{t_1, t_2, \ldots, t_r\}$ respectively. Suppose that $\Phi : E_1 \times E_2 \to E_3$ is a bilinear pairing of algebraic G-modules of complex vectorspaces.

(21.2) Proposition. *Wedge product of differential forms determines a unique extension of* Φ,

$$\tilde\Phi : \mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E_1}) \times \mathbf{W}^q\mathbf{C}^\bullet(\mathbf{E_2}) \to \mathbf{W}^{p+q}\mathbf{C}^\bullet(\mathbf{E_3})$$

(21.3) Proof. The wedge product of two special differential forms is a special differential form. The wedge product of an invariant differential form on Y_P whose S_P weight is equal to s, with an invariant differential form on Y_P whose S_P weight is equal to t, is an invariant differential form with S_P-weight $s + t$. □

Now suppose further that $s_i + t_i = -2\rho|S_{Q_i}$ for each $i = 1, 2, \ldots r$, in other words, that p and q are "dual" weight profiles (§9.6). Suppose that $\Phi : E_1 \times E_2 \to \mathbb{C}$ is a nondegenerate pairing (with values in the trivial G-module $\mathbb{C}$).

(21.4) Theorem. *The extension* $\tilde\Phi : \mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E_1}) \times \mathbf{W}^q\mathbf{C}^\bullet(\mathbf{E_2}) \to \mathbf{W}^d\mathbf{C}^\bullet(\mathbb{C}) = \mathbb{D}_{\bar X}^\bullet[-n]$ *is a Verdier dual pairing and induces a quasi-isomorphism*

$$\mathbf{W}^p\mathbf{C}^\bullet(\mathbf{E_1}) \xrightarrow{\sim} \mathbf{RHom}(\mathbf{W}^q\mathbf{C}^\bullet(\mathbf{E_2}), \mathbb{D}_{\bar X}^\bullet[-n])$$

in the constructible bounded derived category $D^b(\bar X)$ *of sheaves of complex vectorspaces on the reductive Borel-Serre compactification* $\bar X$.

(21.5) Proof . By Prop. 19.4 we have already identified $\mathbf{W}^d\mathbf{C}^\bullet(\mathbb{C})$ as the dualizing sheaf. By Lemma 20.2 we need to show that, for each $x \in \bar{X}$, the induced pairing

$$W^pH^i(U_x;\mathbf{E}_1) \times W^qH_c^{n-i}(U_x,\mathbf{E}_2) \to \mathbb{C}$$

is nondegenerate, where U_x is a fundamental neighborhood of $x \in \bar{X}$. If $x \in X$ is an interior point then this duality is already guaranteed by Poincaré duality for manifolds and by the duality between the local systems $\mathbf{E}_1$ and $\mathbf{E}_2$. Thus we may assume that $x \in X_P$ is in some singular stratum, corresponding to some proper rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$. Denote by $\mathfrak{N}_P = \mathrm{Lie}(\mathscr{U}_P)$ the Lie algebra of the unipotent radical of $\mathbf{P}$ and set $m = \dim_{\mathbb{R}}\mathfrak{N}_P$. Then the stalk cohomology and stalk cohomology with compact supports were identified in §17.2 and §18.13 as $H^i(\mathfrak{N}_P, E_1)_+$ and $H^{n-i-d-r}(\mathfrak{N}_P, E_2)_- = H^{m-i}(\mathfrak{N}_P, E_2)_-$, respectively. Furthermore, the pairing is easily seen to be given by wedge product of differential forms. Therefore, by Lemma 10.9 this pairing is nondegenerate. □

22. Baily-Borel Satake compactification

(22.1) Throughout this section we will assume that X is a *Hermitian* locally symmetric space (so that D is a bounded symmetric domain). We denote by $\hat{X}$ the Baily-Borel Satake compactification of X [BB] [AMRT]. The inclusion $X \to \hat{X}$ extends to a unique continuous map $\Phi : \bar{X} \to \hat{X}$ whose restriction to each stratum of $\bar{X}$ is a smooth fibration over some corresponding stratum in $\hat{X}$. These constructions occur in [Z2] §3.6–3.8, and are well known to the experts in the field. However we will need to study the detailed structure of Φ so we will summarize its properties in this section.

The organization of this section follows closely a lecture given by M. Rapoport at MIT in March, 1991. We wish to thank him for many useful conversations. See also [LR], especially §6.

(22.2) Each boundary stratum F in $\hat{X}$ corresponds to some maximal rational parabolic subgroup $Q \subset G$, which we may take to be standard. The Levi factor of Q decomposes as an almost direct product (commuting factors with finite intersection),

$$M_Q = Q_hQ_l$$

where Q_h is the centralizer of $\mathscr{W}_Q = Z(\mathscr{U}_Q)$ in M_Q. (See [AMRT] III, §3 and §4. Here, $Z(\mathscr{U}_Q)$ denotes the center of $\mathscr{U}_Q$.) The boundary stratum F is an arithmetic quotient of $Q_h/K \cap Q_h$: it is a Hermitian locally symmetric space. (The symmetric space for the other factor Q_l is an open self adjoint homogeneous cone in a vectorspace.)

(22.3) Let $\mathbf{P}_0$ denote the standard rational parabolic subgroup of $\mathbf{G}$. Let us denote the maximal (standard) rational parabolic subgroups of $\mathbf{G}$ which contain $\mathbf{P}_0$ by $\mathbf{Q}_1, \mathbf{Q}_2, \dots \mathbf{Q}_r$. In order to avoid double subscripts, we will denote by $\mathscr{U}_i$ the unipotent radical of Q_i, and by $\mathscr{W}_i \subset \mathscr{U}_i$ the center of the unipotent radical. There is a total ordering $\prec$ on the set of standard maximal rational parabolic subgroups which is given by

$$\mathbf{Q}_i \prec \mathbf{Q}_j \text{ iff } \mathscr{W}_i \subset \mathscr{W}_j \text{ iff } \bar{F}_i \supset F_j.$$

For convenience, let us choose the numbering of the standard parabolic subgroups $\mathbf{Q}_i$ so that $\mathbf{Q}_1 \prec \mathbf{Q}_2 \prec \ldots \prec \mathbf{Q}_r$. Any standard parabolic subgroup $\mathbf{P}$ may be written in a unique way as

$$\mathbf{P} = \mathbf{Q}_{i_1} \cap \mathbf{Q}_{i_2} \cap \ldots \cap \mathbf{Q}_{i_s}$$

where $i_1 < i_2 < \ldots < i_r$. In this case we define

$$\mathbf{P}^\dagger = \mathbf{Q}_{i_s}$$

to be the *last* maximal rational parabolic (with respect to this ordering) which appears in the list. Notice that if $\mathbf{P} \subset \mathbf{Q}$ then $\mathbf{P}^\dagger \succeq \mathbf{Q}^\dagger$.

Fix a rational maximal parabolic subgroup $\mathbf{Q} \subset \mathbf{G}$. Let $\nu : Q \to L_Q$ denote the projection to the Levi quotient with $K_Q = \nu(K_Q(x_0) \cap Q)$ a maximal compact subgroup of M_Q. Then $K_h = Q_h \cap K_Q$ and $K_\ell = Q_\ell \cap K_Q$ are maximal compact subgroups of Q_h and Q_ℓ respectively. The (generalized) symmetric space $D_Q = M_Q/K_Q$ factors as a Cartesian product

$$M_Q/K_Q \cong Q_h/K_h \times Q_\ell/K_\ell.$$

This determines a fibration of locally symmetric spaces,

$$\Phi_Q : X_Q = \Gamma_L \backslash M_Q/K_Q \to F = \Gamma_h \backslash Q_h/K_h$$

with fiber $\Gamma_\ell \backslash Q_\ell/K_\ell$. (Here, $\Gamma_L = \nu(\Gamma \cap Q)$, $\Gamma_\ell = \Gamma_L \cap Q_\ell$, and $\Gamma_h \cong \Gamma_\ell \backslash \Gamma_L$ is the projection of Γ_L into Q_h.) A choice of $q \in Q_h$ therefore determines a diffeomorphism

$$f_q : \Gamma_\ell \backslash Q_\ell/K_\ell \to \Phi_Q^{-1}(\Gamma_h q K_h)$$

by $f_q(\Gamma_\ell y K_\ell) = \Gamma_L q y K_Q \in X_Q$.

(22.4) Suppose $\mathbf{P}$ is a rational parabolic subgroup of $\mathbf{G}$. Set $\mathbf{Q} = \mathbf{P}^+$. The choice of basepoint x_0 determines lifts $M_P(x_0) \subset M_Q(x_0)$ which induces a map on the (generalized) symmetric spaces

$$D_P = M_P/K_P \to M_Q/K_Q \to Q_h/K_h$$

and gives rise to a map on locally symmetric spaces,

$$\Phi_P : X_P = \Gamma_L \backslash M_P/K_P \to F = \Gamma_h \backslash Q_h/K_h$$

to the stratum $F \subset \hat{X}$ which corresponds to the maximal parabolic subgroup $\mathbf{Q} = \mathbf{P}^+$.

(22.5) Theorem. ([Z2], [LR], [R2]) *The mapping Φ_P is the restriction of Φ to the boundary stratum X_P.* □

(22.6) It follows that the boundary strata X_P which map to a fixed stratum $F = \Gamma_h \backslash Q_h/K_h \subset \hat{X}$ correspond to parabolic subgroups $\mathbf{P}'$ which are Γ- conjugate to a standard parabolic subgroup $\mathbf{P}$ with $\mathbf{P}^\dagger = \mathbf{Q}$. Fix $x \in F \subset \hat{X}$ and choose $q \in Q_h$ such that $\Gamma_h q K_h = x \in F$. Then the associated diffeomorphism $f_q : \Gamma_\ell \backslash Q_\ell/K_\ell \cong \Phi_Q^{-1}(x)$ extends to an identification

$$\bar{f}_q : \overline{\Gamma_\ell \backslash Q_\ell/K_\ell} \cong \Phi^{-1}(x) = \overline{\Phi_Q^{-1}(x)}$$

of the reductive Borel-Serre compactification of the locally symmetric space $\Gamma_\ell \backslash Q_\ell / K_\ell$ with the closure in $\bar{X}$ of the fiber $\Phi_Q^{-1}(x)$. In summary, we have:

(22.7) Corollary. *For any stratum $F \subset \hat{X}$, the restriction $(\Phi|\Phi^{-1}(F)) : \Phi^{-1}(F) \to F$ is a fiber bundle with compact fiber $\Phi^{-1}(x)$ which is isomorphic to the reductive Borel-Serre compactification of the (generalized) locally symmetric space $\Gamma_\ell \backslash Q_\ell / K_\ell$.*

(22.8) Corollary. [P] [LR] *Let $i : X \to \bar{X}$ and $\hat{\imath} : X \to \hat{X}$ denote the inclusions of $X = \Gamma \backslash D$ into the reductive Borel-Serre and the Baily-Borel compactifications, respectively. Let $G \to Gl(E)$ be an algebraic representation giving rise to a local system $\mathbf{E}$ on X. Then the stalk cohomology at a point $x \in \hat{X}$ of the direct image sheaf $R\hat{\imath}_*(\mathbf{E})$ is given by group cohomology,*

$$
\begin{aligned}
H_x^j(R\hat{\imath}_*\mathbf{E}) &\cong \bigoplus_{p+q=j} H^p(\overline{\Gamma_\ell \backslash Q_\ell / K_\ell}; i_{\ell *}\mathbf{H}^q(N_Q, \mathbf{E})) \\
&\cong \bigoplus_{w \in W_Q^1} H^{i-\ell(w)}(\Gamma_\ell; V_{w(\lambda+\rho)-\rho})
\end{aligned}
\tag{22.8.1}
$$

where $i_\ell : \Gamma_\ell \backslash Q_\ell / K_\ell \to \overline{\Gamma_\ell \backslash Q_\ell / K_\ell}$ denotes the inclusion, Q is a maximal rational parabolic subgroup corresponding to the stratum of $\hat{X}$ which contains the point x, the Weyl subset W_Q^1 is given by (11.5.3) and V_μ denotes the irreducible L_Q-representation of highest weight μ (restricted to the subgroup $\Gamma_\ell \subset L_\ell \subset L_Q$).

(22.9) Proof. The first isomorphism follows from the fact that the inclusion $\hat{\imath} : X \to \hat{X}$ factors as a composition

$$X \xrightarrow{j} \tilde{X} \xrightarrow{\pi} \bar{X} \xrightarrow{\Phi} \hat{X}$$

and from (11.4) and (11.8). The second isomorphism follows from Kostant's theorem (11.6). □

(22.10) Definition. A stratum $F \subset \hat{X}$ in the Baily-Borel compactification of X has *depth* less than or equal to d if every string of strata,

$$F = F_0 \subsetneqq \bar{F}_1 \subsetneqq \bar{F}_2 \subsetneqq \dots \subsetneqq \bar{F}_r = \hat{X}$$

has length $r \leq d$.

(22.11) Definition. A stratum $X_P \subset \bar{X}$ in the reductive Borel-Serre stratification has *depth* $\leq d$ if the corresponding parabolic subgroup is $\mathbf{G}(\mathbb{Q})$-conjugate to a standard parabolic subgroup $\mathbf{P}$ which may be written as an intersection of maximal parabolic subgroups,

$$\mathbf{P} = \mathbf{Q}_{i_1} \cap \mathbf{Q}_{i_2} \cap \dots \mathbf{Q}_{i_k}$$

with $\{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, \dots, d\}$. (See §16.2.)

