Invent. math. 111, 1–33 (1993)

Inventiones mathematicae

Local contribution to the Lefschetz fixed point formula*

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Oblatum 9-XI-1990 & 29-IV-1992

Summary. For a class of self-correspondences C called *weakly hyperbolic*, we give a computable formula for the contribution of a fixed point component to the Lefschetz number of C. The formula applies to Lefschetz numbers of cohomology with coefficients in a constructible complex of sheaves (such as intersection homology).

1 Introduction

(1.1) Local contributions to Lefschetz numbers. Suppose that we are given a compact stratified space X and a self map $f: X \to X$. The *Lefschetz number* $L(f)$ is defined by

$$L(f) = \sum_i (-1)^i \operatorname{trace}(f^*: H^i(X) \to H^i(X)).$$

The fixed point set of f is $\{x \in X \mid f(x) = x\}$. A fixed point component F is a connected component of the fixed point set. The Lefschetz fixed point theorem states,

Theorem [Lef] *There is a canonical way to associate to each fixed point component F a local contribution, which is a number $n(F)$, so that*

$$\sum_F n(F) = L(f).$$

The object of this paper is to give computable formulas for the local contributions $n(F)$, in terms of local data collected near F.

As an example, consider a graph (or 1-complex) with two vertices a and b and k edges going from a to b. Let f be a self map that fixes a and b and moves each edge along itself from a to b.

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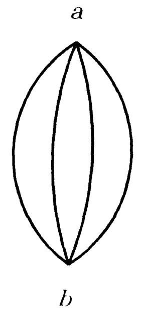
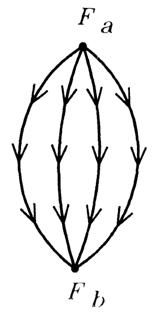
* In memory of J.L. Verdier

** Partially supported by NSF grant # DMS8802638 and DMS9001941
*** Partially supported by NSF grant # DMS8803083 and DMS9106522



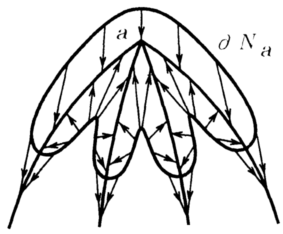
The 1-complex X



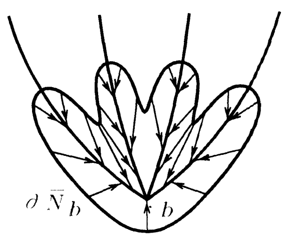
The self map $f: X \to X$

Since f is homotopic to the identity, the Lefschetz number $L(f)$ is the Euler characteristic of X which is $2 - k$. There are two fixed point components $F_a = \{a\}$ and $F_b = \{b\}$. It turns out that $n(F_a) = 1 - k$ and $n(F_b) = 1$.

There are already several known explicit formulas for $n(F)$. Perhaps the most beautiful one for ordinary cohomology is the *degree formula* [D1, D2] which is analogous to the Hopf index theorem: Embed X in $\mathbb{R}^d$. Take a regular neighborhood $\bar{N}$ of F in $\mathbb{R}^d$. Construct a retraction $r: \bar{N} \to \bar{N} \cap X$. Let m be the map $m: \partial\bar{N} \to S^{d-1}$ of the boundary of $\bar{N}$ to the $(n-1)$-sphere obtained by taking the direction of the vector $v(y)$ from y to $f \circ r(y)$. Then $n(F)$ is the degree of the map m. The reader can check visually that this procedure gives the right answer for F_a and F_b from the following picture, which shows the vectors $v(y)$.



Degree formula for $n(F_a)$



Degree formula for $n(F_b)$

Because it requires an embedding, this procedure is impractical for most applications. Another formula, which applies to sheaf cohomology as well, is the *product formula* due to Grothendieck, Illusie, and Verdier [GI, V]. This constructs $n(F)$ as a product in a sheaf cohomology group; it is analogous to the original formula of Lefschetz, which defined $n(F)$ as an intersection product of the graph of f with the diagonal. However, only in special cases has this formula been used to produce computable numbers [I].

(1.2) Expanding and contracting maps. The formula for $n(f)$ that we present here is a *trace formula*, i.e. a formula that expresses $n(F)$ as an alternating sum of traces,

in a way that resembles the definition of $L(f)$. The simplest case is when the map f is either contracting or expanding near F. We choose a regular neighborhood N of F in X.

If f is contracting near F, then

$$n(F) = \sum_i (-1)^i \operatorname{trace}(f^*: H^i(N) \to H^i(N))$$

$$= \sum_i (-1)^i \operatorname{trace}(f^*: H^i(N, \phi) \to H^i(N, \phi)).$$

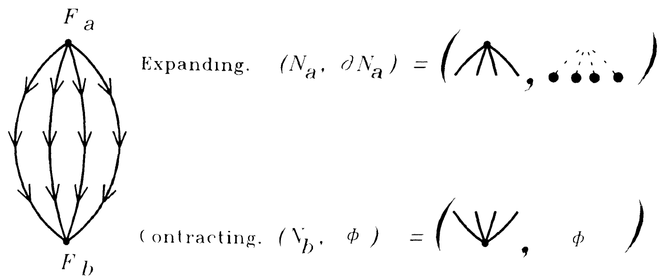
If f is expanding near F, then

$$n(F) = \sum_i (-1)^i \operatorname{trace}(f^*: H^i(X, X - N) \to H^i(X, X - N))$$

$$= \sum_i (-1)^i \operatorname{trace}(f^*: H^i(N, \partial N) \to H^i(N, \partial N)).$$

If f is contracting near F, then N is mapped to itself; and if f is expanding near F, then $X - N$ is mapped to itself. Therefore, the expressions above make sense.

Let's apply the formula to the case of our 1-complex. The map f is expanding at F_a and contracting at F_b. In each case, the local contribution is the alternating trace of maps induced by f on the homology of a particular pair of spaces. The following picture shows this pair:



The fixed point contributions are

$$n(F_a) = \sum_i (-1)^i \operatorname{trace}(f^*: H^i(N_a, \partial N_a) \to H^i(N_a, \partial N_a))$$

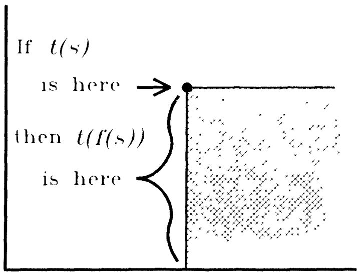
$$= (-1)^0 \times 0 + (-1)^1 \times (k - 1) = 1 - k,$$

$$n(F_b) = \sum_i (-1)^i \operatorname{trace}(f^*: H^i(N_b) \to H^i(N_b))$$

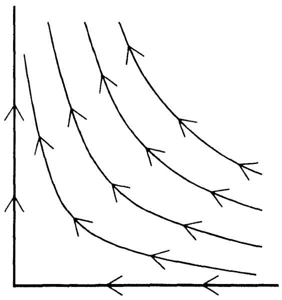
$$= (-1)^0 \times 1 + (-1)^1 \times 0 = 1.$$

(1.3) Weakly hyperbolic maps. The most general class of maps for which we are able to give a trace formula for $n(F)$ is the class of *weakly hyperbolic* maps. These are maps all of whose fixed point components are weakly hyperbolic in the following sense:

Definition (See §3.1) A fixed point component F is called *weakly hyperbolic* if, for some neighborhood $W \subset X$ of F, there exists an "indicator map" $t: W \to \mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$ such that $t^{-1}(0, 0) = F$, $t_1 f(x) \geqq t_1(x)$, and $t_2 f(x) \leqq t_2(x)$. Pictorially,

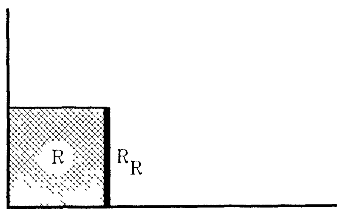


The definition



An example

The figure on the right is an example that justifies the name: Here W is $\mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$ itself, and t is the identity. A contracting fixed point set F is weakly hyperbolic: it suffices to take $t_1(x) = 0$ and take $t_2(x)$ to be the distance from F. Similarly, an expanding fixed point set is weakly hyperbolic taking $t_2(x) = 0$ and $t_1(x)$ as the distance from F. Now, consider a small rectangle R and its right-hand edge R_R:



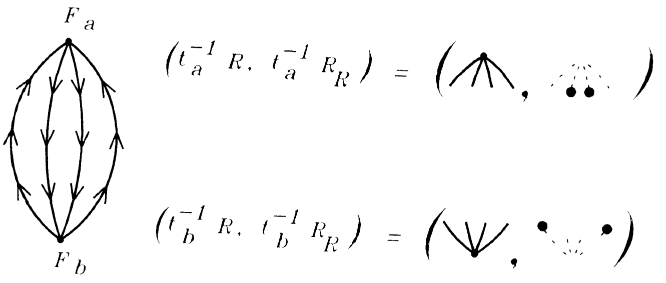
The basic formula is,

Theorem (See §4.7, case $j = 1$) *If F is a weakly hyerbolic fixed point component for f, then f induces a map $f^*: H^i(t^{-1} R, t^{-1} R_R) \to H^i(t^{-1} R, t^{-1} R_R)$ and*

$$n(F) = \sum_i (-1)^i \operatorname{trace}(f^*: H^i(t^{-1} R, t^{-1} R_R) \to H^i(t^{-1} R, t^{-1} R_R)).$$

Perhaps the most familiar case of this formula arises from the negative gradient flow of a Morse function on a compact manifold, for which the local contribution at an isolated fixed point is $(-1)^d$, where d (the Morse index) is the dimension of the expanding space.

In our example, suppose that our 1-complex has a self map that moves some of the edges up and moves the others down. Then at each fixed point component $F = F_a$ or F_b, the indicator map t_a or t_b sends the edges moving out away from F to the t_1 axis and sends the edges moving in towards F to the t_2 axis. So the pair $(t^{-1} R, t^{-1} R_R)$ looks like this:



Then $n(F_a) = \sum_i (-1)^i \operatorname{trace}(f^*: H^i(t^{-1} R, t^{-1} R_R) \to H^i(t^{-1} R, t^{-1} R_R)) = 1 - l$ where l is the number of edges going down. Similarly, $n(F_b) = 1 - (k - l)$, where $k - l$ is the number of edges going up.

(1.4) Dependence on the choices. Suppose that the self map is neither strictly expanding nor strictly contracting, but instead preserves distances to F. For example, let X be the round 2-sphere and let f be a rigid rotation. Then the two intersections of the axis of rotation with X are the two fixed point components. We may treat them either as expanding or as contracting. (That is, the indicator map can go to the t_1 axis or to the t_2 axis.) The individual groups $H^i(t^{-1} R, t^{-1} R_R)$ will be different in these two cases, as may already be seen from the 2-sphere example. However the alternating sum $\sum_i (-1)^i \operatorname{trace}(f^*: H^i(t^{-1} R, t^{-1} R_R) \to H^i(t^{-1} R, t^{-1} R_R))$ will be the same.