Let $\hat{U}_d$ be the open subset of $\hat{X}$ consisting of strata of depth $\leq d$. Let $\bar{U}_d$ be the open subset of $\bar{X}$ consisting of strata of depth $\leq d$.

(22.12) Lemma. *For each d, the map Φ has the property,*

$$\Phi^{-1}(\hat{U}_d) = \bar{U}_d.$$

Proof. The proof is straightforward. □

23. Pushforward of weighted cohomology

(23.1) Let $\Phi : \bar{X} \to \hat{X}$ be the map from the reductive Borel-Serre compactification to the Baily-Borel Satake compactification of X. Let $G \to Gl(E)$ be an algebraic representation on some complex vectorspace E. Fix a weight profile $p = \{\chi(S_P)_\pm\}$.

Definition. The weighted cohomology complex of sheaves on $\hat{X}$ is the direct image sheaf,

$$\mathbf{W}^p\mathbf{C}^\bullet(\hat{X}, \mathbf{E}) = R\Phi_* \mathbf{W}^p\mathbf{C}^\bullet(\bar{X}, \mathbf{E})).$$

Now let μ, ν denote the upper and lower middle weight profiles (§9.5) giving rise to weighted cohomology sheaves

$$\mathbf{W}^\mu\mathbf{C}^\bullet(\bar{X}, \mathbf{E}) \to \mathbf{W}^\nu\mathbf{C}^\bullet(\bar{X}, \mathbf{E}).$$

respectively.

(23.2) Theorem. *The above morphism induces a quasi-isomorphism,*

$$R\Phi_* \mathbf{W}^\mu\mathbf{C}^\bullet(\bar{X}, \mathbf{E}) \xrightarrow{\sim} R\Phi_* \mathbf{W}^\nu\mathbf{C}^\bullet(\bar{X}, \mathbf{E}) \cong \mathbf{IC}^\bullet(\hat{X}, \mathbf{E})$$

on the Baily Borel compactification $\hat{X}$.

(23.3) The proof will occupy the rest of §23. Let $\hat{U}_d$ and $\bar{U}_d$ denote the open subsets of $\hat{X}$ and $\bar{X}$ consisting of strata with depth $\leq d$. We will prove by induction that, for each d, the theorem holds for the restriction of these sheaves to the open set $\hat{U}_d$, in other words, that

$$R\Phi_* \mathbf{W}^\mu\mathbf{C}^\bullet(\mathbf{E})|\hat{U}_d \cong R\Phi_* \mathbf{W}^\nu\mathbf{C}^\bullet(\mathbf{E})|\hat{U}_d \cong \mathbf{IC}^\bullet(\mathbf{E})|\hat{U}_d \qquad (23.3.1)$$

and that the same holds for the contragredient representation $\mathbf{E}^*$, i.e.

$$R\Phi_* \mathbf{W}^\mu\mathbf{C}^\bullet(\mathbf{E}^*)|\hat{U}_d \cong R\Phi_* \mathbf{W}^\nu\mathbf{C}^\bullet(\mathbf{E}^*)|\hat{U}_d \cong \mathbf{IC}^\bullet(\mathbf{E}^*)|\hat{U}_d \qquad (23.3.2)$$

For $d = 0$ the result holds since all the sheaves are quasi-isomorphic to the local system $\mathbf{E}$ (resp. $\mathbf{E}^*$) on the interior X. Suppose we have verified (23.3.1) for all $d \leq k$. Consider the diagram of spaces,

$$\begin{array}{ccc} \bar{U}_k & \xrightarrow{\bar{i}} & \bar{U}_{k+1} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \hat{U}_k & \xrightarrow{\hat{i}} & \hat{U}_{k+1} \end{array}$$

The complement $\hat{U}_{k+1} - \hat{U}_k$ consists of the standard stratum F_{k+1} together with its $\mathbf{G}(\mathbb{Q})$-conjugates. The stratum F_{k+1} is a hermitian locally symmetric space for the hermitian part of the standard maximal parabolic subgroup Q_{k+1}. We denote the corresponding reductive Borel-Serre stratum by X_{k+1}, and the one-dimensional split torus in the center of the Levi by $S_{k+1} = S_{Q_{k+1}}(x_0)$. The complement $\bar{U}_{k+1} - \bar{U}_k$

consists of a partial compactification of the stratum X_{k+1} (namely, the closure of X_{k+1} in $\bar{U}_{k+1}$) together with its $\mathbf{G}(\mathbb{Q})$-conjugates.

In order to show that the quasi-isomorphism

$$R\Phi_* \mathbf{W}^\mu \mathbf{C}^\bullet(\mathbf{E})|\hat{U}_k \xrightarrow{\sim} \mathbf{IC}^\bullet(\mathbf{E})|\hat{U}_k$$

extends to a similar quasi-isomorphism on $\hat{U}_{k+1}$ it suffices to prove, by axioms [AX3] in [GM2] that

(a) $R\Phi_* \mathbf{W}^\mu \mathbf{C}^\bullet(\mathbf{E})|\hat{U}_{k+1}$ is Verdier dual to $R\Phi_* \mathbf{W}^\mu \mathbf{C}(\mathbf{E}^*)|U_{k+1}$

(b) For each $x \in \hat{U}_{k+1} - \hat{U}_k$, the stalk cohomology groups vanish,

$$H^i_x(\mathbf{W}^\mu \mathbf{C}^\bullet(\mathbf{E})) = 0 \quad \forall i \geq m$$

(c) For each $x \in \hat{U}_{k+1} - \hat{U}_k$, the stalk cohomology groups vanish,

$$H^i_x(\mathbf{W}^\mu \mathbf{C}^\bullet(\mathbf{E}^*)) = 0 \quad \forall i \geq m$$

where m is the complex codimension of the stratum F_{k+1}.

We have already shown (§21.4) that $\mathbf{W}^\mu \mathbf{C}^\bullet(\mathbf{E})$ and $\mathbf{W}^\nu \mathbf{C}^\bullet(\mathbf{E}^*)$ are dual, so the same holds for their pushforwards to $\hat{X}$. Thus item (a) above may be replaced by

(a') $R\Phi_* \mathbf{W}^\mu \mathbf{C}^\bullet(\mathbf{E}^*)|U_{k+1} \to R\Phi_* \mathbf{W}^\nu \mathbf{C}^\bullet(\mathbf{E}^*)|U_{k+1}$ is a quasi-isomorphism.

(23.4) By induction, the inclusion of complexes

$$\mathbf{W}^\mu \mathbf{C}^\bullet \to \mathbf{W}^\nu \mathbf{C}^\bullet$$

induces a quasi-isomorphism over $\hat{U}_k$, and hence we have a diagram with quasi-isomorphisms along the top row,

$$\begin{array}{ccccc} R\Phi_* R\bar{\imath}_* \mathbf{W}^\mu \mathbf{C}^\bullet(\mathbf{E})|U_k & \xrightarrow{\cong} & R\Phi_* R\bar{\imath}_* \mathbf{W}^\nu \mathbf{C}^\bullet(\mathbf{E})|U_k & \xrightarrow{\cong} & R\hat{\imath}_* \mathbf{IC}^\bullet|U_k \\ \uparrow & & \uparrow & & \uparrow \\ R\Phi_* \mathbf{W}^\mu \mathbf{C}^\bullet(\mathbf{E})|U_{k+1} & \longrightarrow & R\Phi_* \mathbf{W}^\nu \mathbf{C}^\bullet(\mathbf{E})|U_{k+1} & & \mathbf{IC}^\bullet|U_{k+1} \end{array}$$

Fix a point $x \in F_{k+1}$ and consider the stalk cohomology groups at the point x in this diagram.

$$\begin{array}{ccccc} W^\mu H^i(\Phi^{-1}(\mathscr{L}_x); \mathbf{E}) & \xrightarrow{\cong} & W^\nu H^i(\Phi^{-1}(\mathscr{L}_x); \mathbf{E}) & \xrightarrow{\cong} & IH^i(\mathscr{L}_x; \mathbf{E}) \\ \alpha \uparrow & & \beta \uparrow & & \gamma \uparrow \\ W^\mu H^i_x(\mathbf{E}) & \longrightarrow & W^\nu H^i_x(\mathbf{E}) & & IH^i_x(\mathbf{E}) \end{array}$$

Here, $\mathscr{L}_x$ denotes the link of $x \in F_{k+1}$ in $\hat{X}$ and $W^\mu H^i(\Phi^{-1}(\mathscr{L}_x); \mathbf{E})$ denotes the hypercohomology of the restriction of the weighted cohomology sheaf to the space $\Phi^{-1}(\mathscr{L}_x)$. The torus S_{k+1} acts semi-simply on each of these cohomology groups across the top row.

(23.5) Lemma. *The maps* $\alpha, \beta,$ and γ *are injective with image*

(a) $\mathrm{Im}(\alpha) = W^\mu H^i(\Phi^{-1}(\mathscr{L}_x), \mathbf{E})_{+(\mu)}$

(b) $\mathrm{Im}(\beta) = W^\nu H^i(\Phi^{-1}(\mathscr{L}_x), \mathbf{E})_{+(\nu)}$

(c) $\mathrm{Im}(\gamma) = \begin{cases} IH^i(\mathscr{L}_x, \mathbf{E}) & \text{for } i < m \\ 0 & \text{for } i \geq m \end{cases}$

The subscripts $+(\mu)$ and $+(\nu)$ are used to emphasize that the weight truncation with respect to the torus S_{k+1} is possibly different in cases (a) and (b).

(23.6) Proof. Part (c) is the basic property of intersection cohomology (see [GM2] §2.4). Parts (a) and (b) follow from Cor. 16.5. In fact it is shown there that for any subset of $\bar{X}_{k+1}$ (for example, for $\Phi^{-1}(x)$), the hypercohomology of the restriction of the i_* sheaf to that subset breaks into a direct sum of two subgroups, once of which is the cohomology of the restriction of the weighted cohomology sheaf to that subset. In particular the second cohomology group is a subgroup of the first. It is necessary to check that the open sets U_d considered in §16 agree with the open sets $\bar{U}_d$ considered here, but this is the content of Lemma 22.12.

(23.7) Proposition. ([L]) *The image of γ is the subgroup of the intersection cohomology of the link $\mathscr{L}_x$ with S_{k+1}−weights $> -\rho_{Q_{k+1}}$ (see §9.4). The weight $-\rho_{Q_{k+1}}$ does not occur in the intersection cohomology of the link.*

(23.8) Proof. This is the main technical result in Looijenga's proof of the Zucker conjecture and it is the key step in our proof of Theorem 23.2. It takes a little work to translate Looijenga's language into ours. First, Looijenga defines weights on the stalk intersection cohomology through the action of locally defined Hecke correspondences. We have shown in §15 that, for an appropriate choice of basepoint $x_0 \in D$, the action of these Hecke correspondences agrees with our action of the group S_{k+1}, and in particular that the resulting decomposition of the stalk cohomology is the same with respect to either definition.

Now let $\mathscr{U}$ denote the unipotent radical of Q_{k+1}. Let $\mathscr{W}$ be the center of $\mathscr{U}$ and denote the quotient by $\mathscr{V} = \mathscr{U}/\mathscr{W}$. Then $\mathscr{V}$ is abelian. These groups have corresponding (real) Lie algebras $\mathfrak{v} = \mathfrak{N}/\mathfrak{w}$. The character group of the one dimensional torus S_{k+1} has a unique positive generator χ. Then S_{k+1} acts on $\mathfrak{v}$ with character χ and it acts on $\mathfrak{w}$ with character χ^2. Let $m = \frac{1}{2}(\dim(\mathfrak{v}) + 2\dim(\mathfrak{w}))$. Then $-\rho|S_{k+1} = \chi^{-m}$. Furthermore, m is the complex codimension of the stratum F_{k+1} since the normal slice to this stratum is constructed from a Siegel domain in $\mathfrak{v} \oplus (\mathfrak{w} \otimes \mathbb{C})$.

Looijenga proves ([L] §3.8) that the action of S_{k+1} on the stalk intersection cohomology IH^i_x has weights χ^{-a} with $a \leq i$. Since the stalk intersection cohomology vanishes in degrees $\geq m$ this means that S_{k+1} acts on the stalk cohomology with weights χ^{-a} with $a < m$. It follows that the weights in $IH^i(\mathscr{L}_x, \mathbf{E})$ for $i \geq m$ are of the form χ^{-a} with $a > m$ since $\dim_{\mathbb{R}}(\mathscr{L}_x) = 2m - 1$ and Poincaré duality switches weights about the center $-\rho$. In particular, the weight χ^{-m} does not occur in $IH^*(\mathscr{L}_x, \mathbf{E})$. Thus the stalk intersection cohomology is precisely the subgroup of the intersection cohomology of the link consisting of the weight spaces with weights $> \chi^{-m}$ (or, in Looijenga's terminology, with "weights $< m$"). □

(23.9) Proof of Theorem 23.2. By Prop. 23.7, the image of γ is the subgroup of the intersection cohomology of the link consisting of weight spaces with weights $> -\rho_{Q_{k+1}}$. This is precisely the image of α. Since the weight $-\rho_{Q_{k+1}}$ does not occur, the image of γ is also the subgroup of the intersection cohomology of the link consisting of weight spaces with weights $\geq -\rho_{Q_{k+1}}$. This is precisely the image of β. This proves conditions (a') and (b) of §23.3. Condition (c) follows by replacing $\mathbf{E}$ with $\mathbf{E}^*$, and this completes the proof of the inductive step. □

(23.10) It follows from (23.2) and (22.7) that the local L^2 cohomology (or the local intersection cohomology) of the Baily-Borel compactification may be expressed as a certain sum of weighted cohomology groups of the reductive Borel-Serre compactification $\overline{\Gamma_\ell \backslash Q_\ell / K_\ell}$ of a "linear" locally symmetric space. In a later paper we will describe precisely which weighted cohomology groups occur.

24. The group algebra A[$\mathcal{U}$]

(24.1) In this section we consider a connected unipotent algebraic group $\mathcal{U}$ defined over $\mathbb{Q}$. We denote by $\mathrm{Mod}_{\mathcal{U}}$ the category of inductive limits of finite dimensional rational representations of $\mathcal{U}(\mathbb{Q})$. We call this the category of locally finite $\mathcal{U}/\mathbb{Q}$-modules. We may also consider this to be the category of locally finite modules of the Lie algebra $\mathfrak{N}(\mathbb{Q}) = \mathrm{Lie}(\mathcal{U}(\mathbb{Q}))$.

If $\Gamma_{\mathcal{U}}$ is an arithmetic subgroup of $\mathcal{U}(\mathbb{Q})$, then any locally finite $\mathcal{U}/\mathbb{Q}$ module B is also a $\Gamma_{\mathcal{U}}$ module. One of the goals of this section is to give a purely algebraic proof of the van Est theorem,

$$H^*(\mathcal{U}(\mathbb{Q}), B) \cong H^*(\mathfrak{N}(\mathbb{Q}), B) \cong H^*(\Gamma_{\mathcal{U}}, B)$$

The first two cohomology groups are computed in the category $\mathrm{Mod}_{\mathcal{U}}$ and the first isomorphism is obvious. The third cohomology group is computed in the category of all $\Gamma_{\mathcal{U}}$-modules and this is the reason the theorem is nontrivial.

(24.2) The group $\mathcal{U}(\mathbb{Q})$ has a filtration by normal connected subgroups defined over $\mathbb{Q}$, say

$$\mathcal{U} = \mathcal{U}_0 \supset \mathcal{U}_1 \supset \mathcal{U}_2 \supset \ldots \supset \mathcal{U}_n = \{e\}$$

such that each quotient $\mathcal{U}_i(\mathbb{Q})/\mathcal{U}_{i+1}(\mathbb{Q})$ is isomorphic to the additive group $\mathbb{Q}$. If $\Gamma_{\mathcal{U}} \subset \mathcal{U}(\mathbb{Q})$ is any arithmetic subgroup then such a filtration gives rise to a filtration

$$\Gamma_{\mathcal{U}} \supset \Gamma_{\mathcal{U}_1} \supset \ldots \supset \Gamma_{\mathcal{U}_n} = \{e\}$$

where $\Gamma_{\mathcal{U}_i}/\Gamma_{\mathcal{U}_{i+1}} \cong \mathbb{Z}$. Each locally finite $\mathcal{U}/\mathbb{Q}$-module B has an increasing filtration

$$\{0\} \subset B_1 \subset B_2 \subset \ldots \subset B_n \subset \ldots$$

with $\bigcup B_i = B$ and B_{i+1}/B_i a one-dimensional trivial $\mathcal{U}(\mathbb{Q})$-module.

Let $A[\mathcal{U}]$ denote the group algebra of polynomial functions with rational coefficients on $\mathcal{U}(\mathbb{Q})$. We may also consider $A[\mathcal{U}]$ to be the algebra of polynomial functions with rational coefficients on the real points, $\mathcal{U}(\mathbb{R})$. The following proposition was communicated to us by J. C. Jantzen, and it is the central result of this section.

(24.3) Proposition. *Let B be a locally finite $\mathcal{U}$-module. Then*

(a) The tensor product $B \otimes A[\mathcal{U}]$ is an injective module in the category $\mathrm{Mod}_{\mathcal{U}}$ *and hence $H^i(\mathcal{U}(\mathbb{Q}), B \otimes A[\mathcal{U}]) = H^i(\mathfrak{N}(\mathbb{Q}), B \otimes A[\mathcal{U}]) = 0$ for all $i \geq 1$.*

(b) The $\Gamma_{\mathcal{U}}$-module $B \otimes A[\mathcal{U}]$ is acyclic, i.e.

$$H^i(\Gamma_{\mathcal{U}}, B \otimes A[\mathcal{U}]) = 0 \text{ for all } i \geq 1$$

(24.4) In this section we show that part (b) implies part (a). First note that for any locally finite $\mathcal{U}$-module B, we obtain the same invariants,

$$B^{\mathcal{U}(\mathbb{Q})} = B^{\mathfrak{N}(\mathbb{Q})} = B^{\Gamma_{\mathcal{U}}}$$

where $B^{\mathfrak{N}} = \{b \in B | \mathfrak{N}b = 0\}$.

The proof of part (a) reduces to the following question: Suppose we have a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & Y & \longrightarrow & Y' \\ & & \big\downarrow i & & \\ & & B \otimes A[\mathcal{U}] & & \end{array}$$

where $Y' = Y \oplus \mathbb{Q}y$ as a rational vectorspace, and where $\mathcal{U}(\mathbb{Q})$ acts trivially on Y'/Y. We must show that i extends to a morphism of $\mathcal{U}(\mathbb{Q})$ modules, $\tilde{\imath} : Y' \to B \otimes A[\mathcal{U}]$. Since y generates a finite dimensional $\mathcal{U}(\mathbb{Q})$ module in Y', we may also assume that Y is finite dimensional. Then $\mathrm{Hom}(Y, B \otimes A[\mathcal{U}])$ and $\mathrm{Hom}(Y', B \otimes A[\mathcal{U}])$ are locally finite. So it suffices to show that

$$\mathrm{Hom}(Y', B \otimes A[\mathcal{U}])^{\mathcal{U}(\mathbb{Q})} \to \mathrm{Hom}(Y, B \otimes A[\mathcal{U}])^{\mathcal{U}(\mathbb{Q})}$$

is surjective. We may take invariants under $\Gamma_{\mathcal{U}}$ instead of invariants under $\mathcal{U}(\mathbb{Q})$. However,

$$H^1(\Gamma_{\mathcal{U}}, \mathrm{Hom}(Y'/Y, B \otimes A[\mathcal{U}])) = H^1(\Gamma_{\mathcal{U}}, B \otimes A[\mathcal{U}]) = 0$$

which completes the proof that (b) implies (a).

(24.5) In this section we prove part (b) of Prop. 24.3 by induction on $\dim(\mathcal{U})$. Filtering B as above, we can reduce to the case $\dim(B) = 1$. If $\mathcal{V}$ is a normal subgroup of $\mathcal{U}$ then we have a spectral sequence,

$$H^p(\Gamma_{\mathcal{U}/\mathcal{V}}, H^q(\Gamma_{\mathcal{V}}, A[\mathcal{U}])) \Rightarrow H^{p+q}(\Gamma_{\mathcal{U}}, A[\mathcal{U}]).$$

The projection map $p : \mathcal{U} \to \mathcal{V}/\mathcal{U}$ between the affine varieties admits a section s (which is usually not a group homomorphism), and which gives rise to an isomorphism of affine varieties

$$\mathcal{V} \times (\mathcal{V} \backslash \mathcal{U}) \to \mathcal{U}$$

by $(v, \bar{u}) \mapsto v.s(\bar{u})$, which is compatible with the action of $\mathcal{V}$ from the left on both sides of the isomorphism. Hence,

$$A[\mathcal{U}] \cong A[\mathcal{V}] \otimes A[\mathcal{V} \backslash \mathcal{U}]$$

as $\mathcal{V}$-modules. Now assume that $\mathcal{V}$ is a proper subgroup of $\mathcal{U}$ and apply the induction hypothesis. In the spectral sequence, the terms $H^q(\Gamma_{\mathcal{V}}, A[\mathcal{U}])$ vanish for $q > 0$ and we are left with

$$H^p(\Gamma_{\mathcal{U}/\mathcal{V}}, H^0(\Gamma_{\mathcal{V}}, A[\mathcal{U}])) = H^p(\Gamma_{\mathcal{U}/\mathcal{V}}, A[\mathcal{V}]) = 0 \text{ for } p > 0$$

But these terms also vanish for $p = 0$ by the induction hypothesis. This reduces the problem to the case $\dim(\mathcal{U}) = 1$ and $\dim(B) = 1$, i.e. we must prove that

$$H^i(\mathbb{Z}, \mathbb{Q}[u]) = 0 \text{ for } i > 0$$

where $\mathbb{Z}$ acts on the rational polynomials in one variable, $\mathbb{Q}[u]$ by $u \mapsto u+1$. Since the cohomological dimension of $\mathbb{Z}$ is 1, we need to show that $H^1(\mathbb{Z}, \mathbb{Q}[u]) = 0$ and this is simply the fact that any polynomial $P(u)$ may be written as a first difference,

$$P(u) = Q(u+1) - Q(u)$$

for some polynomial $Q(u) \in \mathbb{Q}[u]$. □

(24.6) Proposition 24.3 implies that we may compute the cohomology of a locally finite $\mathcal{U}$-module B from the following resolution $I^\bullet(B)$,

$$0 \to B \to B \otimes A[\mathcal{U}] \to B_1 \to B_2 \dots$$

where the B_i are locally finite $\mathcal{U}(\mathbb{Q})$ modules and where the second arrow is given by $b \mapsto b \otimes 1$, by taking the homology groups of the resulting sequence of invariants,

$$(B \otimes A[\mathcal{U}])^{\mathcal{U}(\mathbb{Q})} \to (B_1)^{\mathcal{U}(\mathbb{Q})} \to (B_2)^{\mathcal{U}(\mathbb{Q})} \dots$$

This gives

$$H^*(\mathcal{U}(\mathbb{Q}), B) \cong H^*(\mathfrak{N}(\mathbb{Q}), B) \cong H^*(\Gamma_{\mathcal{U}}, B) = H^*(I^\bullet(B)^{\mathcal{U}(\mathbb{Q})})$$

which is van Est's theorem.

(24.7) Now suppose that $\mathbf{P} \subsetneqq \mathbf{Q}$ are rational parabolic subgroups of $\mathbf{G}$. The group $\mathcal{U}_{\mathbf{Q}}(\mathbb{Q}) \subset \mathcal{U}_{\mathbf{P}}(\mathbb{Q})$ acts on $A[\mathcal{U}_P]$ by translation. Let B be a locally finite rational representation of $\mathcal{U}_{\mathbf{Q}}(\mathbb{Q})$.

Corollary. *The $\Gamma_{\mathcal{U}_Q}$ module $B \otimes A[\mathcal{U}_P]$ is acyclic. In other words*

$$\begin{aligned} H^i(\Gamma_{\mathcal{U}_Q}, B \otimes A[\mathcal{U}_P]) &= H^i(\mathcal{U}_Q(\mathbb{Q}), B \otimes A[\mathcal{U}_P]) \\ &= H^i(\mathfrak{N}_Q(\mathbb{Q}), B \otimes A[\mathcal{U}_P]) \\ &= 0 \text{ for } i > 0 \end{aligned}$$

Proof. The module $A[\mathcal{U}_P]$ is locally finite as a $\mathcal{U}_Q$-module. □

(24.8) Fix a rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$. Let $\nu : \mathbf{P} \to \mathbf{L_P}$ denote the projection to the Levi quotient. Choose a basepoint $x_0 \in D$ which is rational for $\mathbf{P}$, in other words, so that the resulting lift $i : \mathbf{L_P} \to \mathbf{P}$ is a morphism of rationally defined algebraic groups.

Definition. The *chosen action* of $\mathbf{P}(\mathbb{Q})$ on $\mathcal{U}_{\mathbf{P}}(\mathbb{Q})$ is the action

$$\gamma.u = \gamma u i \nu(\gamma^{-1}) \tag{24.8.1}$$

for $\gamma \in \mathbf{P}(\mathbb{Q})$, and $u \in \mathcal{U}_{\mathbf{P}}(\mathbb{Q})$. The *chosen representation* of $\mathbf{P}(\mathbb{Q})$ on $A[\mathcal{U}_P]$ is the representation

$$(\gamma.\tilde{g})(u) = \tilde{g}(\gamma^{-1} u i \nu(\gamma)) \tag{24.8.2}$$

for $\gamma \in \mathbf{P}(\mathbb{Q})$ and $\tilde{g} \in A[\mathcal{U}_P]$.

These actions *depend* on the choice of basepoint x_0.

(24.9) Let $\mathbf{P} \subset \mathbf{Q}$ be rational parabolic subgroups of $\mathbf{G}$. Choose a basepoint x_0 which is rational for both $\mathbf{P}$ and $\mathbf{Q}$. Denote by $\nu : \mathbf{Q} \to \mathbf{L_Q} = \mathbf{Q}/\mathcal{U}_\mathbf{Q}$ the projection to the Levi quotient, and by $\bar{\mathbf{P}} = \nu(\mathbf{P})$ the parabolic subgroup of $\mathbf{L_Q}$ which is determined by $\mathbf{P}$. Let $\mathcal{U}_{\bar{\mathbf{P}}}$ denote its unipotent radical. Thus, we have a commutative diagram of groups, projections, and lifts,

$$\begin{array}{ccccccc} \mathcal{U}_\mathbf{Q} & \subset & \mathcal{U}_\mathbf{P} & \subset & \mathbf{P} & \subset & \mathbf{Q} \\ \nu\downarrow & & \nu\downarrow & & \nu\downarrow & & \nu\downarrow\uparrow i \\ 1 & \subset & \mathcal{U}_{\bar{\mathbf{P}}} & \subset & \bar{\mathbf{P}} & \subset & \mathbf{L_Q} \\ & & & & \downarrow\uparrow j & & \\ & & & & \mathbf{L_P} & & \end{array} \tag{24.9.1}$$

Then $i_{x_0}(\mathcal{U}_{\bar{\mathbf{P}}}) \subset \mathcal{U}_\mathbf{P}$ and the group $\mathcal{U}_P$ is the semidirect product of rationally defined algebraic groups,

$$\mathcal{U}_\mathbf{P} = \mathcal{U}_\mathbf{Q} \rtimes i(\mathcal{U}_{\bar{\mathbf{P}}}).$$

(24.10) Define $H : \mathcal{U}_\mathbf{Q} \times \mathcal{U}_{\bar{\mathbf{P}}} \to \mathcal{U}_\mathbf{P}$ by $H(v,u) = v.i(u)$ and define the associated *chosen projection* $\rho = \mathrm{pr}_1 \circ H^{-1} : \mathcal{U}_\mathbf{P} \to \mathcal{U}_\mathbf{Q}$.