Also, there is more than one small rectangle R. Different choices will actually lead to different cohomology groups $H^i(t^{-1} R, t^{-1} R_R)$. However, again the alternating sum is independent of the choice.

The basic formula has been presented in its simplest form for expositional purposes. For most applications, various generalizations and extensions of it are needed:

(1.5) Other cohomology theories. The Lefschetz number $L(f)$ exists not only in ordinary cohomology, but also in cohomology with coefficients in a constructible complex of sheaves $\mathbf{S}^\bullet$, so long as f has a lift $\mathbf{\Phi}$ to $\mathbf{S}^\bullet$ (see §2). One example is intersection cohomology. The basic formula holds for such cohomology theories.

(1.6) A sheaf theoretic version. Consider the inclusions,

$$F \overset{h_L}{\hookrightarrow} t^{-1} R_L \overset{j_L}{\hookrightarrow} W,$$

where R_L is the left hand edge of the rectangle R, (i.e. the t_2 axis). Then, for an appropriate choice of rectangle R, we have

$$H^i(t^{-1} R, t^{-1} R_R; \mathbf{S}^\bullet) = H^i(F; h_L^* j_L^! \mathbf{S}^\bullet).$$

So we get an alternative formula using the sheaf $\mathbf{A}_4^\bullet = h_L^* j_L^!(\mathbf{S}^\bullet)$.

Theorem (see §4.7, case $j = 4$) *Suppose F is a connected component of the fixed point set of a weakly hyperbolic endomorphism $f: X \to X$. Then the local contribution to the Lefschetz number $n(\mathrm{F})$ is*

$$n(F) = \sum_i (-1)^i \operatorname{trace}(f^*: H^i(F; \mathbf{A}_4^\bullet) \to H^i(F; \mathbf{A}_4^\bullet)).$$

The advantage of this formulation is that the set $t^{-1} R_L$ is natural: it has the interpretation as the set on which the map f is contracting towards F. For example, if f is a complex algebraic map, $t^{-1} R_L$ will often be a complex analytic subspace of W.

There is a dual version of this with the expanding and contracting directions reversed. Let R_R denote the right hand edge of the rectangle and consider the inclusion

$$F \overset{h_R}{\hookrightarrow} t^{-1}(R_R) \overset{j_R}{\hookrightarrow} W.$$

Then the complex of sheaves $\mathbf{A}_5^\bullet = h_R^! j_R^* \mathbf{S}^\bullet$ may be used in place of $\mathbf{A}_4^\bullet$. The complexes of sheaves $\mathbf{A}_4^\bullet$ and $\mathbf{A}_5^\bullet$ will generally have different cohomology groups although they will give the same local contribution $n(F)$ (see Theorem 4.7, case $j = 5$).

(1.7) A version that is local on F. When F is larger than a point, it may be useful to write $n(F)$ as a sum over strata of F. The action of f^* on $H^i(F; h^* j^! \mathbf{S}^\bullet)$ is induced by a self map called $\tilde{\Phi}^*$ on the sheaf $h^* j^! \mathbf{S}^\bullet$. For $x \in F$, define the alternating sum of traces on the stalk cohomology groups,

$$n(x) = \sum_i (-1)^i \operatorname{trace}(\tilde{\Phi}^*: H^i(x; h^* j^! \mathbf{S}^\bullet) \to H^i(x; h^* j^! \mathbf{S}^\bullet)).$$

Then $n(x)$ is a constructible function on F. Suppose $F = F_1 \cup F_2 \cup \ldots \cup F_r$ is a stratification of F so that $n(x)$ is constant on each stratum.

Theorem (See §10.3) *Suppose F is a connected component of the fixed point set of a weakly hyperbolic endomorphism $f: X \to X$, as above. Then the local contribution to the Lefschetz number $n(F)$ can be written as a sum over the strata of F as follows:*

$$n(F) = \sum_{j=1}^{r} \chi_c(F_j) n(x_j),$$

where $x_j \in F_j$, and χ_c denotes the Euler characteristic with compact supports.

(1.8) Correspondences. A *self correspondence* of X is a diagram $C \rightrightarrows X$ whose two maps are the "source map" c_1 and the "target map" c_2. A fixed point is a point p on C (not on X) such that $c_1(p) = c_2(p)$. Two fixed point components in C may have the same image in X. The whole theory that we have been discussing goes through for correspondences. A lot of the complexity of this paper stems from the care that must be taken in choosing the neighborhoods W in this context.

(1.9) Applications and examples. There will probably never be an "ultimate" Lefschetz fixed point formula. There is always a trade-off between generality of the

situation covered and computability of the formula for $n(F)$. Therefore, which formula is the most useful depends on the intended application. See §13 and §14 for a number of interesting special cases and examples.

The main application that we have in mind is to a Hecke correspondence operating on an appropriate compactification of a modular variety. The cohomology theory of interest is either the intersection cohomology of the compactification or the ordinary cohomology of the variety. For this application, the formula given above is general enough: the Hecke correspondences are weakly hyperbolic (although they are neither expanding nor contracting, they cannot, even locally, be written as the graph of an endomorphism, and the fixed point set of Hecke correspondence may be a very complicated subvariety). This formula for the local contribution $n(F)$ to the Lefschetz number of a Hecke correspondence is quite computable, and it can be evaluated in terms of roots and weights (see [GM4] where this calculation is carried out explicitly).

(1.10) Remarks on the proof. The ideas in the proof are very simple and are best illustrated by considering the case of a (weakly hyperbolic) endomorphism $f: X \to X$, rather than a correspondence. Let us suppose that X is a compact stratified subanalytic space, $\mathbf{S}^\bullet$ is a complex of (cohomologically constructible) sheaves on X, and that $\mathbf{\Phi}: f^*(\mathbf{S}^\bullet) \to \mathbf{S}^\bullet$ is a lift of f to the sheaf level. Let $F \subset X$ be a connected component of the fixed point set of f, and let $U \subset X$ be a weakly hyperbolic neighborhood of F with indicator map $t: U \to \mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$. Assume that U contains no fixed points other than those in F. Choose a (closed) rectangle

$$R = R(x_0, y_0) \subset \mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0},$$

whose upper right hand corner is at the point (x_0, y_0), and let R_T, R_B, R_R, R_L denote the top, bottom, right, and left sides of this rectangle, respectively. We assume that x_0 and y_0 are chosen so small that $t^{-1}(R) \subset U$.

The first observation is that the self map f induces an endomorphism on the cohomology groups

$$A_1^i(x_0, y_0) = H^i(t^{-1}(R), t^{-1}(R_R); \mathbf{S}^\bullet).$$

Suppose ξ is a cocycle in $t^{-1}(R)$ which is compactly supported near $t^{-1}(R_R)$, i.e. $|\xi| \cap t^{-1}(R_R) = \emptyset$. Then $f^*(\xi)$ is a cocycle in $f^{-1}(t^{-1}(R))$. Let ξ' be the cochain which is obtained first by restricting ξ to $f^{-1}(t^{-1}(R)) \cap t^{-1}(R)$ and then extending (by 0) to $t^{-1}(R)$. Extending a cocycle by 0 in this way does not normally give a cocycle: ξ' could have boundary along the pre-image of the top, $f^{-1}t^{-1}(R_T) \cap t^{-1}(R)$ or right hand side, $f^{-1}t^{-1}(R_R) \cap t^{-1}(R)$ of the rectangle. However, because of the hyperbolic assumption on f, the top part $f^{-1}(t^{-1}(R_T))$ has empty intersection with the interior of $t^{-1}(R)$. And although the right hand side, $f^{-1}t^{-1}(R_R)$ may intersect the interior of $t^{-1}(R)$, the cocycle $f^*(\xi)$ is compactly supported near this part of the boundary. Thus the hyperbolic assumptions on f together with the support assumptions on ξ are precisely what is needed to guarantee that ξ' is a cocycle in $t^{-1}(R)$. Furthermore, ξ' is again compactly supported near $t^{-1}(R_R)$. So f^* gives an endomorphism of the local group $A_1^i(x_0, y_0)$.

This whole procedure actually works on the sheaf level. It is possible to restrict the sheaf $\mathbf{S}^\bullet$ to the region $t^{-1}(R)$ so as to have compact supports near $t^{-1}(R_R)$. This restricted sheaf is called $\mathbf{A}_1^\bullet(x_0, y_0)$ in §4.5. It coincides with $\mathbf{S}^\bullet$ near the fixed point component F, it vanishes outside $t^{-1}(R)$, and its cohomology is precisely the local group, i.e.

$$H^i(X; \mathbf{A}_1^\bullet(x_0, y_0)) = A_1^i(x_0, y_0).$$

Furthermore, the lift $\mathbf{\Phi}: f^* \mathbf{S}^\bullet \to \mathbf{S}^\bullet$ determines a unique lift $\tilde{\mathbf{\Phi}}: f^* \mathbf{A}_1^\bullet(x_0, y_0) \to \mathbf{A}_1^\bullet(x_0, y_0)$ which agrees with $\mathbf{\Phi}$ near the fixed point component F and which gives the above endomorphism f^* on cohomology. The existence of the lift $\tilde{\mathbf{\Phi}}$ is just the argument given above, although it is made rigorous using the formal (derived category) properties of h^* and $j^!$. In this way, we have isolated the fixed point component F along with "part" of the sheaf $\mathbf{S}^\bullet$, from the rest of the space.

Now apply the Grothendieck–Verdier–Illusie Lefschetz fixed point formula to the sheaf $\mathbf{A}_1^\bullet(x_0, y_0)$. Since this sheaf agrees with $\mathbf{S}^\bullet$ near the fixed point component F (and since the lifts $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ also agree near F), they have the same local contribution to the Lefschetz formula. But in the case of the sheaf $\mathbf{A}_1^\bullet$, there is only one term in the Lefschetz formula and it is precisely the Lefschetz number on the global cohomology of $\mathbf{A}_1^\bullet$. In summary, we have shown that the local contribution from the fixed point component F to the global Lefschetz number of $(f, \mathbf{\Phi}): (X, \mathbf{S}^\bullet) \to (X, \mathbf{S}^\bullet)$ is

$$\sum_i (-1)^i \operatorname{Tr}(f^*: H^i(X, \mathbf{A}_1^\bullet) \to H^i(X, \mathbf{A}_1^\bullet))$$

$$= \sum_i (-1)^i \operatorname{Tr}(f^*: A_1^i(x_0, y_0) \to A_1^i(x_0, y_0)),$$

which is Theorem 4.7 for the case $j = 1$.

In the final sheaf theoretic version of our formulation of the local contribution, we consider the endomorphism f^* induced on the group

$$A_4^i = H^i(F; h_L^* j_L^! \mathbf{S}^\bullet)$$

and show that the local contribution is

$$n(F) = \sum_i (-1)^i \operatorname{Tr}(f^*: A_4^i \to A_4^i).$$

The key observation in comparing this group with the local group $A_1^i(x_0, y_0)$ above is the formula

$$A_4^i = \lim_{x_0 \to 0} \lim_{y_0 \to 0} A_1^i(x_0, y_0),$$

which follows directly from the definition of h_L^* and of $j_L^!$. In fact, because of the subanalytic stratifiable nature of the space X and the indicator map t, the limits are attained for sufficiently small, finite values of x_0 and y_0, corresponding to a rectangle which is small but very thin, i.e. $y_0 \ll x_0 \ll 1$. The rest of the proof consists of showing that the endomorphism $f^*: A_4^i \to A_4^i$ agrees with the homomorphism induced on cohomology by a lift $\tilde{\mathbf{\Phi}}: \mathbf{A}_4^\bullet \to \mathbf{A}_4^\bullet$ of the self map, to the sheaf level.