(24.10) Proposition. *The mappings H and ρ are polynomial mappings with rational coefficients. The chosen action of $\mathbf{P}(\mathbb{Q})$ on $\mathcal{U}_\mathbf{P}(\mathbb{Q})$ is given, in these coordinates, by*

$$\gamma.(v,u) = (\gamma v i \nu(\gamma^{-1}), \nu(\gamma) u \nu(\gamma^{-1})) \tag{24.10.1}$$

where $v \in \mathcal{U}_\mathbf{P}(\mathbb{Q}), u \in \mathcal{U}_{\bar{\mathbf{P}}}(\mathbb{Q}), \gamma \in \mathbf{P}(\mathbb{Q})$.

Proof. The proof follows from the direct computation that $H(\gamma.(v,u)) = \gamma.H(v,u)$. □

(24.11) Corollary. *The chosen projection $\rho : \mathcal{U}_\mathbf{P} \to \mathcal{U}_\mathbf{Q}$ determines an embedding*

$$\rho^* : A[\mathcal{U}_Q] \hookrightarrow A[\mathcal{U}_P]$$

The image $\rho^*(A[\mathcal{U}_Q])$ is an invariant subspace under the chosen $\mathbf{P}(\mathbb{Q})$ action on $A[\mathcal{U}_P]$, and the resulting representation of $\mathbf{P}(\mathbb{Q})$ on $A[\mathcal{U}_Q]$ coincides with the restriction to $\mathbf{P}(\mathbb{Q})$ of the chosen $\mathbf{Q}(\mathbb{Q})$ action on $A[\mathcal{U}_Q]$.

(24.12) Proof. The proof follows from direct calculation from (24.10.1) and the fact that ρ is a rational polynomial mapping. □

25. Locally flat polynomial functions

(25.1) Throughout this section we will refer to the notation of §7: $\mathbf{P}$ denotes a rational parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathcal{U}_\mathbf{P}$ and nilmanifold $N_P = \Gamma_\mathcal{U} \backslash \mathcal{U}_P$. The Borel-Serre boundary stratum corresponding to $\mathbf{P}$ is denoted Y_P and the nilmanifold fibration $\pi : Y_P \to X_P$ projects the Borel-Serre stratum to the reductive Borel-Serre stratum.

(25.2) Suppose $U \subset N_P$ is an open, connected, simply connected subset. Let $f : U \to \mathbb{R}$ be a smooth function. We will say that f is *polynomial* if for some (and hence for any) lift $\tilde{U} \subset \mathcal{U}_P$ of the open set U, the resulting map $\tilde{f} : \tilde{U} \to \mathbb{R}$ is the restriction to $\tilde{U}$ of a polynomial function $p \in A[\mathcal{U}_P]$ (with rational coefficients). The polynomial functions on N_P form a locally constant sheaf $\mathbf{A}[\mathcal{U}_P]$ of rational vectorspaces which is associated to the locally finite representation of $\boldsymbol{\mathcal{U}}_{\mathbf{P}}(\mathbb{Q})$ on $A[\mathcal{U}_P]$ induced by translation, i.e. $u.p(x) = p(u^{-1}x)$.

(25.3) We wish to extend this notion of polynomial function to obtain a sheaf of functions on the Borel Serre stratum Y_P, which are polynomial when restricted to each nilmanifold fiber of the fibration $\pi : Y_P \to X_P$. In order to do this we will insist that the functions should be locally constant with respect to the canonical integrable connection (§7.10) on Y_P and then we will check that the connection preserves polynomials with rational coefficients.

For this purpose we fix a basepoint $x_0 \in D$ so that the resulting lift $\mathbf{L}_{\mathbf{P}}(x_0)$ of the Levi factor of $\mathbf{P}$ is a rationally defined subgroup of $\mathbf{P}$. Our definition will use this choice of basepoint but the resulting sheaf of functions is independent of the choice of basepoint.

Let $F : \mathcal{U}_P \times D_P \to e_P$ denote the resulting diffeomorphism (§7.6). Let $V_{\mathcal{U}} \subset \mathcal{U}_P$ and $V_D \subset D_P$ be open subsets and let $V = F(V_{\mathcal{U}} \times V_D) \subset e_P$.

(25.4) Definition. A function $\tilde{f} : V \to \mathbb{R}$ is a locally flat $\mathcal{U}_P$-polynomial function if

(a) $\tilde{f} \circ F(u, z)$ is constant in z
(b) $\tilde{g}(u) = \tilde{f} \circ F(u, z)$ is the restriction to $V_{\mathcal{U}}$ of a polynomial with rational coefficients, $\tilde{g} : \mathcal{U}_P \to \mathbb{R}$.

(25.5) The locally flat $\mathcal{U}_P$-polynomial functions on e_P form a (trivial) sheaf, which is independent of the choice of basepoint x_0 (among all choices which give rise to rational lifts of the Levi factor) by §7.14: both properties (a) and (b) are preserved by change of basepoint. The action of the discrete group Γ_P on e_P takes locally flat $\mathcal{U}_P$-polynomial functions to locally flat $\mathcal{U}_P$-polynomial functions because (a) Γ_P preserves the flat connection (§7.11) and (b) the resulting action of Γ_P on functions $\tilde{g} \in A[\mathcal{U}_P]$ is given by the "chosen representation" (24.8.2), and this takes polynomials with rational coefficients to polynomials with rational coefficients. Let $\mu : e_P \to Y_P = \Gamma_P \backslash e_P$ denote the projection to the Borel-Serre stratum.

(25.6) Definition. The sheaf $\mathbf{A}[\mathcal{U}_P, Y_P]$ of locally flat, locally $\mathcal{U}_P$-polynomial functions on Y_P is the local system of rational vectorspaces whose sections over a sufficiently small ball $B \subset Y_P$ consists of functions $f : B \to \mathbb{R}$ such that for some (and hence for any) lift, $\tilde{B} \subset e_P$ of the ball B, the resulting function $\tilde{f} : \tilde{B} \to \mathbb{R}$ is a locally flat $\mathcal{U}_P$-polynomial function on $\tilde{B} \subset e_P$.

(25.7) The sheaf $\mathbf{A}[\mathcal{U}_P, Y_P]$ is well defined by (25.5) and does not depend on the choice of basepoint $x_0 \in D$. It is a locally constant sheaf and the pullback $\mu^*\mathbf{A}[\mathcal{U}_P, Y_P]$ is the sheaf of locally flat $\mathcal{U}_P$-polynomial functions on e_P. The sheaf $\mathbf{A}[\mathcal{U}_P, Y_P]$ has a unique extension, which we denote by $\mathbf{A}[\mathcal{U}_P, \bar{Y}_P]$, to the closure $\bar{Y}_P$ of Y_P in the Borel-Serre compactification $\tilde{X}$. This is because the closure $\bar{Y}_P$ is obtained by adding collared corners to the manifold Y_P.

(25.8) The choice of basepoint $x_0 \in D$ determines a "chosen" representation (24.8.2) of $\mathbf{P}(\mathbb{Q})$ on the group algebra $A[\mathscr{U}_P]$ and hence gives a local system of rational vectorspaces,

$$A[\mathscr{U}_P] \times_{\Gamma_P} e_P \tag{25.8.1}$$

on $Y_P = \Gamma_P \backslash e_P$. Different basepoints give different but isomorphic local systems. The basepoint x_0 also determines a trivialization (§7.6), $F : \mathscr{U}_P \times D_P \to e_P$ which allows one to identify the locally flat $\mathscr{U}_P$-polynomial functions on e_P with elements in $A[\mathscr{U}_P]$.

(25.9) Proposition. *The choice of basepoint x_0 determines an isomorphism between local systems*

$$h : \mathbf{A}[\mathscr{U}_P, Y_P] \cong A[\mathscr{U}_P] \times_{\Gamma_P} e_P.$$

(25.10) Proof. The proof is the same as that of Prop. 12.13. □

26. Locally flat $\mathscr{U}_P$ -polynomial functions on a geodesic neighborhood

(26.1) We continue with the notation of §25. Let $\tilde{U}_P \subset \tilde{X}$ be the collared geodesic neighborhood of the closure $\bar{Y}_P$ of the Borel-Serre boundary stratum Y_P which corresponds to a rational parabolic subgroup $\mathbf{P}$ (see Eq. (6.6.1)). The geodesic retraction $r_P : \tilde{U}_P \to \bar{Y}_P$ is a fibration whose fibers are diffeomorphic to a product of intervals, $[0, 1)^{\mathrm{rank}(P)}$. Let $U_P = \tilde{U}_P \cap X$.

(26.2) Definition. The sheaf $\mathbf{A}[\mathscr{U}_P, \tilde{U}_P]$ of locally flat, geodesic invariant, $\mathscr{U}_P$-polynomial functions on the geodesic neighborhood $\tilde{U}_P$ is the sheaf

$$\mathbf{A}[\mathscr{U}_P, \tilde{U}_P] = r_P^* \mathbf{A}[\mathscr{U}_P, \bar{Y}_P].$$

We denote by $\mathbf{A}[\mathscr{U}_P, U_P] = r^* \mathbf{A}[\mathscr{U}_P, Y_P] = \mathbf{A}[\mathscr{U}_P, \tilde{U}_P] | U_P$ its restriction to U_P.

(26.3) Remark. Thus, sections of this sheaf over a small open ball $V \subset \tilde{U}_P$ are functions $f : \tilde{U}_P \to \mathbb{R}$ which are invariant under the geodesic action of A_P, so $f = r_P^*(\bar{f})$ where $\bar{f}$ is a locally defined function on $\bar{Y}_P$ which is locally constant with respect to the flat connection, and whose restriction to the nilmanifold fibers is given locally by a polynomial with rational coefficients.

The neighborhood $\tilde{U}_P$ has a lot of structure. It is foliated by the pre-image $r_P^{-1}(N_x)$ of the nilmanifolds $N_x = \pi^{-1}(x) \subset Y_P$, and also by the pre-image $r_P^{-1}(H)$ of the horizontal submanifolds $H \subset Y_P$ which are determined by the flat connection. The locally flat geodesic invariant $\mathscr{U}_P$-polynomial functions are functions which are polynomial along the leaves of the first foliation, and constant along the leaves of the second foliation.

27. Gluing the sheaves $\mathbf{A}[\mathcal{U}_P, \tilde{U}_P]$

(27.1) In this section we will show how to glue the sheaves $\mathbf{A}[\mathcal{U}_P, \tilde{U}_P]$ together on the various geodesic neighborhoods, in order to construct a single sheaf $\tilde{\mathbf{A}}$ of rational vectorspaces, on the space $\tilde{X}$. Then we will show how to resolve any local system of finite dimensional vectorspaces $\mathbf{E}$ on X by tensoring with the sheaf $\mathbf{A} = \tilde{\mathbf{A}}|X$.

(27.2) Glue. Suppose Y is an *open* subset of a topological space X, with inclusion $i : Y \to X$. Let $\mathbf{S}$ be a sheaf on X and let $\mathbf{R}$ be a sheaf on Y. Suppose there is an injective sheaf morphism, $\beta : \mathbf{S}|Y \hookrightarrow \mathbf{R}$.

Definition. The sheaf union, $\mathbf{S} \cup \mathbf{R}$ is the sheafification of the presheaf $\mathbf{T}$ whose sections over an open subset $U \subset X$ are given by

$$\Gamma(U, \mathbf{T}) = \begin{cases} \Gamma(U, \mathbf{R}) & \text{if} \quad U \subset Y \\ \Gamma(U, \mathbf{S}) & \text{if} \quad U \not\subset Y \end{cases}$$

There are canonical injective sheaf maps, $i_! i^* \mathbf{S} \xrightarrow{i_! \beta} i_! \mathbf{R} \hookrightarrow \mathbf{S} \cup \mathbf{R}$. Let $\epsilon : i_! i^* \mathbf{S} \to \mathbf{S}$ be the canonical morphism and let $\alpha = i_! \beta \oplus \epsilon$. We obtain a short exact sequence of sheaves on X,

$$0 \to i_! i^* \mathbf{S} \xrightarrow{\alpha} i_! \mathbf{R} \oplus \mathbf{S} \to \mathbf{S} \cup \mathbf{R} \to 0$$

which identifies the sheaf union as the cokernel of α.

(27.3) Let us fix rational parabolic subgroups $\mathbf{P} \subset \mathbf{Q}$ of $\mathbf{G}$, and a collection of geodesic neighborhoods of the boundary strata in the Borel-Serre compactification $\tilde{X}$ of X. Then $\tilde{U}_P \subset \tilde{U}_Q$. The sheaves $\mathbf{A}[\mathcal{U}_P, \tilde{U}_P]$ and $\mathbf{A}[\mathcal{U}_Q, \tilde{U}_Q]|\tilde{U}_P$ are both sub-sheaves of the sheaf of real valued smooth functions defined on the geodesic neighborhood $\tilde{U}_P$.

(27.4) Proposition. *The restriction* $\mathbf{A}[\mathcal{U}_Q, \tilde{U}_Q]|\tilde{U}_P$ *is a sub-sheaf of the sheaf* $\mathbf{A}[\mathcal{U}_P, \tilde{U}_P]$.

Proof. The basepoint x_0 determines (see Prop. 25.9) an isomorphism

$$h : \mathbf{A}[\mathcal{U}_P, \tilde{U}_P] \cong r_P^*(A[\mathcal{U}_P] \times_{\Gamma_P} e_P) \tag{27.4.1}$$

between the sheaf of sections of the local system (25.8.1) with the sheaf $\mathbf{A}[\mathcal{U}_P, \tilde{U}_P]$. It also determines an isomorphism

$$h : \mathbf{A}[\mathcal{U}_Q, \tilde{U}_Q]|\tilde{U}_P \cong r_P^*(A[\mathcal{U}_Q] \times_{\Gamma_P} e_P). \tag{27.4.2}$$

The "chosen embedding" (§24.11) $\rho^* : A[\mathcal{U}_Q] \to A[\mathcal{U}_P]$ identifies (27.4.2) as a sub-local system of (27.4.1). It is, however, necessary to check that the resulting embedding

$$\mathbf{A}[\mathcal{U}_Q, \tilde{U}_Q]|\tilde{U}_P \hookrightarrow \mathbf{A}[\mathcal{U}_P, \tilde{U}_P]$$

does not depend on the choice of basepoint, and is in fact given by restriction of functions. This is a tedious verification and we will leave it until §27.11 to §27.16.

(27.5) Let $\mathbb{Q}_{\tilde{X}}$ denote the ("constant") sheaf on $\tilde{X}$ of locally constant functions on $\tilde{X}$ with values in the rational numbers. A constant function obviously satisfies the

invariance requirements for functions $f \in \mathbf{A}[\mathcal{U}_Q, \tilde{U}_Q]$, so the constant sheaf is a subsheaf of each of the sheaves $\mathbf{A}[\mathcal{U}_Q, \tilde{U}_Q]$. It follows that the *union* of the sheaves $\mathbf{A}[\mathcal{U}_Q, \tilde{U}_Q]$ forms a single sheaf on the space $\tilde{X}$.

(27.6) Definition. The sheaf of (locally) geodesic invariant, (locally) flat, (locally) $\mathcal{U}$-polynomial functions $\tilde{\mathbf{A}}$ on the Borel-Serre compactification $\tilde{X}$ is the union

$$\tilde{\mathbf{A}} = \mathbb{Q}_{\tilde{X}} \cup \bigcup \mathbf{A}[\mathcal{U}_P, \tilde{U}_P]$$

of the constant sheaf with the sheaves $\mathbf{A}[\mathcal{U}_P, \tilde{U}_P]$ which are defined on the various geodesic neighborhoods. The sheaf $\mathbf{A} = \tilde{\mathbf{A}}|X$ is the restriction of this sheaf to X.

(27.7) Remark. If $j : X \to \tilde{X}$ is the inclusion, then $\tilde{\mathbf{A}} = j_*(\mathbf{A})$ because the same is true for each $\mathbf{A}[\mathcal{U}_P, \tilde{U}_P]$.

(27.8) Suppose E is a rational representation of $\mathbf{G}(\mathbb{Q})$. Let $\mathbf{E} = E \times_\Gamma D$ denote the corresponding local system of rational vectorspaces on the locally symmetric space $X = \Gamma \backslash D$. The inclusion of the constant sheaf $\mathbb{Q} \hookrightarrow \mathbf{A}$ induces a sheaf map,

$$\partial^{-1} : \mathbf{E} \to \mathbf{E} \otimes \mathbf{A}$$

Let $\mathbf{C}^0$ denote the coker sheaf, and let ∂^0 be the composition,

$$\mathbf{E} \otimes \mathbf{A} \to \mathbf{C}^0 \to \mathbf{C}^0 \otimes \mathbf{A}.$$

Continuing in this way, we obtain a resolution of $\mathbf{E}$,

$$\mathbf{E} \otimes \mathbf{A} \xrightarrow{\partial^0} \mathbf{C}^0 \otimes \mathbf{A} \xrightarrow{\partial^1} \mathbf{C}^1 \otimes \mathbf{A} \longrightarrow \dots \qquad (27.8.1)$$

where $\mathbf{C}^j = \operatorname{coker}(\partial^{j-1})$.

(27.9) Definition. The complex of sheaves $\mathbf{I}^\bullet(\mathbf{E})$ on the space X, is the above resolution (27.8.1),

$$\mathbf{I}^j(\mathbf{E}) = \mathbf{C}^{j-1} \otimes \mathbf{A}$$

of the sheaf $\mathbf{E}$, and we will call it the *standard rational resolution of* $\mathbf{E}$.

(27.10) Remark.

1. The resolution $\mathbf{I}^\bullet$ does not depend on the choice of basepoint, and the quasi-isomorphism class of $\mathbf{I}^\bullet$ does not depend on the choice of geodesic neighborhoods.
2. The local system $\mathbf{E}$ has a unique extension $\tilde{\mathbf{E}}$ to $\tilde{X}$ and hence the standard rational resolution has a canonical extension $\tilde{\mathbf{I}}^\bullet(\tilde{\mathbf{E}})$ to $\tilde{X}$ which is obtained by using the sheaf $\tilde{\mathbf{A}}$ instead of $\mathbf{A}$ in the above construction. If $j : X \to \tilde{X}$ denotes the inclusion, then by (27.7),
$$\tilde{\mathbf{I}}^\bullet(\tilde{\mathbf{E}}) = j_*(\mathbf{I}^\bullet(\mathbf{E})).$$
3. Since the sheaf $\mathbf{A}$ (respectively, $\tilde{\mathbf{A}}$) is a union of local systems $\mathbf{A}[\mathcal{U}_Q, U_Q]$ (resp. $\mathbf{A}[\mathcal{U}_Q, \tilde{U}_Q]$), the resolution $\mathbf{I}^\bullet(\mathbf{E})$ (resp. $\tilde{\mathbf{I}}^\bullet(\tilde{\mathbf{E}})$) is a union of complexes of local systems, $\mathbf{I}^k_Q(\mathbf{E}) = \tilde{\mathbf{I}}^k_Q(\tilde{\mathbf{E}})|U_Q$. Each $\tilde{\mathbf{I}}^k_Q(\tilde{\mathbf{E}})$ is defined on the geodesic neighborhood $\tilde{U}_Q \cong \tilde{Y}_Q \times [0,1)^{\operatorname{rank}(Q)}$. By (25.9), the choice of basepoint $x_0 \in D$ determines an isomorphism between the complex $\tilde{\mathbf{I}}^\bullet_Q(\tilde{\mathbf{E}})$ and the local system on $\tilde{U}_Q$ which

corresponds to the following complex of rational representations of the parabolic subgroup Q,

$$E \xrightarrow{\partial^{-1}} E \otimes A[\mathcal{U}_Q] \xrightarrow{\partial^0} \operatorname{coker}(\partial^{-1}) \otimes A[\mathcal{U}_Q] \xrightarrow{\partial^1} \dots \qquad (27.10.1)$$

4. If $\mathbf{P} \subset \mathbf{Q}$ then the local system $\tilde{\mathbf{I}}^k_Q(\tilde{\mathbf{E}})$ is, locally near Y_P, equal to the pullback via the geodesic retraction, of the restriction $\tilde{\mathbf{I}}^k_Q(\tilde{\mathbf{E}})|Y_P$. This follows from Prop. 27.4.

(27.11) The rest of §27 consists of the proof of Prop. 27.4. We shall show that $\mathbf{A}[\mathcal{U}_Q, U_Q]|U_P$ is a subsheaf of $\mathbf{A}[\mathcal{U}_P, U_P]$, but the same method applies to the extended sheaves $\mathbf{A}[\mathcal{U}_Q, \tilde{U}_Q]|\tilde{U}_P$. Suppose $B \subset U_P$ is a sufficiently small ball, and that $f : B \to \mathbb{R}$ is a section, $f \in \Gamma(B, \mathbf{A}[\mathcal{U}_Q, U_Q])$. Since the function f is invariant under the A_Q-geodesic action, we may suppose it is given by a function, which we also denote by f, which is defined in a neighborhood of some point in the Borel-Serre boundary stratum Y_Q. As the question is local, we can lift this neighborhood into the Borel-Serre boundary component e_Q. Then f is a locally defined function on e_Q and we need to check the following three invariance properties with respect to the parabolic subgroup $\mathbf{P}$:

1. f is invariant under the geodesic action of A_P and is hence the pullback $f = r_P^*(\bar{f})$ of a locally defined function on Y_P.
2. $\bar{f}$ is locally flat with respect to the canonical connection on $\pi : Y_P \to X_P$.
3. $\bar{f}$ is polynomial when restricted to the fibers of the fibration π.

(27.12) Properties (1), (2), and (3) above are independent of the choice of basepoint x_0, however we need to choose a basepoint in order to verify them. Choose the basepoint so as to be rational for the parabolic subgroups $\mathbf{P} \subset \mathbf{Q}$. We refer to §24.9 and §24.10 for the definition of the diffeomorphism, $H : \mathcal{U}_Q \times \mathcal{U}_{\bar{P}} \to \mathcal{U}_P$. The geodesic projection $\tau : D_Q \to \mathcal{U}_{\bar{P}} \times D_P$ may be expressed as follows. Set

$$D_Q = L_Q/K_Q A_Q = \bar{P}/K_{\bar{P}}(x_0)A_Q(x_0)$$

and $D_P = L_P/K_P(x_0)A_P(x_0)$. Any element $s \in \bar{P}$ may be written $s = uj_{x_0}(l)$, where $u \in \mathcal{U}_{\bar{P}}$ and $l \in L_P$. Then

$$\tau(uj_{x_0}(l)K_{\bar{P}}(x_0)A_Q) = (u, lK_P A_P)$$

which is easily seen to be well defined.

(27.13) Lemma. *The following diagram commutes:*

$$\begin{array}{ccccc}
\mathcal{U}_Q \times D_Q & & \xrightarrow[F_Q]{\cong} & & e_Q \\
\Big\downarrow{\scriptstyle I \times \tau} & & & & \Big\downarrow{\scriptstyle r} \\
\mathcal{U}_Q \times \mathcal{U}_{\bar{P}} \times D_P & \xrightarrow[H \times I]{\cong} & \mathcal{U}_P \times D_P & \xrightarrow[F_P]{\cong} & e_P
\end{array}$$

where r is the geodesic projection from the Borel-Serre boundary component for Q to the Borel-Serre boundary component for P.

Proof. The proof is a computation with double cosets. □

(27.14) Proof of property (1). The function f is the pullback via the geodesic retraction of a locally defined function (which we also denote by f) on the Borel-Serre boundary component $e_Q \cong \mathcal{U}_Q \times D_Q$. It is, by assumption, constant in the D_Q-variables. However the geodesic action of A_P on e_Q is given by the projection τ, which affects only the D_Q-variables (see diag. 27.13). Thus f is invariant under the geodesic action of A_P, and it passes to a locally defined function $\tilde{f}$ on e_P. □

(27.15) Proof of property (2). We consider $\tilde{f}$ to be a locally defined function on $e_P \cong \mathcal{U}_P \times D_P$ and it will be "flat" if it is constant in the D_P variables. In fact, it is constant in the $\mathcal{U}_{\bar{P}} \times D_P$ variables (see diag. 27.13). □

(27.16) Proof of property (3). We need to check that $\tilde{f}$ is locally a polynomial (with rational coefficients) in $\mathcal{U}_P$. In other words, we have to check that a function $\tilde{f}$ which is defined on the group

$$\mathcal{U}_P \xleftarrow[H]{\cong} \mathcal{U}_Q \times \mathcal{U}_{\bar{P}} \tag{27.16.1}$$

which is polynomial on $\mathcal{U}_Q$ and which is constant on $\mathcal{U}_{\bar{P}}$, is a polynomial on $\mathcal{U}_P$. If $g = \tilde{f}|\mathcal{U}_Q$, then $\tilde{f} = \rho^*(g)$, where $\rho = \mathrm{pr}_1 \circ H^{-1} : \mathcal{U}_P \to \mathcal{U}_Q$ is the chosen projection (§24.9). Since ρ is a polynomial mapping with rational coefficients (Prop. 24.10), the function $\tilde{f}$ is also. □

28. Weighted cohomology over $\mathbb{Q}$

(28.1) We use the notation of §27 and fix a collection $\{U_P = \tilde{U}_P \cap X\}$ of geodesic neighborhoods of the strata in the Borel-Serre compactification of X. Let E be a rational representation of $\mathbf{G}(\mathbb{Q})$ and let $\mathbf{I}^\bullet(\mathbf{E})$ be the standard rational resolution (§27.9) of the associated local system of rational vectorspaces $\mathbf{E}$ on $X = \Gamma\backslash D$. Let $\bar{X}$ denote the reductive Borel-Serre compactification with inclusion $i : X \to \bar{X}$. Let $\mathbf{Q}$ be a rational parabolic subgroup of $\mathbf{G}$. For each rational parabolic subgroup $\mathbf{P} \subset \mathbf{Q}$ it is possible to find a basepoint $x_0 \in D$ so that the lifts $\mathbf{S_Q}(x_0) \subset \mathbf{L_P}(x_0) \subset \mathbf{L_Q}(x_0)$ are rationally defined algebraic groups. Hence the torus $\mathbf{S_Q}(x_0)(\mathbb{Q})$ acts on the polynomial functions with rational coefficients, $A[\mathcal{U}_P]$. Let $p = \{\chi(S_Q)_+\}$ denote a weight profile.