(1.11) Historical comments. As discussed above, our theorem relies on the Grothendieck–Illusie–Verdier version [GI] of the Lefschetz fixed point theorem

(which was preceded by work of Artin and Verdier [V]). Computations of the local terms were made in [I] for special cases, including the case of algebraic curves. See also [D1, D2].

The literature on Lefschetz fixed point theorems contains a number of articles in which attention has been directed, either implicitly or explicitly, towards expanding and contracting behavior of self maps. In [L] (Proposition 7.12), Langlands indicated that the local contribution at the origin for the one dimensional correspondence $f(z) = z^{b/a}$ (on $\mathbb{C}$) has two different answers, depending on whether $a > b$ or $a < b$. A detailed proof was furnished in [I, p. 144]. Expanding and contracting maps are considered also in [GM2, BS], and probably in other places as well. The sets $t^{-1}(R)$ are closely related to the "isolating blocks" of Conley and Easton [CE]. The local complex of sheaves $\mathbf{A}_5^\bullet = h_R^! j_R^* \mathbf{S}^\bullet$ arises in Kirwan's paper [K].

Kashiwara and Schapira have recently published very important work on the Lefschetz fixed point theorem in [K] and [KS] (the latter of which we received after this paper was submitted). Their techniques (which are quite different from ours) apply to a correspondence $C \rightrightarrows X$ which is embedded in a correspondence of smooth manifolds $\bar{C} \rightrightarrows \bar{X}$. They get both an intersection formula, involving intersections of Lagrangian submanifolds of $T^* C$, and a trace formula, in the same spirit as ours, for the local contributions. See also the discussion in §13.

(1.12) Open problems. We do not know to what extent the local contributions $n(F)$ are uniquely determined. This issue is elegantly addressed in [D1, D2] for the case of ordinary cohomology.

We have conjectured [GM2] that for any algebraic endomorphism $f: X \to X$ of a complex algebraic variety X, and for each fixed point component $F \subset X$, there exists a locally defined constructible function $n(x)$ on F such that the local contribution $n(F)$ is equal to the Euler characteristic $\chi(F; n)$ with coefficients in this constructible function. Theorem 10.3 verifies this conjecture for the case of a weakly hyperbolic self map (or correspondence).

2 Notation and terminology

In this section we summarize the main results from [V] and [GI] but translated into the subanalytic category. By "space" we will mean "subanalytic set" and by "map" we will mean "subanalytic map". Many of the results in this paper are valid for arbitrary continuous maps between paracompact Hausdorff spaces, however the proofs become much more technical in this generality and, so far as we know, the main applications will be in the subanalytic category.

Let R denote a commutative Noetherian ring of finite cohomological dimension. (We will be mainly concerned with the case $R = \mathbf{Q}$ or $R = \mathbf{Z}$.) By a "sheaf" $\mathbf{S}^\bullet$ on a space X, we will mean a bounded complex of sheaves of R-modules which are cohomologically constructible with respect to some subanalytic Whitney stratification [GM1, B, GM3, KS]. The hypercohomology of such a complex of sheaves will be denoted simply $H^*(X; \mathbf{S}^\bullet)$ and its stalk cohomology at a point $x \in X$ will be denoted $H_x^*(\mathbf{S}^\bullet)$. The derived category $D(X)$ of (cohomologically) constructible sheaves on X supports the standard operations listed in §1 of [GM1] (see also [B]).

We will write f_* instead of Rf_* to denote the induced functor on the derived category. We shall often use the identifications

(2.1a) $$\mathrm{Hom}_{D(x)}(f^* \mathbf{B}^\bullet, \mathbf{A}^\bullet) \cong \mathrm{Hom}_{D(Y)}(\mathbf{B}^\bullet, f_* \mathbf{A}^\bullet),$$

(2.1b) $$\mathrm{Hom}_{D(Y)}(f_! \mathbf{A}^\bullet, \mathbf{B}^\bullet) \cong \mathrm{Hom}_{D(X)}(\mathbf{A}^\bullet, f^! \mathbf{B}^\bullet),$$

[B, V, §10.3, 10.4] whenever $f: X \to Y$ is a map, and $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ are sheaves on X and Y respectively. By taking $\mathbf{A}^\bullet = f^* \mathbf{B}^\bullet$ (resp. $\mathbf{A}^\bullet = f^! \mathbf{B}^\bullet$, $\mathbf{B}^\bullet = f_* \mathbf{A}^\bullet$, $\mathbf{B}^\bullet = f_! \mathbf{A}^\bullet$) we obtain natural "adjunction" morphisms in $D(Y)$,

(2.2) $$\mathbf{B}^\bullet \to f_* f^* \mathbf{B}^\bullet, \qquad f_! f^! \mathbf{B}^\bullet \to \mathbf{B}^\bullet$$

and

(2.3) $$f^* f_* \mathbf{A}^\bullet \to \mathbf{A}^\bullet, \qquad \mathbf{A}^\bullet \to f^! f_! \mathbf{A}^\bullet$$

in $D(X)$. Throughout this paper we will label any of these four morphisms by "Ad". If

(*) $$\begin{array}{ccc} W & \xrightarrow{a} & X \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

is a fibre square, and if $\mathbf{A}^\bullet \in D(X)$, then there are canonical isomorphisms in $D(Y)$ [B, §V, 10.7]

(2.4a) $$b_* a^! \mathbf{A}^\bullet \cong g^! f_* \mathbf{A}^\bullet,$$

(2.4b) $$b_! a^* \mathbf{A}^\bullet \cong g^* f_! \mathbf{A}^\bullet.$$

The adjunction morphisms (2.3) and the identifications (2.1) give rise to natural morphisms in $D(Y)$.

(2.5a) $$g^* f_* \mathbf{A}^\bullet \to b_* a^* \mathbf{A}^\bullet,$$

(2.5b) $$b_! a^! \mathbf{A}^\bullet \to g^! f_! \mathbf{A}^\bullet,$$

which are isomorphisms if (*) is a fiber square and if f is proper (in which case $f_* = f_!$ and $b_* = b_!$). If (*) is a fiber square then the adjunction morphism (2.2) when combined with the canonical isomorphism (2.4) gives natural morphisms in $D(W)$ and $D(Z)$,

(2.6a) $$a^* f^! \mathbf{B}^\bullet \to b^! g^* \mathbf{B}^\bullet,$$

(2.6b) $$f_! a_* \mathbf{C}^\bullet \to g_* b_! \mathbf{C}^\bullet,$$

and by symmetry we obtain morphisms

(2.7a) $$b^* g^! \mathbf{B}^\bullet \to a^! f^* \mathbf{B}^\bullet,$$

(2.7b) $$g_! b_* \mathbf{C}^\bullet \to f_* a_! \mathbf{C}^\bullet.$$

A *correspondence* C from a space X to a space Y is a space C together with a *proper* map

$$c = (c_1, c_2): C \to X \times Y.$$

If $\mathbf{S}^\bullet$ is a complex of sheaves on X and $\mathbf{T}^\bullet$ is a complex of sheaves on Y, we define a *lift* $\mathbf{\Phi}$ of the correspondence C to the sheaf level to be a sheaf morphism,

(2.8) $$\mathbf{\Phi}: c_2^*(\mathbf{T}^\bullet) \to c_1^!(\mathbf{S}^\bullet).$$

By applying the functor $s_! c_{2*} = s_! \pi_{2*} c_!$ and adjunction, we see that the lift $\mathbf{\Phi}$ determines an element of $\mathrm{Hom}_{D(pt)}(s_! \mathbf{T}^\bullet, s_* \mathbf{S}^\bullet)$ (where s denotes the constant map to a point) [GI, §3.4.1 and Corollary 3.5] and therefore gives a ("pullback") homomorphism

(2.9 $$\mathbf{\Phi}_*^i: H_c^i(Y; \mathbf{T}^\bullet) \to H^i(X; \mathbf{S}^\bullet)$$

Now suppose that $X = Y$ and $\mathbf{S}^\bullet = \mathbf{T}^\bullet$. A point $p \in C$ is *fixed* if $c_1(p) = c_2(p)$.

(2.10) Theorem [GI, V] *Suppose X is compact and R is a field. Then the Lefschetz number*

$$L(C, \mathbf{\Phi}, \mathbf{S}^\bullet) = \sum_i (-1)^i \operatorname{Tr}(\mathbf{\Phi}_*^i: H^i(X; \mathbf{S}^\bullet) \to H^i(X; \mathbf{S}^\bullet))$$

can be expressed as a sum over the connected components $F_j \subset C$ of the set of fixed points,

$$L(C, \mathbf{\Phi}, \mathbf{S}^\bullet) = \sum_j n(F_j)$$

of local contributions $n(F_j) \in R$.

The local contributions $n(F_j)$ are locally defined in the following sense: suppose that $\mathbf{T}^\bullet$ is another complex of sheaves on X, and $\mathbf{\Psi}: c_2^*(\mathbf{T}^\bullet) \to c_1^!(\mathbf{T}^\bullet)$ is a lift of the correspondence C to the sheaf $\mathbf{T}^\bullet$. Let $F_j \subset C$ denote a connected component of the fixed point set, and let $F'_j = c_1(F_j) = c_2(F_j)$ denote its image in X. Choose open neighborhoods $U \subset C$ of F_j and $U' \subset X$ of F'_j such that $U \subset c_1^{-1}(U') \cap c_2^{-1}(U')$.

Theorem (continued) *Suppose $h: \mathbf{S}^\bullet | U \to \mathbf{T}^\bullet | U$ is a quasi-isomorphism, and suppose that the resulting diagram*

(2.11) $$\begin{array}{ccc} c_2^*(\mathbf{S}^\bullet)|U & \xrightarrow{\mathbf{\Phi}|U} & c_1^!(\mathbf{S}^\bullet)|U \\ {\scriptstyle c_2^*(h)}\downarrow & & \downarrow{\scriptstyle c_1^!(h)} \\ c_2^*(\mathbf{T}^\bullet)|U & \xrightarrow{\mathbf{\Psi}|U} & c_1^!(\mathbf{T}^\bullet)|U \end{array}$$

commute (in the constructible bounded derived category of sheaves on U). Then $n(F_j, \mathbf{S}^\bullet) = n(F_j, \mathbf{T}^\bullet)$.

(2.12) Remarks. A construction of the local contributions $n(F_j)$ is given in [GI] but it is quite complicated and difficult to compute in particular examples. In order to prove that our formula for the local contribution agrees with the construction in [GI], it is not necessary to follow their construction: it suffices to know that the terms are locally defined in the above sense. Although this is not explicitly stated in [GI] it follows immediately from the fact that all their constructions are given on the level of sheaves.

(2.13) The assumption that R is a field is not strictly necessary, however the trace of an endormorphism $\mathbf{\Phi}$ of an R-module M is usually defined only when M is a free

R-module. (See also [D1, D2, Proposition 6.3, p. 208.) This assumption may be replaced by the hypothesis that R is an integral domain, in which case the trace of $\Phi: M \to M$ is defined to be the trace of the induced homomorphism on $M/\mathrm{Tor}(M)$ or (equivalently) by tensoring with the field of fractions of R.

3 Hyperbolic fixed points

Let $c: C \to X \times X$ be a correspondence. Take a connected component $F \subset C$ of the set of fixed points of C. Let $W \subset C$ be a neighborhood of F which contains no fixed points except those in F.

(3.1) Definition. The neighborhood $W \subset C$ of F is called *weakly hyperbolic* if there exists a neighborhood $W' \subset X$ of $F' = c_1(F) = c_2(F)$ and a subanalytic map $t: W' \to \mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$ (called the indicator map) such that

(a) $W \subset c_1^{-1}(W') \cap c_2^{-1}(W')$
(b) $F' = t^{-1}(0,0)$
(c) For all $w \in W$ we have

(i) $t_1\, c_1(w) \leqq t_1\, c_2(w)$,

(ii) $t_2\, c_1(w) \leqq t_2\, c_2(w)$.

(3.2) Examples. If $f: M \to M$ has a weakly hyperbolic fixed point set F, and if $X \subset M$ is preserved by f, then $F \cap X$ is a hyperbolic fixed set in X because the same indicator map t may be used. Many examples of weakly hyperbolic self maps arise this way, especially when M is a smooth manifold and f is a smooth map.

Suppose $f_\lambda: \mathbb{C}^n \to \mathbb{C}^n$ is the time λ flow of a "linear" $\mathbb{C}^*$ action, say

$$f_\lambda(x_1, x_2, \ldots, x_n) = (\lambda^{a_1} x_1, \lambda^{a_2} x_2, \ldots, \lambda^{a_n} x_n).$$

Then f_λ is a weakly hyperbolic self map. The fixed point set is the span of the vectors $\{e_i | a_i = 0\}$ (where $e_1, e_2, \ldots, e_n$ are the standard basis vectors of $\mathbb{C}^n$). Fix $\lambda \in \mathbb{C}$ with $|\lambda| \geqq 1$. An indicator map for f_λ given by

$$t_1 = \sum \{|x_i|^2 \mid |a_i| \geqq 1\},$$
$$t_2 = \sum \{|x_i|^2 \mid |a_i| < 1\},$$

although the coordinates for which $|a_i| = 1$ could be mapped either to the t_1 axis or to the t_2 axis.

More generally, if $M = \mathbb{R}^n$ and if f is a linear self-map with no generalized eigenvalues equal to 1, then f is weakly hyperbolic and the Jordan normal form for f may be used to define the indicator map in a similar manner.

Suppose $C = \{(x, f(x)) | x \in X\}$ is the graph of a self map $f: X \to X$, and let F denote a connected component of the fixed point set. Consider the contracting and expanding sets associated to F,

$$F^+ = \{x \in X | f^{(n)}(x) \to F \text{ as } n \to \infty\},$$
$$F^- = \{x \in X | f^{(-n)}(x) \to F \text{ as } n \to \infty\}.$$

Proposition. *For any indicator map $t: W' \to \mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$ which is defined on a neighborhood W' of the fixed point component F, we have $t_1(F^+ \cap W') = 0$ and*

$t_2(F^- \cap W') = 0$ so the t_2 and t_1 axes in $\mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$ represent the contracting and expanding directions of the self map f.

Proof. This follows immediately from the equations $t_1(x) \leqq t_1 f(x)$ and $t_2(x) \geqq t_2 f(x)$. □

§4. The local groups and sheaves

(4.1) Parts of the rectangle. Fix $x_0 > 0$ and $y_0 > 0$ and let $R = R(x_0, y_0)$ denote the half open rectangle,

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqq x < x_0, 0 \leqq y < y_0\},$$

with partial closure

$$Q = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqq x \leqq x_0, 0 \leqq y < y_0\}.$$

Let $\bar{R}$ denote the closure of R and define R_B, R_L, and R_R to be the bottom, left and right sides,

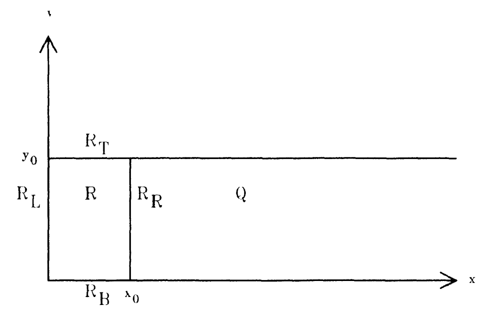
$$R_B = \{(x, 0) \mid 0 \leqq x \leqq x_0\},$$

$$R_L = \{(x_0, y) \mid 0 \leqq y < y_0\},$$

$$R_R = \{(x_0, y) \mid 0 \leqq y < y_0\}.$$

We will also make use of the bottom right corner of the rectangle, $R_{BR} = \{(x_0, 0)\}$.

(4.2) Diagram



Suppose F is a connected component of the fixed point set of a correspondence $C \to X \times X$, with indicator map $t: W' \to \mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$ and suppose that $t^{-1}(\bar{R}) \subset W'$. Let $F' = c_1(F) = t^{-1}(0, 0)$ be the image in X of the fixed point set. These parts of the rectangle give rise to subsets of the space X:

$$(4.3) \qquad \begin{array}{ccc} t^{-1}(R_L) & \xrightarrow{j_L} & X \\ h_L \uparrow & & \uparrow j_B \\ F' & \xrightarrow[h_B]{} & t^{-1}(R_B) \end{array}$$

and

$$t^{-1}(R) \xrightarrow{\alpha} t^{-1}(Q) \xrightarrow{k} X$$

(4.4) Definition. We define the five local groups, for $\mathbf{S}^\bullet \in D(X)$,

$$A_1^i = A_1^i(x_0, y_0) = H^i(t^{-1}(Q), t^{-1}(R_R); \mathbf{S}^\bullet)$$
$$A_2^i = A_2^i(y_0) = H^i(t^{-1}(R_L); j_L^!(\mathbf{S}^\bullet))$$
$$A_3^i = A_3^i(x_0) = H^i(t^{-1}(R_B), t^{-1}(R_{BR}); j_B^*(\mathbf{S}^\bullet))$$
$$A_4^i = H^i(F'; h_L^* j_L^!(\mathbf{S}^\bullet))$$
$$A_5^i = H^i(F'; h_B^! j_B^*(\mathbf{S}^\bullet)).$$

(4.5) Proposition. *If X is compact then each of these groups is finite dimensional and is the cohomology of X with coefficients in a corresponding local sheaf,*

$$\mathbf{A}_1^\bullet = \mathbf{A}_1^\bullet(x_0, y_0) = k_* \alpha_!(\mathbf{S}^\bullet | t^{-1}(R)),$$

which is supported on $t^{-1}(\bar{R})$;

$$\mathbf{A}_2^\bullet = \mathbf{A}_2^\bullet(y_0) = j_{L*} j_L^!(\mathbf{S}^\bullet),$$

which is supported on $t^{-1}(\bar{R}_L)$;

$$\mathbf{A}_3^\bullet = \mathbf{A}_3^\bullet(x_0) = j_{B!} j_B^*(\mathbf{S}^\bullet),$$

which is supported on $t^{-1}(\bar{R}_B)$;

$$\mathbf{A}_4^\bullet = j_{L*} h_{L*} h_L^* j_L^!(\mathbf{S}^\bullet),$$

which is supported on F', and

$$\mathbf{A}_5^\bullet = j_{B*} h_{B*} h_B^! j_B^*(\mathbf{S}^\bullet),$$

which is supported on F'. Furthermore, there are canonical sheaf morphisms

(4.5.1) $$\Psi_{21}: \mathbf{A}_2^\bullet(y_0) \to \mathbf{A}_1^\bullet(x_0, y_0),$$

which, for sufficiently small x_0, induce isomorphisms $A_2^i(y_0) \cong A_1^i(x_0, y_0)$ on hypercohomology;

(4.5.2) $$\Psi_{31}: \mathbf{A}_3^\bullet(x_0) \to \mathbf{A}_1^\bullet(x_0, y_0),$$

which, for sufficiently small y_0, induce isomorphisms $A_3^i(x_0) \cong A_1^i(x_0, y_0)$ on hypercohomology;

(4.5.3) $$\Psi_{24}: \mathbf{A}_2^\bullet(y_0) \to \mathbf{A}_4^\bullet,$$

which, for sufficiently small y_0, induce isomorphisms $A_2^i(y_0) \cong A_4^i$ on hypercohomology;

(4.5.4) $$\Psi_{34}: \mathbf{A}_3^\bullet(x_0) \to \mathbf{A}_4^\bullet,$$

which, for sufficiently small x_0, induces isomorphisms $A_3^i(x_0) \cong A_4^i$ on hypercohomology.

Proof. The finite dimensionality of the local groups follows from the subanalytic constructibility of the sheaves and the fact that the closed rectangle $\bar{R}$ is a subanalytic set (see [B, V, 10.3 p. 164]). The proof of the "furthermore" part will be delayed until §6.1, §7.1, and §8.1. □

(4.6) Remarks. Although we are primarily interested in the (complex of) sheaves $\mathbf{A}_1^\bullet$ and $\mathbf{A}_4^\bullet$. there is no canonical sheaf map between them. It is for this reason that we have introduced the intermediate complex of sheaves $\mathbf{A}_2^\bullet$, which maps to both. Similarly $\mathbf{A}_3^\bullet$ is intermediate between $\mathbf{A}_1^\bullet$ and $\mathbf{A}_5^\bullet$.

The main technical problem which is addressed in §5–8 is the construction of a lift of the correspondence C to each of these sheaves $\mathbf{A}_1^\bullet$, $\mathbf{A}_2^\bullet$, $\mathbf{A}_3^\bullet$, $\mathbf{A}_4^\bullet$, and $\mathbf{A}_5^\bullet$, which is "compatible" with the given lift $\mathbf{\Phi}$ of the correspondence C to the sheaf $\mathbf{S}^\bullet$. Rather than discourage the reader by starting with these involved constructions we will formalize the existence of the lifts here, and then refer to this proposition in §5–8 where the constructions are actually given.