(28.2) Proposition. *The action of the torus $\mathbf{S_Q}(x_0)(\mathbb{Q})$ on the functions $A[\mathcal{U}_P]$ (where $\mathbf{P} \subseteq \mathbf{Q}$) determines a semisimple action of $\mathbf{S_Q}$ on each of the sheaves*

$$\mathbf{J}^k_Q = i_*\mathbf{I}^k(\mathbf{E})|X_Q$$

which decomposes into a sum of sub-sheaves,

$$\mathbf{J}^k_Q = \bigoplus_{\lambda \in \chi(\mathbf{S_Q})} (J^k_Q)_\lambda.$$

The resulting subsheaf,

$$(\mathbf{J}^k_Q)_+ = \bigoplus_{\lambda \in \chi(\mathbf{S_Q})_+} (\mathbf{J}^k_Q)_\lambda$$

is independent of the choice of basepoint x_0.

(28.3) It is easy to check that the collection of subsheaves $(\mathbf{J}^k_Q)_+ \subset i_*\mathbf{I}^k(\mathbf{E})|X_Q$ satisfy the compatibility condition (2.2). Hence there is a unique subsheaf $\mathbf{W}^p\mathbf{I}^\bullet(\mathbf{E}) \subset i_*\mathbf{I}^k(\mathbf{E})$ such that, for each boundary stratum X_Q we have:

$$\mathbf{W}^p\mathbf{I}^\bullet(\mathbf{E})|X_Q = (\mathbf{J}^k_Q)_+ \tag{28.3.1}$$

(28.4) Definition. The rational weighted cohomology complex of sheaves,

$$\mathbf{W}^p\mathbf{I}^\bullet(\mathbf{E})$$

is the complex of sheaves which is obtained from the complex $i_*\mathbf{I}^\bullet(\mathbf{E})$ by truncating each $i_*\mathbf{I}^k(\mathbf{E})$ to the subsheaf $(\mathbf{J}^k_Q)_+$ along the singular stratum X_Q. The rational weighted cohomology group $W^pH^k(\bar{X}, \mathbf{E})$ is the hypercohomology group of this complex of sheaves.

(28.5) *Caution:* The sheaves $\mathbf{I}^k(\mathbf{E}), i_*\mathbf{I}^k(\mathbf{E}), \mathbf{W}^p\mathbf{I}^k(\mathbf{E})$ are not fine. Their hypercohomology groups cannot be computed as the cohomology of the complex of global sections.

The rest of §28 is devoted to the proof of Prop. 28.2.

(28.6) Each sheaf $\tilde{\mathbf{I}}^k(\tilde{\mathbf{E}}) = \tilde{\mathbf{C}}^{k-1} \otimes \tilde{\mathbf{A}}$ is a union of local systems, $\tilde{\mathbf{I}}^k_Q(\tilde{\mathbf{E}}) = \tilde{\mathbf{C}}^{k-1}_Q \otimes \mathbf{A}[\mathscr{U}_Q, \tilde{U}_Q]$ which are defined on geodesic neighborhoods $\tilde{U}_Q$ and which are glued together via canonical, inductively defined embeddings,

$$\tilde{\alpha}_{QP} : \tilde{\mathbf{I}}^k_Q(\tilde{\mathbf{E}})|\tilde{U}_P \hookrightarrow \tilde{\mathbf{I}}^k_P(\tilde{\mathbf{E}}) \tag{28.6.1}$$

whenever $\mathbf{P} \subset \mathbf{Q}$ (see Prop. 27.4). It follows that the same holds for the sheaf $i^*_Q i_*\mathbf{I}^k(\mathbf{E})$, in other words,

$$i^*_Q i_*\mathbf{I}^k(\mathbf{E}) = \bigcup_{\mathbf{P}\subseteq\mathbf{Q}} i^*_Q i_*\mathbf{I}^k_P(\mathbf{E}) \tag{28.6.2}$$

where $i^*_Q i_*\mathbf{I}^k_P(\mathbf{E})$ is supported on the geodesic neighborhood $\bar{U}_P \cap X_Q$ of X_P in X_Q, and it is glued to the local system $i^*_Q i_*\mathbf{I}^k_Q(\mathbf{E})$ via the embedding $i^*_Q i_*(\alpha_{QP})$.

(28.7) We must show that

1. each of these local systems admits a decomposition by S_Q-weights,
2. these decompositions are compatible with respect to the embedding α_{PQ}, and
3. the subsheaf $(\mathbf{J}^k_Q)_+$ is independent of the choice of basepoint.

If $\mathbf{P} \subseteq \mathbf{Q}$ are rational parabolic subgroups of $\mathbf{G}$, denote by $\nu : \mathbf{Q} \to \mathbf{Q}/\mathscr{U}_\mathbf{Q}$ the projection to the Levi quotient, $\bar{\mathbf{P}} = \nu(\mathbf{P})$, and $\Gamma_{\bar{P}} = \nu(\Gamma_P)$. The intersection $\bar{U}_P \cap X_Q$ is a geodesic neighborhood of X_P in X_Q. (For $\mathbf{P} = \mathbf{Q}$ we have $\bar{\mathbf{Q}} = \mathbf{L}_\mathbf{Q}$ and $\bar{U}_Q \cap X_Q = X_Q$.)

(28.8) Lemma. *Let $\mathbf{P} \subseteq \mathbf{Q}$ be rational parabolic subgroups. The $A_{\bar{P}}$-geodesic invariant local systems on the geodesic neighborhood $\bar{U}_P \cap X_Q$ of X_P in X_Q are in canonical one to one correspondence with representations of the group $\Gamma_{\bar{P}} = \nu(\Gamma_P)$.*

(28.9) Proof. Let $\tilde{X}_Q$ denote the *Borel-Serre* compactification of the boundary stratum X_Q. The rational parabolic subgroup $\mathbf{P} \subset \mathbf{Q}$ corresponds to a boundary stratum,

$$Y_{\bar{P}} = \Gamma_{\bar{P}} \backslash \bar{P} / K_{\bar{P}} A_{\bar{P}} = \Gamma_{\bar{P}} \backslash e_P / \mathcal{U}_P.$$

The $A_{\bar{P}}$-geodesic projection defines a homeomorphism, $\bar{U}_P \cap X_Q \cong Y_{\bar{P}} \times [0,1)^{\mathrm{rank}(\bar{P})}$ so a geodesic-invariant local system on $\bar{U}_P \cap X_Q$ is the pullback of a local system on $Y_{\bar{P}}$, which is given by a representation of the covering group, $\Gamma_{\bar{P}}$. □

(28.10) Proposition. *Let* $\mathbf{P} \subseteq \mathbf{Q}$ *be rational parabolic subgroups. The choice of basepoint* $x_0 \in D$ *determines an isomorphism between the local systems* $i_Q^* i_* \mathbf{I}_P^k(\mathbf{E})$ *on the geodesic neighborhood* $\bar{U}_P \cap X_Q$*, and the geodesic-invariant local system which corresponds to the rational representation* $(I_P^k(E))^{\mathcal{U}_Q}$ *of the parabolic subgroup* $\bar{\mathbf{P}} \subset \mathbf{L}_\mathbf{Q}$*. (Here we include the possibility that* $\mathbf{P} = \mathbf{Q}$*.)*

(28.11) Proof. The problem is that the sheaf $i_Q^* i_* \mathbf{I}_P^k(\mathbf{E})$ on $\bar{U}_P \cap X_Q$ is *not* the pullback, via the geodesic retraction, of a local system on the stratum X_P. However, it is the pullback of a local system on the boundary stratum $Y_{\bar{P}}$ of the Borel-Serre compactification of X_Q.

The inclusion $i : X \to \bar{X}$ factors through the Borel-Serre compactification, $j : X \to \tilde{X}$. Let us denote the inclusions of boundary strata by $j_Q : Y_Q \to \tilde{X}$ and $i_Q : X_Q \to \bar{X}$, so we have a diagram of spaces and inclusions,

$$\begin{array}{ccccc} X & \xrightarrow[j]{} & \tilde{X} & \xleftarrow[j_Q]{} & Y_Q \\ \| & & \pi \big\downarrow & & \pi \big\downarrow \\ X & \xrightarrow[i]{} & \bar{X} & \xleftarrow[i_Q]{} & X_Q \end{array} \qquad (28.11.1)$$

The geodesic neighborhoods are chosen (§6.4, §6.6) so that $\tilde{U}_Q = \pi^{-1}\pi(\tilde{U}_Q)$ is saturated by the fibers of π. For each parabolic subgroup $\mathbf{P} \subseteq \mathbf{Q}$, the local system $\tilde{\mathbf{I}}_P^k(\tilde{\mathbf{E}})$ on $\tilde{U}_P$ is the pullback via the A_P−geodesic retraction $r_P : \tilde{U}_P \to \tilde{Y}_P$ of a local system $\tilde{\mathbf{I}}_P^k(\tilde{\mathbf{E}})_Y$ on $\tilde{Y}_P$. The local system

$$\begin{aligned} i_Q^* i_* \mathbf{I}_P^k(\mathbf{E}) &= i_Q^* \pi_* j_* \mathbf{I}_P^k(\mathbf{E}) \\ &= \pi_* j_Q^* \tilde{\mathbf{I}}_P^k(\tilde{\mathbf{E}}) \end{aligned} \qquad (28.11.2)$$

is the pushdown by $\pi : Y_Q \to X_Q$ of a local system on Y_Q. Consider the following commutative diagram of neighborhoods and geodesic retractions, where $\pi = \beta \circ \alpha$,

$$\begin{array}{ccccccc} \tilde{Y}_Q & \hookleftarrow & Y_Q \cap \tilde{U}_P & \xrightarrow[\tilde{r}_P]{} & Y_P & = & \Gamma_P \backslash e_P \\ \big\downarrow \alpha & & \big\downarrow & & \big\downarrow \alpha & & \\ \tilde{X}_Q & \hookleftarrow & X_Q \cap \bar{U}_P & \xrightarrow[r_{\bar{P}}]{} & Y_{\bar{P}} & = & \Gamma_{\bar{P}} \backslash e_P / \mathcal{U}_Q \\ \big\downarrow \beta & & \| & & \big\downarrow \beta & & \\ \bar{X}_Q & \hookleftarrow & X_Q \cap \bar{U}_P & \xrightarrow[\bar{r}_P]{} & X_P & = & \Gamma_L \backslash e_P / \mathcal{U}_P \end{array} \qquad (28.11.3)$$

The choice of basepoint determines an isomorphism between local system $\tilde{\mathbf{I}}^k_P(\tilde{\mathbf{E}})_Y$ and the local system $I^k_P(E) \times_{\Gamma_P} e_P$ which arises from the "chosen" representation of $\mathbf{P}(\mathbb{Q})$ on $A[\mathcal{U}_P]$. So,

$$\begin{aligned} \pi_* j_Q^* \tilde{\mathbf{I}}^k_P(\tilde{\mathbf{E}}) &\cong \pi_* \tilde{r}_P^*(I^k_P(E) \times_{\Gamma_P} e_P) \qquad (28.11.4)\\ &= \alpha_* \tilde{r}_P^*(I^k_P(E) \times_{\Gamma_P} e_P)\\ &= r_{\bar{P}}^* \alpha_*(I^k_P(E) \times_{\Gamma_P} e_P)\\ &= r_{\bar{P}}^*((I^k_P(E))^{\mathcal{U}_Q} \times_{\Gamma_{\bar{P}}} (e_P/\mathcal{U}_Q)) \end{aligned}$$

which is the pullback via the geodesic retraction of the local system which corresponds to the $\bar{\mathbf{P}}(\mathbb{Q})$ representation on the $\mathcal{U}_{\mathbf{Q}}(\mathbb{Q})$-invariants in $I^k_P(E)$ as claimed. □

(28.12) Proof of Proposition 28.2. The torus $\mathbf{S_Q}$ is in the center of $\bar{\mathbf{P}}$. Therefore the local system on $Y_{\bar{P}}$,

$$(I^k_P(E))^{\mathcal{U}_Q} \times_{\Gamma_{\bar{P}}} (e_P/\mathcal{U}_P)$$

breaks into a sum of local systems,

$$\bigoplus_{\lambda \in \chi(\mathbf{S_Q})} (I^k_P(E))^{\mathcal{U}_Q}_{\lambda} \times_{\Gamma_{\bar{P}}} (e_P/\mathcal{U}_P)$$

This proves statement (1) of §28.7. The compatibility between these weight-isotypical sub local systems for different $P \subseteq Q$ is a consequence of Cor. 24.11, which proves (2) of §28.7.

A change of basepoint modifies the isomorphism (28.11.4) by $\mathcal{U}_{\mathbf{P}}(\mathbb{Q})$-conjugacy, so just as in Prop. 10.4, the weight *filtration* is unchanged. This proves statement (3) of §28.7 and completes the proof of Prop. 28.2 □

29. International agreement

(29.1) Let E be a rational representation of $\mathbf{G}(\mathbb{Q})$ and let p be a weight profile. We now have two constructions of the weighted cohomology groups: as the hypercohomology of the complex of sheaves, $\mathbf{W}^p\mathbf{I}^\bullet(\mathbf{E})$ (which gives the rational weighted cohomology groups) and as the hypercohomology of the complex of sheaves $\mathbf{W}^p\mathbf{C}^\bullet(\bar{X}, \mathbf{E} \otimes \mathbb{C})$ (which gives the complex weighted cohomology groups).

(29.2) Theorem. *The inclusion* $\mathbf{E} \otimes \mathbb{C} \to \boldsymbol{\Omega}^0(X, \mathbf{E} \otimes \mathbb{C})$ *of sheaves on* X *induces a quasi-isomorphism*

$$\mathbf{W}^p\mathbf{I}^\bullet(\mathbf{E}) \otimes \mathbb{C} \xrightarrow{\sim} \mathbf{W}^p\mathbf{C}^\bullet(\bar{X}, \mathbf{E} \otimes \mathbb{C})$$

of sheaves on $\bar{X}$*, and hence they have the same hypercohomology.*

The rest of this paper consists of a proof of Theorem 29.2.

30. Double complexes of sheaves

(30.1) Suppose $(\mathbf{A}^{pq}, \partial', \partial'')$ is a first quadrant double complex of sheaves of complex vectorspaces on a topological space X. This means that $\mathbf{A}^{pq} = 0$ for $p < 0$ or $q < 0$, that $\partial' : \mathbf{A}^{pq} \to \mathbf{A}^{p+1,q}$ and $\partial'' : \mathbf{A}^{pq} \to \mathbf{A}^{p,q+1}$ satisfy $\partial'\partial' = \partial''\partial'' = 0$ and $\partial'\partial'' = \partial''\partial'$. The hypercohomology of this double complex is defined to be the hypercohomology of the associated single complex, $\mathbf{B}^j = \oplus_{p+q=j}\mathbf{A}^{pq}$ with differential $d|\mathbf{A}^{pq} = \partial' \oplus (-1)^p\partial''$. Consider the kernel complex, $(\mathbf{K}^q, \partial'') = \mathbf{ker}(\partial' : \mathbf{A}^{0q} \to \mathbf{A}^{1q})$. The following result is standard:

(30.2) Proposition. *Suppose that, for each point $x \in X$, and for all $q \geq 0$ the following sequence of stalks is exact:*

$$0 \to \mathbf{K}^q_x \to \mathbf{A}^{0q}_x \to \mathbf{A}^{1q}_x \to \dots$$

Then the sheaf map $\mathbf{K}^\bullet \to \mathbf{B}^\bullet = \oplus_{p+q=\bullet}\mathbf{A}^{pq}$ is a quasi-isomorphism. □

31. Special differential forms with coefficients in the standard resolution

(31.1) In this section we will make sense of the double complex of sheaves

$$\bar{\boldsymbol{\Omega}}^\bullet_{\mathrm{sp}}(\bar{X}, \mathbf{I}^\bullet(\mathbf{E}) \otimes \mathbb{C})$$

of special differential forms on the reductive Borel-Serre compactification, with coefficients in the standard resolution $\mathbf{I}^\bullet(\mathbf{E}) \otimes \mathbb{C}$ of the local system $\mathbf{E}$. We would like to thank J. Steenbrink for suggesting (during breakfast at the Motives conference) that we might use a double complex of this sort in order to compare our two constructions of weighted cohomology.

(31.2) As in §6.6 and §26.1 we fix a compatible collection $\{\tilde{U}_Q\}$ of geodesic neighborhoods of the Borel-Serre boundary strata in $\tilde{X}$. Set $U_Q = \tilde{U}_Q \cap X$ and $\bar{U}_Q = \pi(\tilde{U}_Q) \subset \bar{X}$. Each sheaf $\mathbf{I}^k(\mathbf{E})$ is a union of local systems, $\mathbf{I}^k_Q(\mathbf{E})$ defined on the neighborhood U_Q of the stratum $X_Q \subset \bar{X}$. It is the restriction to U_Q of a local system $\tilde{\mathbf{I}}^k_Q(\mathbf{E})$ on $\tilde{U}_Q$ and it is an inductive limit of finite dimensional local systems.

(31.3) A differential form ω on U_Q with values in $\mathbf{I}^k_Q(\mathbf{E}) \otimes \mathbb{C}$ is *special* if, for any rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ (with the property that $U_P \cap U_Q \neq \phi$), there is a neighborhood of the Borel-Serre stratum Y_P in $\tilde{X}$ such that within this neighborhood, (1) the form ω is the pullback of a form $\omega_P \in \Omega^\bullet(Y_P, \mathbf{I}^k_Q(\mathbf{E}) \otimes \mathbb{C})$ via the geodesic action, and (2) the form ω_P is $\mathcal{U}_P$-invariant. Let $\boldsymbol{\Omega}^\bullet_{\mathrm{sp}}(U_P, \mathbf{I}^k_Q(\mathbf{E}) \otimes \mathbb{C})$ denote the presheaf of special differential forms on U_Q.

(31.4) Let $i_Q : U_Q \hookrightarrow \bar{U}_Q = \pi(\tilde{U}_Q) \subset \bar{X}$ denote the inclusion. Define the sheaf,

$$\bar{\boldsymbol{\Omega}}^j_{\mathrm{sp}}(\bar{U}_Q, \mathbf{I}^k_Q(\mathbf{E}) \otimes \mathbb{C}) = \mathrm{Sh}(i_{Q*}\boldsymbol{\Omega}^j_{\mathrm{sp}}(U_Q, \mathbf{I}^k_Q(\mathbf{E}))) \qquad (31.4.1)$$

on $\bar{U}_Q$, to be the sheafification of the direct image presheaf. The differentials $\partial : \mathbf{I}^k_Q(\mathbf{E}) \to \mathbf{I}^{k+1}_Q(\mathbf{E})$ and $d : \boldsymbol{\Omega}^j \to \boldsymbol{\Omega}^{j+1}$ make the sheaves (31.4.1) into a double complex of sheaves on $\bar{U}_Q$.

(31.5) It is tedious but straightforward to check that if $\mathbf{P} \subset \mathbf{Q}$ are rational parabolic subgroups, then the injection (27.4, 28.6.1) of local systems, $\mathbf{I}^k_Q(\mathbf{E})|U_P \to \mathbf{I}^k_P(\mathbf{E})$ induces an injection of sheaves,

$$\bar{\boldsymbol{\Omega}}^j_{\mathrm{sp}}(\bar{U}_Q, \mathbf{I}^k_Q(\mathbf{E}) \otimes \mathbb{C})|\bar{U}_P \hookrightarrow \bar{\boldsymbol{\Omega}}^j_{\mathrm{sp}}(\bar{U}_P, \mathbf{I}^k_P(\mathbf{E}) \otimes \mathbb{C}) \tag{31.5.1}$$

on $\bar{U}_P$, which commutes with the differentials of the double complex. We include the case $\mathbf{Q} = \mathbf{G}$ for which we have an injection of sheaves,

$$\bar{\boldsymbol{\Omega}}^j_{\mathrm{sp}}(\bar{X}, \mathbf{E} \otimes \mathbb{C}) \hookrightarrow \bar{\boldsymbol{\Omega}}^j_{\mathrm{sp}}(\bar{U}_P, \mathbf{I}^0_P(\mathbf{E}) \otimes \mathbb{C}). \tag{31.5.2}$$

It follows that the union (27.2) of the sheaves (31.4.1) form a double complex of sheaves on $\bar{X}$:

(31.6) Definition. The double complex of sheaves $\bar{\boldsymbol{\Omega}}^\bullet_{\mathrm{sp}}(\bar{X}, \mathbf{I}^\bullet(\mathbf{E}) \otimes \mathbb{C})$ is the union,

$$\bar{\boldsymbol{\Omega}}^\bullet_{\mathrm{sp}}(\bar{X}, \mathbf{I}^\bullet(\mathbf{E}) \otimes \mathbb{C}) = \bar{\boldsymbol{\Omega}}^\bullet_{\mathrm{sp}}(\bar{X}, \mathbf{E} \otimes \mathbb{C}) \cup \bigcup_{X_P} \bar{\boldsymbol{\Omega}}^\bullet_{\mathrm{sp}}(\bar{U}_P, \mathbf{I}^\bullet_P(\mathbf{E}) \otimes \mathbb{C})$$

with respect to the injections (31.5.1) and (31.5.2). (Here, the union is taken over all the boundary strata $X_P \subset \bar{X}$ of the reductive Borel-Serre compactification.)

In order to simplify notation, let us denote this double complex by $\boldsymbol{\Omega}^\bullet(\mathbf{I}^\bullet)$ with differentials $\partial : \boldsymbol{\Omega}^p(\mathbf{I}^q) \to \boldsymbol{\Omega}^p(\mathbf{I}^{q+1})$ and $d : \boldsymbol{\Omega}^p(\mathbf{I}^q) \to \boldsymbol{\Omega}^{p+1}(\mathbf{I}^q)$.

32. Morphisms from weighted cohomology to the double complex

(32.1) In this section we find a quasi-isomorphism between the direct image $i_*\mathbf{I}^\bullet(\mathbf{E})$ of the standard resolution and the double complex of sheaves from §31. We also find a quasi-isomorphism between the sheaf $\bar{\boldsymbol{\Omega}}^\bullet_{\mathrm{sp}}(\bar{X}; \mathbf{E} \otimes \mathbb{C})$ of special differential forms and the double complex of sheaves from §31.

(32.2) Consider the two kernel complexes,

$$\begin{aligned} \mathbf{K}^\bullet &= \mathbf{ker}(\partial : \boldsymbol{\Omega}^\bullet(\mathbf{I}^0) \to \boldsymbol{\Omega}^\bullet(\mathbf{I}^1)) \\ \mathbf{J}^\bullet &= \mathbf{ker}(d : \boldsymbol{\Omega}^0(\mathbf{I}^\bullet) \to \boldsymbol{\Omega}^1(\mathbf{I}^\bullet)) \end{aligned}$$

(32.3) Lemma. *The complex $\mathbf{K}^\bullet$ is canonically isomorphic to the complex $\bar{\boldsymbol{\Omega}}^\bullet_{\mathrm{sp}}(\bar{X}, \mathbf{E} \otimes \mathbb{C})$ of special differential forms on $\bar{X}$. The complex $\mathbf{J}^\bullet$ is canonically isomorphic to the direct image of the standard resolution, $i_*(\mathbf{I}^\bullet(\mathbf{E}) \otimes \mathbb{C})$.*

(32.4) Proof. The proof consists of untangling the definitions. □

(32.5) Proposition. *For every point $x \in \bar{X}$, and for all i, the following sequence of stalks is exact:*

$$0 \to \mathbf{K}^i_x \xrightarrow{\partial^{-1}} \boldsymbol{\Omega}^i_x(\mathbf{I}^0) \xrightarrow{\partial^0} \boldsymbol{\Omega}^i_x(\mathbf{I}^1) \xrightarrow{\partial^1} \boldsymbol{\Omega}^i_x(\mathbf{I}^2) \to \dots \tag{32.5.1}$$

(32.6) Proof. If $x \in X$ is an interior point then $\boldsymbol{\Omega}_x^*(\mathbf{I}^j)$ consists of germs (at x) of smooth differential forms with coefficients in $\mathbf{I}^j(\mathbf{E})$ and exactness of (32.5.1) follows from the fact that the coefficient sequence

$$\mathbf{E} \to \mathbf{I}^0(\mathbf{E}) \to \mathbf{I}^1(\mathbf{E}) \to \dots \qquad (32.6.1)$$

is exact. If $x \in X_P$ is in a boundary stratum, then $\boldsymbol{\Omega}_x^\bullet(\mathbf{I}^j)$ is noncanonically isomorphic to the tensor product,

$$\bigoplus_{p+q=\bullet} \boldsymbol{\Omega}_x^p(X_P) \otimes \wedge^q(\mathfrak{N}_P, \mathbf{I}^j) \qquad (32.6.2)$$

and exactness again follows from (32.6.1). □

(32.7) Proposition. *For every point $x \in \bar{X}$ and for all $j \geq 0$, the following sequence of stalks is exact:*

$$0 \to \mathbf{J}_x^j \xrightarrow{d^{-1}} \boldsymbol{\Omega}_x^0(\mathbf{I}^j) \xrightarrow{d^0} \boldsymbol{\Omega}_x^1(\mathbf{I}^j) \xrightarrow{d^1} \boldsymbol{\Omega}_x^2(\mathbf{I}^j) \to \dots \qquad (32.7.1)$$

(32.8) Proof. If $x \in X$ is an interior point, then the sequence is exact by the Poincaré lemma. Now suppose that $x \in X_Q$ is a point in a boundary stratum. Let $\mathbf{P} \subseteq \mathbf{Q}$ denote the smallest rational parabolic subgroup such that x is in the geodesic neighborhood of X_P, in other words, $x \in \bar{U}_P \cap X_Q$. Let $\nu : \mathbf{Q} \to \mathbf{L_Q}$ denote the projection to the Levi quotient, $\bar{\mathbf{P}} = \nu(\mathbf{P})$, and $\Gamma_{\bar{P}} = \nu(\Gamma_P)$. Fix a $\mathbf{P}$–rational basepoint $x_0 \in D$ and hence a "chosen" action of $\mathbf{P}(\mathbb{Q})$ on $A[\mathcal{U}_P]$ and on $I_P^j(E)$. This establishes (27.10.1) an isomorphism between the local system $\mathbf{I}_P^j(\mathbf{E})$ on U_P with the local system corresponding to the "chosen" representation of $\mathbf{P}(\mathbb{Q})$ on $I_P^j(E) = \mathrm{coker}(\partial^{j-1}) \otimes A[\mathcal{U}_P]$. Now apply Prop. 17.2 to conclude that the stalk cohomology at x of the complex $\boldsymbol{\Omega}_{\mathrm{sp}}^\bullet(\bar{X}, \mathbf{I}_P^j(\mathbf{E}) \otimes \mathbb{C})$ is isomorphic to the Lie algebra cohomology group, $H^*(\mathfrak{N}_Q; I_P^j(E) \otimes \mathbb{C})$. By Cor. 24.7, these groups vanish in all positive degrees. □

(32.9) Remark. The acyclicity of the modules $A[\mathcal{U}_P]$ is the fundamental reason that the complex $\mathbf{I}^\bullet(\mathbf{E})$ may be used. Notice that if the module $I_P^j(E)$ was an inductive limit of representations of the full group G, then Kostant's theorem would guarantee that its $\mathfrak{N}_P$– cohomology was nonzero! For this reason the local systems $\mathbf{A}[\mathcal{U}_P, \tilde{U}_P]$ exist only on the geodesic neighborhood and do not extend to local systems on all of $\tilde{X}$.