(4.7) Theorem. *Suppose X is compact, C is a correspondence on X and F is a connected component of the fixed point set, having a weakly hyperbolic neighborhood W (which contains no fixed points other than those in F), and indicator map $t: W' \to \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. Suppose $\mathbf{\Phi}$ is a lift of this correspondence to a complex of sheaves $\mathbf{S}^\bullet$ on X. If x_0 and y_0 are chosen sufficiently small then for each $j = 1, 2, 3, 4, 5$, the lift $\mathbf{\Phi}: c_2^*(\mathbf{S}^\bullet) \to c_1^!(\mathbf{S}^\bullet)$ induces a lift*

$$\tilde{\mathbf{\Phi}}_j : c_2^*(\mathbf{A}_j^\bullet) \to c_1^!(\mathbf{A}_j^\bullet)$$

of the local sheaf $\mathbf{A}_j^\bullet$ to the correspondence, with the following properties:

(a) The lift $\tilde{\mathbf{\Phi}}_j$ *is the zero morphism outside the open neighborhood W.*

(b) *The lifts $\tilde{\mathbf{\Phi}}_j$ are compatible with the four morphisms $\mathbf{\Psi}_{ij}$ described above, in the sense that the following diagrams commute:*

$$\begin{array}{ccc} c_2^*(\mathbf{A}_1^\bullet) & \xrightarrow{\tilde{\mathbf{\Phi}}_1} & c_1^!(\mathbf{A}_1^\bullet) \\ {\scriptstyle c_2^*(\mathbf{\Psi}_{ij})}\downarrow & & \downarrow{\scriptstyle c_1^!(\mathbf{\Psi}_{ij})} \\ c_2^*(\mathbf{A}_j^\bullet) & \xrightarrow[\tilde{\mathbf{\Phi}}_j]{} & c_1^!(\mathbf{A}_j^\bullet) \end{array}$$

(c) *The morphism $\tilde{\mathbf{\Phi}}_1$ agrees with the morphism $\mathbf{\Phi}$ in the neighborhood $W \cap c_1^{-1}(R) \cap c_2^{-1}(R)$ of F, and is zero outside W.*

(d) *The lift $\tilde{\mathbf{\Phi}}_j$ induces a homomorphism on cohomology, $\tilde{\mathbf{\Phi}}_{j*}^i : A_j^i \to A_j^i$ and the local contribution $n(F)$ to the Lefschetz number $L(C, \mathbf{\Phi}; \mathbf{S}^\bullet)$ is equal to the alternating sum of the traces of these homomorphisms on the local groups,*

$$n(F) = \sum_i (-1)^i \operatorname{Tr}(\tilde{\mathbf{\Phi}}_{j*}^i).$$

In particular, this number is independent of j (i.e. *is independent of which of the local groups is used), and is also independent of the choice of* (x_0, y_0).

(4.8) Remarks. If the rectangle R is chosen sufficiently small, then the partially closed box Q may be replaced with the closed box, $\bar{R}$ in the definition of the groups A_1^i, in other words,

$$H^i(t^{-1}(Q), t^{-1}(R_R); \mathbf{S}^\bullet) \cong H^i(t^{-1}(\bar{R}), t^{-1}(\bar{R}_R); \mathbf{S}^\bullet).$$

The local groups may also be described as the cohomology groups (with appropriate supports) of the complex of sheaves $\mathbf{T}^\bullet = t_*(\mathbf{S}_i^\bullet \mid W')$ on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. For example, $A_1^i = H^i(Q, R_R; \mathbf{T}^\bullet)$. The local groups and sheaves are determined from data on X and the indicator map, and do not otherwise involve the correspondence C. The purpose of the correspondence is to determine an endomorphsim of each local group. The local groups are usually different even though the alternating sum of the traces on each is the same. In fact, the local groups $A_1^*(x_0, y_0)$ may even vary as (x_0, y_0) changes (but see §9). In §10 we will decompose $n(F)$ into a sum over the strata of F'.

5 The local group A_1

(5.1) In this section we prove Theorem 4.7 for the case $j = 1$. We refer to §3 and 4 for the definitions of C, R, Q, A_1, $\mathbf{\Phi}$, and W. The local group A_1^i is the hypercohomology of the local sheaf, $\mathbf{A}_1^\bullet = k_* \alpha_! (\mathbf{S}^\bullet | R)$, i.e.

$$A_1^i = H^i(t^{-1}(Q), t^{-1}(R_R); \mathbf{S}^\bullet) = H^i(X; \mathbf{A}_1^\bullet).$$

Part (d) of Theorem 4.7 follows from parts (a) and (c) by applying the Lefschetz fixed point theorem to the local sheaf $\mathbf{A}_1^\bullet$. According to part (a), the correspondence C has a lift $\tilde{\mathbf{\Phi}}_1$ to the sheaf $\mathbf{A}_1^\bullet$, which is the zero morphism outside W. Applying the fixed point theorem of Grothendieck and Verdier to this lift $\tilde{\mathbf{\Phi}}_1$, we see that the Lefschetz number $L(C, \tilde{\mathbf{\Phi}}_1, \mathbf{A}_1^\bullet)$ is equal to the single local contribution $n(F; \mathbf{A}_1^\bullet)$ since W contains no fixed points other than F. According to part (c), this lift $\tilde{\mathbf{\Phi}}_1$ agrees with the original lift $\mathbf{\Phi}$ in a neighborhood of F. Thus the fixed point theorem of Grothendieck and Verdier (continued) says that the local contributions are the same: $n(F; \mathbf{A}_1^\bullet) = n(F; \mathbf{S}^\bullet)$. In summary,

$$n(F; \mathbf{S}^\bullet) = n(F; \mathbf{A}_1^\bullet) = L(C, \tilde{\mathbf{\Phi}}_1; \mathbf{A}_1^\bullet) = \sum (-1)^i \, Tr\,(\tilde{\mathbf{\Phi}}_{1*}^i : A_1^i \to A_1^i).$$

Thus we must prove parts (a) and (c), i.e. we must construct a lift $\tilde{\mathbf{\Phi}}_1$ of the correspondence to the local sheaf $\mathbf{A}_1^\bullet$, which is compatible with the original lift $\mathbf{\Phi}$ and which vanishes outside W.

(5.2) We will use the notation $R_i = W \cap c_i^{-1} t^{-1}(R)$ and $Q_i = W \cap c_i^{-1} t^{-1}(Q)$ (for $i = 1, 2$). The "weakly hyperbolic" assumption (§3(c) (ii)) implies $Q_1 \subset Q_2$ so we obtain the following commutative diagram of subsets and inclusions:

(5.3)
$$\begin{array}{ccccc} & & C & & \\ & \overset{k_1}{\nearrow} & & \overset{k_2}{\nwarrow} & \\ Q_1 & & \overset{b}{\longrightarrow} & & Q_2 \\ \alpha_1 \uparrow & \overset{\delta}{\nwarrow} & & & \uparrow \alpha_2 \\ R_1 & \underset{a_1}{\longleftarrow} & R_1 \cap R_2 & \underset{a_2}{\longrightarrow} & R_2 \end{array}$$

The hyperbolic assumption §3.1(c)(i) implies $R_1 \cap R_2 = Q_1 \cap R_2$ so the lower right hand corner of this diagram is a fibre square. Using the adjunction morphisms (§1) it is now possible to define a natural morphism,

$$\Omega : k_{2*} \alpha_{2!} a_{2*} (\mathbf{B}^\bullet) \to k_{1*} \alpha_{1!} a_{1!} (\mathbf{B}^\bullet)$$

for any sheaf $\mathbf{B}^\bullet$ on $R_1 \cap R_2$, as follows:

(5.4) Definition. The morphsim Ω is the composition

$$k_{2*}\,\alpha_{2!}\,a_{2*}(\mathbf{B}^\bullet)\xrightarrow{\mathrm{Ad}}k_{2*}\,b_*\,b^*\,\alpha_{2!}\,a_{2*}(\mathbf{B}^\bullet)=k_{2*}\,b_*\,\delta_!\,a_2^*\,a_{2*}(\mathbf{B}^\bullet)$$

$$=k_{1*}\,\delta_!(\mathbf{B}^\bullet)=k_{1*}\,\alpha_{1!}\,a_{1!}(\mathbf{B}^\bullet).$$

(5.5) Introducing the further notation, $\tilde{R}_j=c_j^{-1}\,t^{-1}(R)$ and $\tilde{Q}_j=c_j^{-1}\,t^{-1}(Q)$ (for $j=1,2$), we have a diagram of inclusions (for $j=1$ or 2),

$$\text{(5.6)}\qquad \begin{array}{ccccccc} R_1\cap R_2 & \xrightarrow{a_j} & R_j & \xrightarrow{\alpha_j} & Q_j & \longrightarrow & W\\ {\scriptstyle i}\downarrow & & {\scriptstyle i_j}\downarrow & & {\scriptstyle i_j}\downarrow & & {\scriptstyle i_j}\downarrow\\ \tilde{R}_1\cap\tilde{R}_2 & \xrightarrow[\tilde{a}_j]{} & \tilde{R}_j & \xrightarrow[\tilde{\alpha}_j]{} & \tilde{Q}_j & \xrightarrow[\tilde{k}_j]{} & C \end{array}$$

where each box is a fiber square since the top row is obtained from the bottom row by intersecting with W.

(5.7) Proposition. *If $R=R(x_0,y_0)$ is chosen sufficiently small, then for any sheaf $\mathbf{B}^\bullet$ on $R_1\cap R_2$, and for $j=1$ or 2, the morphism*

$$i_{j!}\,\alpha_{j!}\,a_{j!}(\mathbf{B}^\bullet)\rightarrow i_{j*}\,\alpha_{j!}\,a_{j!}(\mathbf{B}^\bullet)$$

is a quasi-isomorphism.

(5.8) Proof. The proof will be delayed until Sect. 5.13.

(5.9) Proof of Theorem 4.7 The desired morphism $\tilde{\mathbf{\Phi}}_1$ may be obtained as the following composition, the intuition for which is described in §1.10 of the introduction:

$$c_2^*\,k_*\,\alpha_!(\mathbf{S}^\bullet|t^{-1}R)\xrightarrow{(2.5)}\tilde{k}_{2*}\,c_2^*\,\alpha_!(\mathbf{S}^\bullet|t^{-1}R)\overset{(2.4)}{=}\tilde{k}_{2*}\,\tilde{\alpha}_{2!}\,c_2^*(\mathbf{S}^\bullet|t^{-1}R)$$

$$=\tilde{k}_{2*}\,\tilde{\alpha}_{2!}(c_2^*\,\mathbf{S}^\bullet|c_2^{-1}\,t^{-1}R)\xrightarrow{(2.2)}\tilde{k}_{2*}\,\tilde{\alpha}_{2!}\,i_{2*}\,a_{2*}(c_2^*\,\mathbf{S}^\bullet|R_1\cap R_2)$$

$$\xrightarrow{(2.6)}k_{2*}\,i_{2*}\,\alpha_{2!}\,a_{2*}(c_2^*\,\mathbf{S}^\bullet|R_1\cap R_2)=k_{2*}\,\alpha_{2!}\,a_{2*}(c_2^*\,\mathbf{S}^\bullet|R_1\cap R_2)$$

$$\xrightarrow{\Omega}k_{1*}\,\alpha_{1!}\,a_{1!}(c_2^*\,\mathbf{S}^\bullet|R_1\cap R_2)\xrightarrow{\mathbf{\Phi}}k_{1*}\,\alpha_{1!}\,a_{1!}(c_1^!\,\mathbf{S}^\bullet|R_1\cap R_2)$$

$$=\tilde{k}_{1*}\,i_{1*}\,\alpha_{1!}\,a_{1!}(c_1^!\,\mathbf{S}^\bullet|R_1\cap R_2)\overset{(5.7)}{=}\tilde{k}_{1*}\,i_{1!}\,\alpha_{1!}\,a_{1!}(c_1^!\,\mathbf{S}^\bullet|R_1\cap R_2)$$

$$=\tilde{k}_{1*}\,\tilde{\alpha}_{1!}\,a_{1!}\,i_{1!}(c_1^!\,\mathbf{S}^\bullet|R_1\cap R_2)=\tilde{k}_{1*}\,\tilde{\alpha}_{1!}\,\tilde{a}_{1!}\,i_{1!}\,i_1^!(c_1^!\,\mathbf{S}^\bullet|\tilde{R}_1\cap\tilde{R}_2)$$

$$\xrightarrow{(2.2)}\tilde{k}_{1*}\,\tilde{\alpha}_{1!}\,\tilde{a}_{1!}(c_1^!\,\mathbf{S}^\bullet|\tilde{R}_1\cap\tilde{R}_2)=\tilde{k}_{1*}\,\tilde{\alpha}_{1!}\,\tilde{a}_{1!}\,\tilde{a}_1^!(c_1^!\,\mathbf{S}^\bullet|\tilde{R}_1)$$

$$\xrightarrow{(2.2)}\tilde{k}_{1*}\,\tilde{\alpha}_{1!}(c_1^!\,\mathbf{S}^\bullet|\tilde{R}_1)\xrightarrow{(2.5)}\tilde{k}_{1*}\,c_1^!\,\alpha_!(\mathbf{S}^\bullet|t^{-1}R)\overset{(2.4)}{=}c_1^!\,k_*\,\alpha_!(\mathbf{S}^\bullet|t^{-1}R).$$

This composition agrees with Φ in a neighborhood of F because each of the above morphisms (except for Φ) is the identity in a neighborhood of F. In fact these morphisms only change the supports of the sheaf $\mathbf{S}^\bullet$ at the edges of the rectangle and at the edges of the intersections of rectangles. The composition $\tilde{\Phi}$ vanishes outside W because it factors through $c_2^* \mathbf{S}^\bullet|(R_1 \cap R_2)$. □

The remainder of this section is devoted to the proof of Proposition 5.7. In general, even by choosing $R = R(x_0, y_0)$ very small, we cannot guarantee that $c_1^{-1} t^{-1}(R) \cap c_2^{-1} t^{-1}(R) \subset W$ because there may be several different fixed point components in the correspondence C which project to F', and which may have different expanding-contracting behavior. This difficulty may be overcome using the following result:

(5.10) Lemma. *If $R = R(x_0, y_0)$ is chosen sufficiently small then there is a unique connected component R_{12} of $c_1^{-1} t^{-1}(R) \cap c_2^{-1} t^{-1}(R)$ which contains F, and its closure $\overline{R_{12}}$ is contained in W. (Hence $R_{12} = R_1 \cap R_2$.)*

(5.11) Proof. It suffices to find a single value of (x_0, y_0) with the desired properties because they are clearly inherited by subsets. We have the following arrangement of sets and maps:

$$\begin{array}{ccccc} C & \overset{c_1}{\underset{c_2}{\rightrightarrows}} & X & & \\ \cup & & \cup & & \\ W & & W' & \overset{t}{\longrightarrow} & \mathbb{R}_+ \times \mathbb{R}_+ \\ \cup & & \cup & & \cup \\ F & & F' & & (0, 0) \end{array} \tag{5.12}$$

By choosing W' sufficiently small, we may assume that the indicator map $t: W' \to \mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$ extends to the closure $\bar{W}'$. Hence $t|\bar{W}'$ is proper. Now suppose such an $R(x_0, y_0)$ does not exist. Then there is a sequence $R^1 \supset R^2 \supset \ldots$ of rectangles so that $t^{-1}(R^j)$ converge to F', in other words, so that

$$\bigcap_j t^{-1}(R^j) = F'$$

and so that the closure $\bar{R}^j_{12}$ is not contained in W. (Here R^j_{12} denotes the unique component of $c_1^{-1} t^{-1}(R^j) \cap c_2^{-1} t^{-1}(R^j)$ which contains F.) Let $z_j \in R^j_{12} - W$. Since c is proper, the sequence $\{z_j\}$ contains a convergent subsequence. Thus we may assume the z_j converge to some z_0 and hence $z_0 \in c_1^{-1}(F') \cap c_2^{-1}(F')$. However the z_j were chosen to be in the component which contains F, so in fact $z_0 \in F$. This contradicts the assumption that $z_j \notin W$. □

(5.13) Proof of Proposition 5.7. We now assume that (x_0, y_0) has been chosen as in Lemma 5.10. The idea is that the sheaf of $i_{j*} \alpha_{j!} a_{j!} \mathbf{B}^\bullet$ is supported on the closure $\overline{R_1 \cap R_2 \cap \tilde{Q}_j}$ which coincides with $\overline{R_1 \cap R_2} \cap \tilde{Q}_j$ by the Lemma 5.10. The inclusion of this closure into $\tilde{Q}_j$ is proper, so $i_{j*} = i_{j!}$. Here are the details:

By intersecting with W we obtain a fibre square (for each $j = 1, 2$) of inclusions,

$$\begin{array}{ccccc} R_1 \cap R_2 & \xrightarrow{m_j} & \overline{R_1 \cap R_2} \cap Q_j & \xrightarrow{r_j} & Q_j \\ & & {\scriptstyle i_j} \downarrow & & {\scriptstyle i_j} \downarrow \\ & & \overline{R_1 \cap R_2} \cap \tilde{Q}_j & \xrightarrow{\tilde{r}_j} & \tilde{Q}_j \end{array}$$

Here, r_j and $\tilde{r}_j$ are closed embeddings, and the left vertical arrow i_j is the identity (by Lemma 5.10). Furthermore the composition $r_j m_j = \alpha_j a_j$ is just the inclusion into Q_j. Therefore for any sheaf $\mathbf{B}^\bullet$ on $R_1 \cap R_2$, we have

$$i_{j!}\,\alpha_{j!}\,a_{j!}(\mathbf{B}^\bullet) = i_{j!}\,r_{j!}\,m_{j!}(\mathbf{B}^\bullet) = \tilde{r}_{j!}\,i_{j!}\,m_{j!}(\mathbf{B}^\bullet) = \tilde{r}_{j*}\,i_{j*}\,m_{j!}(\mathbf{B}^\bullet)$$

$$= i_{j*}\,r_{j*}\,m_{j!}(\mathbf{B}^\bullet) = i_{j*}\,r_{j!}\,m_{j!}(\mathbf{B}^\bullet) = i_{j*}\,\alpha_{j!}\,a_{j!}(\mathbf{B}^\bullet). \qquad \square$$

6 The local group A_2

(6.1) In this section we prove Theorem 4.7 in the case $j = 2$. We will (1) construct a sheaf map $\boldsymbol{\Psi}_{21}: \mathbf{A}_2^\bullet(y_0) \to \mathbf{A}_1^\bullet(x_0, y_0)$ which, for sufficiently small x_0, induces isomorphisms on cohomology, and (2) find a lift $\tilde{\boldsymbol{\Phi}}_2$ of the correspondence $(C, \boldsymbol{\Phi})$ to the sheaf $\mathbf{A}_2^\bullet$ (which is the zero morphism outside the neighborhood W of the fixed point component F), which commutes with the lift to $\mathbf{A}_1^\bullet$.

By taking x_0 sufficiently small, we conclude that the Lefschetz number $L(C, \boldsymbol{\Phi}, \mathbf{A}_1^\bullet)$ is equal to the Lefschetz number $L(C, \boldsymbol{\Phi}, \mathbf{A}_2^\bullet)$ so Theorem 4.7 (case $j = 2$) follows from Theorem 4.7 (case $j = 1$). $\square$

(6.2) Corollary. *The alternating sum of the traces of the action of the correspondence* $(C, \boldsymbol{\Phi})$ *on the cohomology of the sheaf* $\mathbf{A}_2^\bullet$ *is independent of the choice of* x_0 *and* y_0.

Here are the details for step (1) and step (2). We refer to § 3 and § 4 for the definitions of R, Q, W, t, A_1, and A_2.

(6.3) Step (1) Since the indicator map t is subanalytic, it may be stratified. The resulting stratifications of $\mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$ and of X may be chosen so that the sheaf $\mathbf{S}^\bullet$ is cohomologically locally constant on each stratum and so that the restriction of t to each stratum of X is a smooth submersion to a stratum of $\mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$. The strata in $\mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$ are subanalytic and consist of finitely many points (0-dimensional strata) and finitely many analytic curves (1-dimensional strata).

(6.4) Lemma. *Suppose* x_0 *is chosen so that*

(1) *the region* $\bar{R}$ *contains no 0-dimensional strata except those in* $\bar{R}_L$,

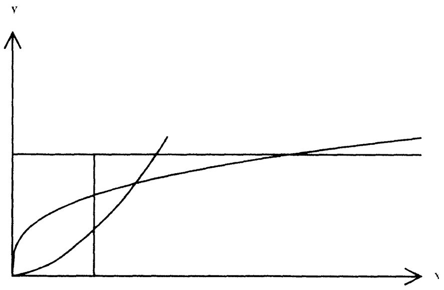
(2) *the top* R_T *of the box does not intersect any 1-dimensional strata except* R_L, *and*

(3) *for each* $0 < x < x_0$ *the vertical segment* $\{x\} \times (0, y_0]$ *is transverse to the 1-dimensional strata. (This may be achieved by making* x_0 *sufficiently small).*

Then for any $x \leqq x_0$ *the inclusion* $R(x, y_0) \to R(x_0, y_0)$ *induces an isomorphism on the cohomology groups*

$$A_1^i(x, y_0) \cong A_1^i(x_0, y_0).$$

(6.5) Diagram.