(32.10) Corollary. *The sheaf mappings from §32.2,*

$$\bar{\boldsymbol{\Omega}}_{\mathrm{sp}}^\bullet(\bar{X}, \mathbf{E} \otimes \mathbb{C}) \xrightarrow{\Phi} \mathbf{B}^\bullet = \bigoplus_{p+q=\bullet} \bar{\boldsymbol{\Omega}}_{\mathrm{sp}}^p(\bar{X}, \mathbf{I}^q(\mathbf{E}) \otimes \mathbb{C}) \xleftarrow{\Psi} i_*\mathbf{I}^\bullet(\mathbf{E}) \otimes \mathbb{C}$$

are quasi-isomorphisms, so each of these sheaves represents the direct image sheaf $Ri_*(\mathbf{E}) \otimes \mathbb{C}$.

(32.11) Proof. In §32.5 and §32.7, we have verified the stalk acyclicity hypothesis of §30.2. □

33. Truncation of the double complex

(33.1) In this section we show that the complex of sheaves

$$\mathbf{B}^\bullet = \bigoplus_{p+q=\bullet} \bar{\Omega}^p_{\mathrm{sp}}(\bar{X}, \mathbf{I}^q(\mathbf{E}) \otimes \mathbb{C})$$

may be truncated by A-weights in a way which is compatible with the quasi-isomorphism Φ, Ψ, and that the resulting sheaf maps give a quasi-isomorphism between the two constructions of weighted cohomology. We only give an outline since this section is a combination of sections §14 and §28.

(33.2) Each $\tilde{\mathbf{I}}^q(\mathbf{E})$ is a union of local systems $\tilde{\mathbf{I}}^q_P(\mathbf{E})$ defined on the geodesic neighborhood, $\tilde{U}_P \subset \tilde{X}$. Let $\pi : Y_Q \longrightarrow X_Q$ be the projection from a Borel-Serre stratum to a reductive Borel-Serre stratum. Each fiber $N_x = \pi^{-1}(x)$ is a homogeneous space for $\mathscr{U}_Q$. Suppose $\pi^{-1}(x) \subset \tilde{U}_P$. Then we have a complex of $\mathscr{U}_Q$-invariant forms on $\pi^{-1}(x)$ with coefficients in the flat vectorbundle $\tilde{\mathbf{I}}^q_P \otimes \mathbb{C}|(Y_Q \cap \tilde{U}_P)$. By §12.5 these complexes are the fibers of a complex of flat vectorbundles, $C^\bullet(N_Q; \tilde{\mathbf{I}}^q_P(\mathbf{E}) \otimes \mathbb{C})$ on $X_Q \cap \tilde{U}_P$.

(33.3) Definition. The double complex $\boldsymbol{\Omega}^\bullet(X_Q; \mathbf{C}^\bullet(N_Q; \tilde{\mathbf{I}}^q_P(\mathbf{E}) \otimes \mathbb{C}))$ is the double complex of smooth differential forms on the geodesic neighborhood $X_Q \cap \tilde{U}_P$ with coefficients in the flat vectorbundle $C^\bullet(N_P; \tilde{\mathbf{I}}^q_P(\mathbf{E}) \otimes \mathbb{C})$.

(33.4) If $\mathbf{R} \subset \mathbf{P} \subseteq \mathbf{Q}$ are rational parabolic subgroups, then the inclusion

$$\tilde{\mathbf{I}}^q_P(\tilde{\mathbf{E}})|\tilde{U}_R \hookrightarrow \tilde{\mathbf{I}}^q_R(\tilde{\mathbf{E}}) \qquad (33.4.1)$$

induces inclusions

$$C^\bullet(N_Q; \tilde{\mathbf{I}}^q_P(\mathbf{E}) \otimes \mathbb{C})|(X_Q \cap \tilde{U}_R) \hookrightarrow C^\bullet(N_Q; \tilde{\mathbf{I}}^q_R(\mathbf{E}) \otimes \mathbb{C}) \qquad (33.4.2)$$

(33.5) Definition. The sheaf $\boldsymbol{\Omega}^\bullet(X_Q; \mathbf{C}^\bullet(N_Q; \tilde{\mathbf{I}}^q(\mathbf{E}) \otimes \mathbb{C}))$ is the union of sheaves,

$$\boldsymbol{\Omega}^\bullet(X_Q; \mathbf{C}^\bullet(N_Q; \tilde{\mathbf{I}}^q(\mathbf{E}) \otimes \mathbb{C})) = \bigcup_{X_P \subset \bar{X}_Q} \boldsymbol{\Omega}^\bullet(X_Q; \mathbf{C}^\bullet(N_Q; \tilde{\mathbf{I}}^q_P(\mathbf{E}) \otimes \mathbb{C}))$$

taken over all strata X_P which is contained in the closure of X_Q.

(33.6) Proposition. *The restriction $\bar{\Omega}^\bullet_{\mathrm{sp}}(X; \mathbf{I}^q(\mathbf{E}) \otimes \mathbb{C})|X_Q$ is canonically isomorphic to the sheaf $\boldsymbol{\Omega}^\bullet(X_Q; \mathbf{C}^\bullet(N_Q; \tilde{\mathbf{I}}^q(\mathbf{E}) \otimes \mathbb{C}))$.* □

(33.7) Fix rational parabolic subgroups $\mathbf{P} \subseteq \mathbf{Q}$. The torus $S_Q(x_0)$ acts semisimply on the local system $C^\bullet(N_Q; \tilde{\mathbf{I}}^q_P(\tilde{\mathbf{E}}) \otimes \mathbb{C})$ (which is supported on $\tilde{U}_P \cap X_Q$), and it decomposes into a sum of weight subbundles,

$$C^\bullet(N_Q; \tilde{\mathbf{I}}^q_P(\mathbf{E}) \otimes \mathbb{C}) = \bigoplus_{\lambda \in \chi(S_Q)} (C^\bullet(N_Q; \tilde{\mathbf{I}}^q_P(\mathbf{E}) \otimes \mathbb{C}))_\lambda$$

If $\mathbf{R} \subset \mathbf{P} \subseteq \mathbf{Q}$ are rational parabolic subgroups, then the inclusions (33.4.1) are compatible with this decomposition. Fix a weight profile p.

(33.8) Definition. The weight subcomplex is the complex of local systems,

$$(C^\bullet(N_Q;\tilde{\mathbf{I}}^q_P(\mathbf{E})\otimes\mathbb{C}))_+ = \bigoplus_{\lambda\in\chi(S_Q)_+} C^\bullet(N_Q;\tilde{\mathbf{I}}^q_P(\mathbf{E})\otimes\mathbb{C})_\lambda$$

(33.9) Proposition. *The weight subcomplex is independent of the choice of lift $S_Q(x_0)$.*

Proof. The proof follows from §10.4. □

(33.10) It follows that the union of the weight subcomplexes forms a subsheaf on X_Q,

$$C^\bullet(N_Q;\tilde{\mathbf{I}}^q(\mathbf{E})\otimes\mathbb{C})_+ = \bigcup_{X_P\subset\bar{X}_Q} C^\bullet(N_Q;\tilde{\mathbf{I}}^q_P(\mathbf{E})\otimes\mathbb{C}) \subset \bar{\boldsymbol{\Omega}}^\bullet(\bar{X};\mathbf{I}^q(\mathbf{E})\otimes\mathbb{C})|X_Q \tag{33.10.1}$$

(33.11) Definition. The weighted cohomology double complex,

$$\mathbf{W}^p\mathbf{C}^\bullet(\bar{X};\mathbf{I}^q(\mathbf{E})\otimes\mathbb{C})$$

is the subcomplex of $\bar{\boldsymbol{\Omega}}^\bullet_{\mathrm{sp}}(\bar{X};\mathbf{I}^q(\mathbf{E})\otimes\mathbb{C}))$ whose sections over an open subset $U\subset\bar{X}$ consist of special differential forms ω on $U\cap X$, with the property that for each stratum X_Q, the corresponding form

$$\omega_Q \in \Gamma(U\cap X_Q, \boldsymbol{\Omega}^\bullet(X_Q;\mathbf{C}^\bullet(N_Q;\tilde{\mathbf{I}}^q(\mathbf{E})\otimes\mathbb{C})))$$

lies in the weight subcomplex, $\Gamma(U\cap X_Q, \boldsymbol{\Omega}^\bullet(X_Q;\mathbf{C}^\bullet(N_Q;\tilde{\mathbf{I}}^q(\mathbf{E})\otimes\mathbb{C})_+))$.

(33.12) Remark. It is not hard to check that the collection of subsheaves, (33.10.1) satisfy the compatibility conditions (2.2). The weighted cohomology double complex is therefore isomorphic to the complex obtained from the double complex $\bar{\boldsymbol{\Omega}}^\bullet_{\mathrm{sp}}(\bar{X},\mathbf{I}^\bullet(\mathbf{E})\otimes\mathbb{C})$ by truncating it to the sheaves (33.10.1) along the singular strata X_P.

34. Proof of Theorem 29.2

(34.1) Theorem 29.2 states that the two constructions of weighted cohomology agree. This is an immediate consequence of the following result:

(34.2) Proposition. *The quasi-isomorphism Φ and Ψ of §32.2 and §32.10 restrict to quasi-isomorphisms,*

$$\mathbf{WC}^\bullet(\mathbf{E}\otimes\mathbb{C}) \xrightarrow{\Phi} \bigoplus_{i+j=\bullet} \mathbf{W}^p\mathbf{C}^i(\bar{X};\mathbf{I}^j(\mathbf{E})\otimes\mathbb{C})) \xleftarrow{\Psi} \mathbf{W}^p\mathbf{I}^\bullet(\mathbf{E})$$

(34.3) Proof. The proof consists of checking that the three weight truncations are compatible with the maps Φ and Ψ. □

35. Variations and generalizations

(35.1) Weighted cohomology is also defined in the adelic setting. Let $\mathbb{A}$ denote the adeles of $\mathbb{Q}$ and let $K^f \subset \mathbf{G}^f(\mathbb{A})$ be a compact open subgroup of the finite adelic points of $\mathbf{G}$. Then $X = \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K^f.K$ is a finite union of locally symmetric spaces of the type considered in §3. The weighted cohomology of a disjoint union of locally symmetric spaces is the direct sum of the weighted cohomology of the connected components. Thus, weighted cohomology is well defined for such a space X.

(35.2) Suppose $X = \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K^f.K$ is a Hermitian locally symmetric space and is the complex points of a quasiprojective scheme. Let $\hat{X}$ be the Baily-Borel Satake compactification of the associated Shimura variety. We conjecture that in this case each weighted cohomology complex on $\hat{X}$ has an étale analogue and therefore the weighted cohomology groups may be defined algebraically in étale cohomology. In [P] a construction is given for an étale version of the restriction $Ri_*(\mathbf{E})|F_Q$ of the direct image sheaf to a boundary stratum $F_Q \subset \hat{X}$, and it is even possible to form weight truncations of this restriction. However, we do not know how to "glue" a truncation of this restriction to the sheaf $\mathbf{E}$ on the interior of $\hat{X}$ in order to obtain a globally defined étale sheaf.

(35.3) J. Franke [F] has recently defined a class of weighted L^2-cohomology spaces as the $(\mathfrak{g}, K)$-cohomology of certain modules of functions. It is an interesting question to determine the relationship between his construction and ours. We do not know if our weighted cohomology groups may be expressed as $(\mathfrak{g}, K)$-cohomology.

(35.4) In this paper we have chosen to work with the most restrictive, and therefore the simplest, notion of weight profile. We have not considered weight truncations $\chi(A_P)_+$ on singular strata X_P other than those which are induced from truncations $\chi(A_Q)_+$ on *maximal* boundary strata X_Q. Nor have we made use of the finer structure of the flat vectorbundle $C^\bullet(N_P, \mathbf{E})$ which is obtained from its decomposition into L_P-irreducible components. Although weighted cohomology groups may be defined using more sophisticated truncation procedures, the applications we have in mind have not necessitated the use of these more general constructions.

(35.5) The referee has kindly pointed out that, except in §17.8 and §17.9, it is not necessary to assume that the discrete group Γ is neat. However, in case Γ is not neat, the locally symmetric space X and the strata of the reductive Borel-Serre compactification may no longer be manifolds: they are "V-manifolds". The smooth differential forms which are used in this paper would have to be replaced by differential forms on V-manifolds.

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