Stratification of $\mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$

(6.6) Proof of Lemma 6.4 Let $\mathbf{T}^\bullet = t_*(\mathbf{S}^\bullet | W)$ denote the pushforward to $\mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$ of the sheaf $\mathbf{S}^\bullet$. Since the map t was stratified and since $\mathbf{S}^\bullet$ is (cohomologically) constructible with respect to the stratification of X, the sheaf $\mathbf{T}^\bullet$ is (cohomologically) constructible with respect to the stratification of $\mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$. Furthermore the local group

$$A_1^i = H^i(Q, R_R; \mathbf{T}^\bullet)$$

is the cohomology of $\mathbf{T}^\bullet$ with appropriate supports in the rectangle R. It follows from "moving the wall" [GM3, (§4.4)] that this group does not change as x_0 shrinks. For completeness we also give the following direct argument: The bottom $\bar{R}_B$ of the box is the closed interval $[0, x_0]$ which is canonically stratified by its endpoints. Assumptions (a), (b), and (c) on the stratification of the box $\bar{R}$ guarantee that the projection to the x axis,

$$F: \bar{R} \to \bar{R}_B$$

is a stratified map. Therefore the sheaf $\tilde{\mathbf{T}}^\bullet = F_*(\mathbf{T}^\bullet | (\bar{R} - R_T))$ is (cohomologically) constructible on $[0, x_0]$. The local group

$$A_1^i = H^i([0, x_0], \{x_0\}; \tilde{\mathbf{T}}^\bullet) = H_c^i([0, x_0); \tilde{\mathbf{T}}^\bullet)$$

is now the cohomology of an interval. The E_2 term of the spectral sequence for this hypercohomology is $E_2^{pq} = H_c^p([0, x_0); H^q(\tilde{\mathbf{T}}^\bullet))$ which is independent of x_0 since the cohomology sheaf $H^q(\tilde{\mathbf{T}}^\bullet)$ is constant on the open interval $(0, x_0)$. The spectral sequence comparison theorem implies that the hypercohomology is also independent of x_0. $\square$

(6.7) The inclusion $j_L: t^{-1}(R_L) \to X$ factors as

$$t^{-1}(R_L) \xrightarrow{\beta} t^{-1}(R) \xrightarrow{\alpha} t^{-1}(Q) \xrightarrow{k} X,$$

where β is a closed embedding and α is an open inclusion.

(6.8) Proposition. *For $x_0 > 0$ sufficiently small, the adjunction map*

$$\alpha_! \beta_! \beta^! (\mathbf{S}^\bullet | t^{-1} R) \xrightarrow{\text{Ad}} \alpha_! (\mathbf{S}^\bullet | t^{-1}(R))$$

extends to a map on the local sheaves, $\mathbf{A}_2^\bullet(y_0) \to \mathbf{A}_1^\bullet(x_0, y_0)$ *and induces an isomorphism on the cohomology groups,*

$$A_2^i = H^i(t^{-1}Q;\ \alpha_! \beta_! \beta^!(\mathbf{S}^\bullet|R)) \to H^i(t^{-1}Q; \alpha_!(\mathbf{S}^\bullet|R)) = A_1^i.$$

(6.9) Proof. Since β is a closed embedding, α is an open embedding, and $t^{-1}(\bar{R}_L) \cap Q \subset t^{-1}(R) \cap Q$, we have $\beta_* = \beta_!$, $\alpha^* = \alpha^!$, and $\alpha_* \beta_! = \alpha_! \beta_!$. Applying k_* to the above morphism we obtain

$$\begin{aligned}\mathbf{A}_2^\bullet &= j_{L*} j_L^! \mathbf{S}^\bullet = k_* \alpha_* \beta_* \beta^! \alpha^! k^! \mathbf{S}^\bullet = k_* \alpha_* \beta_* \beta^! (\mathbf{S}^\bullet|R)\\ &= k_* \alpha_* \beta_! \beta^! (\mathbf{S}^\bullet|R) = k_* \alpha_! \beta_! \beta^! (\mathbf{S}^\bullet|R) \xrightarrow{\mathrm{Ad}} k_* \alpha_! (\mathbf{S}^\bullet|R)\\ &= \mathbf{A}_1^\bullet.\end{aligned}$$

We now show the induced map on cohomology is an isomorphism. The functor k_* does not change the cohomology of the sheaves so it suffices to compute the induced map on the cohomology of the sheaves $\alpha_! \beta_! \beta^! (\mathbf{S}^\bullet|t^{-1}R) \to \alpha_!(\mathbf{S}^\bullet|t^{-1}(R))$. The first group is just $A_2^i = H^i(t^{-1}(R_L); \beta^!(\mathbf{S}^\bullet|R))$ which equals the cohomology of $\mathbf{S}^\bullet$ with supports in $t^{-1}(R_L)$, i.e.

$$A_2^i = \lim_{x_0 \to 0} H^i(t^{-1}Q; \alpha_!(\mathbf{S}^\bullet|R))$$

(see [B, V § 1.8 p. 51 and VI § 3.11 p. 204]). By Lemma 6.4, this limit stabilizes for sufficiently small x_0. □

(6.10) Step (2) We will denote the inclusion $R_{LK} = W \cap c_k^{-1} t^{-1}(R_L) \to C$ by j_k (for $k = 1, 2$). In the following fibre square,

$$(6.11)\qquad \begin{array}{ccccc} R_{L1} \cap R_{L2} & \xrightarrow{a_2} & R_{L2} & & \\ a_1 \downarrow & & \downarrow j_2 & & \\ R_{L1} & \xrightarrow[j_1]{} & W & \xrightarrow[i]{} & C \end{array}$$

the inclusion a_2 is open and the inclusion a_1 is closed, by the expanding-contracting hypothesis (§3(c)). For any complex of sheaves $\mathbf{B}^\bullet$ supported on W, we obtain a quasi-isomorphism (which is the analogue of the map Ω of § 5.4),

$$\Omega: j_{2*} a_{2*} a_2^* j_2^! (\mathbf{B}^\bullet) \cong j_{1*} a_{1!} a_1^! j_1^! (\mathbf{B}^\bullet)$$

as the composition

$$j_{2*} a_{2*} a_2^* j_2^! (\mathbf{B}^\bullet) = j_{2*} a_{2*} a_2^! j_2^! (\mathbf{B}^\bullet) = j_{1*} a_{1*} a_1^! j_1^! (\mathbf{B}^\bullet) = j_{1*} a_{1!} a_1^! j_1^! (\mathbf{B}^\bullet).$$

We will also make use of the fibre squares, (for $k = 1, 2$)

$$(6.12)\qquad \begin{array}{ccccc} R_{L1} \cap R_{L2} & \xrightarrow[a_k]{} & R_{Lk} & \xrightarrow[j_k]{} & W \\ \downarrow i & & \downarrow i_k & & \downarrow i \\ \tilde{R}_{L1} \cap \tilde{R}_{L2} & \xrightarrow[\tilde{a}_k]{} & \tilde{R}_{Lk} & \xrightarrow[\tilde{j}_k]{} & C \\ & & \downarrow c_k & & \downarrow c_k \\ & & t^{-1}(R_L) & \xrightarrow[j]{} & X \end{array}$$

where $\tilde{R}_{Lk} = c_k^{-1} t^{-1}(R_L)$. The following proposition is the analogue of Proposition 5.7.

(6.13) Proposition. *For any complex of sheaves* $\mathbf{B}^\bullet$ *on* $R_{L1} \cap R_{L2}$ *the morphism*

$$i_{k!}\, a_{k!}\, (\mathbf{B}^\bullet) \to i_{k*}\, a_{k!}(\mathbf{B}^\bullet)$$

is a quasi-isomorphism.

The proof is the same as that in § 5.13. □

(6.14) Definition. The lift $\tilde{\Phi}_2$ of (C, Φ) to the sheaf $\mathbf{A}_2^\bullet = j_* j^! \mathbf{S}^\bullet$ is obtained as the composition

$$c_2^* \mathbf{A}_2^\bullet = c_2^* j_* j^! \mathbf{S}^\bullet \xrightarrow{(2.5)} \tilde{j}_{2*}\, c_2^* j^! \mathbf{S}^\bullet \xrightarrow{(2.6)} \tilde{j}_{2*} \tilde{j}_2^!\, c_2^* \mathbf{S}^\bullet$$

$$\xrightarrow{(2.2)} \tilde{j}_{2*} \tilde{j}_2^!\, i_* i^* c_2^* \mathbf{S}^\bullet \overset{(2.4)}{=} \tilde{j}_{2*}\, i_{2*} j_2^!\, i^* c_2^* \mathbf{S}^\bullet = i_* j_{2*} j_2^!\, i^* c_2^* \mathbf{S}^\bullet$$

$$\xrightarrow{(2.2)} i_* j_{2*}\, a_{2*}\, a_2^* j_2^!\, i^* c_2^* \mathbf{S}^\bullet \overset{\Omega}{\cong} i_* j_{1*}\, a_{1!}\, a_1^! j_1^!\, i^! c_2^* \mathbf{S}^\bullet$$

$$\xrightarrow{\Phi} i_* j_{1*}\, a_{1!}\, a_1^! j_1^!\, i^! c_1^! \mathbf{S}^\bullet = \tilde{j}_{1*}\, i_{1*}\, a_{1!}\, a_1^!\, i_1^! \tilde{j}_1^!\, c_1^! \mathbf{S}^\bullet$$

$$\overset{(6.3)}{=} \tilde{j}_{1*}\, i_{1!}\, a_{1!}\, a_1^!\, i_1^! \tilde{j}_1^!\, c_1^! \mathbf{S}^\bullet \xrightarrow{(2.2)} \tilde{j}_{1*} \tilde{j}_1^!\, c_1^! \mathbf{S}^\bullet$$

$$\overset{(2.4)}{=} c_1^!\, j_* j^! \mathbf{S}^\bullet = c_1^!\, (\mathbf{A}^\bullet).$$

The morphism $\tilde{\Phi}_2$ vanishes outside W because it factors through $R_{L1} \cap R_{L2}$. It is a straightforward exercise in diagram chasing to verify that the induced map on cohomology is compatible with the homomorphism $\mathbf{A}_2^* \to \mathbf{A}_1^*$. □

7 The local group A_4

(7.1) In this section we prove Theorem 4.7 for the case $j = 4$. We will

(1) construct a sheaf map $\Psi_{24}: \mathbf{A}_2^\bullet(y_0) \to \mathbf{A}_4^\bullet$ which, for sufficiently small y_0, induces an isomorphism on cohomology, and

(2) find a lift $\tilde{\Phi}_4$ of the correspondence (C, Φ) to the sheaf $\mathbf{A}_4^\bullet$ which vanishes outside a neighborhood W of the fixed point set F, and which commutes with the map from $\mathbf{A}_2^\bullet$.

By choosing y_0 sufficiently small we conclude that the Lefschetz number $L(C, \Phi, \mathbf{A}_2^\bullet)$ is equal to the Lefschetz number $L(C, \Phi, \mathbf{A}_4^\bullet)$, so Theorem 4.7 (case $j = 4$) follows from Theorem 4.7 (case $j = 2$). □

Here are the details for steps 1 and 2. As in § 4.3 we have inclusions

$$t^{-1}(0) \xrightarrow{h_L} t^{-1}(R_L) \xrightarrow{j_L} X.$$

(7.2) Step 1. Since $j^! \mathbf{S}^\bullet$ is supported on $t^{-1}(R_L)$ we have

$$\begin{aligned} \mathbf{A}_4^\bullet &= j_{L*}\, h_{L*}\, h_L^* j_L^! \mathbf{S}^\bullet = j_{L*}\, h_{L*}\, h_L^* j_L^* j_{L*} j_L^! \mathbf{S}^\bullet \\ &= (j_L h_L)_* (j_L h_L)^* (j_{L*} j_L^! \mathbf{S}^\bullet) \\ &= (j_L h_L)_* (j_L h_L)^* \mathbf{A}_2^\bullet . \end{aligned}$$

Thus the natural map $\mathbf{A}_2^\bullet \to \mathbf{A}_4^\bullet$ is simply the adjunction map. Now consider the induced map on cohomology. The group A_4^i is

$$A_4^i = H^i(t^{-1}(0); h_L^* j_L^! \mathbf{S}^\bullet) = \lim_{y_0 \to 0} H^i(t^{-1} R_L; j_L^! \mathbf{S}^\bullet).$$

But as in § 6, this limit stabilizes for sufficiently small y_0, using the fact that the map t may be stratified so that $\mathbf{S}^\bullet$ is cohomologically locally constant on each stratum. In fact, it suffices to choose y_0 so small that $R_L = R_L(y_0)$ contains no 0-dimensional strata other than $\{0\}$. In summary, for $y_0 > 0$ sufficiently small, the above adjunction map induces isomorphisms on cohomology, $A_2^i \cong A_4^i$.

(7.3) Step 2. Let $\tilde{F} = c_1^{-1} t^{-1}(0) \cap c_2^{-1} t^{-1}(0)$ denote the set of fixed points which map to $F' = c_1(F) = c_2(F)$. We have a diagram of fibre squares:

$$\begin{array}{ccccc} F & \xrightarrow{h_{Lk}} & R_{Lk} & \xrightarrow{j_{Lk}} & W \\ {\scriptstyle i}\downarrow & & {\scriptstyle i_k}\downarrow & & {\scriptstyle i}\downarrow \\ \tilde{F} & \xrightarrow{\tilde{h}_{Lk}} & \tilde{R}_{Lk} & \xrightarrow{\tilde{j}_{Lk}} & C \\ {\scriptstyle c_k}\downarrow & & {\scriptstyle c_k}\downarrow & & {\scriptstyle c_k}\downarrow \\ F' & \xrightarrow{h_L} & t^{-1}(R_L) & \xrightarrow{j_L} & X. \end{array}$$

Since F' is fixed under the correspondence C, we have

$$\tilde{j}_{L1}\, \tilde{h}_{L1} = \tilde{j}_{L2}\, \tilde{h}_{L2} : F \to C.$$

The correspondence (C, Φ) then lifts to $\mathbf{A}_4^\bullet$ as follows:

$$\begin{aligned} c_2^* \mathbf{A}_4^\bullet &\overset{(7.2)}{=} c_2^* (j_L h_L)_* (j_L h_L)^* (\mathbf{A}_2^\bullet) \xrightarrow{(2.5)} (\tilde{j}_{L2}\, \tilde{h}_{L2})_* (\tilde{j}_{L2}\, \tilde{h}_{L2})^* c_2^* \mathbf{A}_2^\bullet \\ &\xrightarrow{(6.14)} (\tilde{j}_{L2}\, \tilde{h}_{L2})_* (\tilde{j}_{L2}\, \tilde{h}_{L2})^* c_1^! \mathbf{A}_2^\bullet = (\tilde{j}_{L1}\, \tilde{h}_{L1})_* (\tilde{j}_{L1}\, \tilde{h}_{L1})^* c_1^! \mathbf{A}_2^\bullet \\ &\xrightarrow{(2.6)} c_1^! (j_L h_L)_* (j_L h_L)^* \mathbf{A}_2^\bullet \overset{(7.2)}{=} c_1^! \mathbf{A}_4^\bullet . \end{aligned}$$

It is a straightforward exercise to check that this lift is compatible with the lift to $\mathbf{A}_2^\bullet$. □

8 The local groups A_3 and A_5

In this section we prove Theorem 4.7 for the cases $j = 3$ and 5. This section is dual to § 6 and § 7 so we will only sketch the procedure. The analogy to lemma 6.4 is:

(8.1) Lemma. *Suppose y_0 is chosen so that*

(1) *$\bar{R}(x_0, y_0)$ contains no 0-dimensional strata except those in $\bar{R}_B$;*

(2) *the region R_R does not intersect any 1-dimensional strata except R_B;*

(3) *for each $y < y_0$ the horizontal segment $(0, x_0] \times \{y\}$ is transverse to every 1-dimensional stratum in $\bar{R}$. (This may be achieved by choosing y_0 sufficiently small.)*

Then the inclusion $\bar{R}(x_0, y) \to \bar{R}(x_0, y_0)$ induces an isomorphism

$$A_1^i(x_0, y) \cong A_1^i(x_0, y_0)$$

for all $y < y_0$.

(8.2) Proof. The proof is the same as that in § 6. □

In summary, there are natural maps obtained from adjunction,

$$\mathbf{A}_1^\bullet = \mathbf{A}_1^\bullet(x_0, y_0) \xrightarrow{\omega} \mathbf{A}_3^\bullet = \mathbf{A}_3^\bullet(x_0) \xleftarrow{\tau} \mathbf{A}_5^\bullet$$

such that ω induces an isomorphism on cohomology for sufficiently small y_0, and τ induces isomorphisms on cohomology for sufficiently small x_0. The correspondence $(C, \mathbf{\Phi})$ lifts to the sheaves $\mathbf{A}_3^\bullet$ and $\mathbf{A}_5^\bullet$ in a way which is compatible with the maps ω and τ and the lift of $(C, \mathbf{\Phi})$ to $\mathbf{A}_1^\bullet$. Since the Lefschetz number $L(C, \tilde{\mathbf{\Phi}}, \mathbf{A}_1^\bullet)$ (i.e. the alternating sum of the traces of the induced map on cohomology) is independent of x_0 and y_0, we conclude that it is equal to the Lefschetz number $L(C, \tilde{\mathbf{\Phi}}, \mathbf{A}_3^\bullet)$, which is hence also independent of x_0. We conclude again that this same number is equal to $L(C, \tilde{\mathbf{\Phi}}, \mathbf{A}_5^\bullet)$. □

9 Equality of the local groups

(9.1) In this section we give a sufficient condition for equality among the local groups. As in § 3 we suppose W' is an open subset of X and $t: W' \to \mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$ is an indicator map for a certain hyperbolic component of the fixed point set of a correspondence $c: C \to X \times X$. It is possible to choose Whitney stratifications of W' and of $\mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$ so that t is a stratified map, i.e. it takes each stratum of W' submersively onto a stratum of $\mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$. The "singular" strata in the target will consist of finitely many points and curves.

(9.2) Proposition. *Suppose the stratification of $\mathbb{R}_{\geqq 0} \times \mathbb{R}_{\geqq 0}$ contains no 0 or 1 dimensional strata in the region $[0, \varepsilon] \times [0, \varepsilon]$ except for the origin, the x-axis and the y-axis. Let $0 < x_0, y_0 < \varepsilon$. Then the local groups $A_j^i(x_0, y_0)$ are independent of (x_0, y_0) and they are all canonically isomorphic.*

(9.3) Proof. It suffices to show that $A_1^i(x_0, y_0)$ does not depend on the point (x_0, y_0) since the other groups are obtained from A_1 by various limit procedures. Hypothesis (3) of lemma 6.4 is satisfied because there are no 1-dimensional strata. Thus $A_1^i(x_0, y_0)$ does not depend on x_0. The hypotheses of lemma 8.1 are similarly satisfied, so $A_1^i(x_0, y_0)$ does not depend on y_0. □

10 The local contribution is a sum over strata

(10.1) We assume R is a field. Suppose $c: C \to X \times X$ is a correspondence, $\mathbf{S}^\bullet$ is a complex of sheaves of R-modules on X, and $\mathbf{\Phi}: c_2^*(\mathbf{S}^\bullet) \to c_1^!(\mathbf{S}^\bullet)$ is a lift of the

correspondence to $\mathbf{S}^\bullet$. Let $F \subset C$ be a connected component of the fixed point set and suppose that F has a weakly hyperbolic neighborhood W with indicator map t. Let $F' = c_1(F) = c_2(F) = t^{-1}(0)$ be the image in X of the fixed component and let

$$\mathbf{A}^\bullet = \mathbf{A}_4^\bullet = i_L^* j_L^!(\mathbf{S}^\bullet)$$

denote the sheaf on F' whose cohomology is the local group

$$A_4^i = H^i(F'; \mathbf{A}^\bullet).$$

By Theorem 4.7, the correspondence $(C, \mathbf{\Phi})$ induces a self map $\mathbf{\Phi}_*^i$ on A_4^i whose Lefschetz number is the local contribution,

$$n(F) = \sum (-1)^i \operatorname{Tr}(\mathbf{\Phi}_*^i : A_4^i \to A_4^i) \in R.$$

In this section we will show that if F is compact then $n(F)$ is the Euler characteristic of a constructible function [M] on F' and hence can be written as a sum over the strata of F'.

(10.2) The sheaf $\mathbf{A}^\bullet$ is supported on F'. By § 9 the correspondence $\mathbf{\Phi}$ lifts to a sheaf map

$$\tilde{\mathbf{\Phi}} : c_2^* \mathbf{A}^\bullet \to c_1^! \mathbf{A}^\bullet$$

which is the zero morphism outside the neighborhood W of F. Since F is compact we have $\pi_{1!} c_* = c_{2*} = c_{1!}$ when restricted to F. Applying this functor to $\tilde{\mathbf{\Phi}}$ we obtain a sheaf map $\mathbf{\Phi}' : \mathbf{A}^\bullet \to \mathbf{A}^\bullet$ as the composition

$$\mathbf{A}^\bullet \to c_{2*} c_2^* \mathbf{A}^\bullet \xrightarrow{c_{2*}\tilde{\mathbf{\Phi}}} c_{1!} c_1^! \mathbf{A}^\bullet \to \mathbf{A}^\bullet$$

Lemma. The local contribution $n(F)$ to the Lefschetz number $L(C, \mathbf{\Phi}; \mathbf{S}^\bullet)$ is equal to the alternating sum of traces,

$$n(F) = \sum_i (-1)^i \operatorname{Tr}(\mathbf{\Phi}'_* : H^i(F'; \mathbf{A}^\bullet) \to H^i(F'; \mathbf{A}^\bullet))$$

Proof. The proof follows immediately from 4.7(d), 4.5 and 2.9. □

For each fixed point $x \in F'$ let $n(x) = n(x, \mathbf{\Phi}') \in R$ denote the local Lefschetz number at x, i.e. the alternating sum of traces of the induced homomorphism on the stalk cohomology at x:

$$n(x, \mathbf{\Phi}') = \sum (-1)^i \operatorname{Tr}(\mathbf{\Phi}'_x : H_x^i(\mathbf{A}^\bullet) \to H_x^i(\mathbf{A}^\bullet)).$$

Suppose that $F' = F'_1 \cup F'_2 \cup \ldots \cup F'_r$ is a stratification of F' such that the local Lefschetz number $n(x)$ is constant along each stratum F'_j and denote this number by n_j. Thus,

$$n_j = \sum (-1)^i \operatorname{Tr}(\mathbf{\Phi}'_* : H_x^i(i_L^* j_L^! \mathbf{S}^\bullet) \to H_x^i(i_L^* j_L^! \mathbf{S}^\bullet)),$$

where $x \in F_j$.

(10.3) Theorem. *The local contribution $n(F) = L(\mathbf{\Phi}')$ is equal to the Euler characteristic (see* [M]*) of the constructible function $n(x)$, in other words,*

$$n(F) = \chi(F'; n(x)) = \sum_{j=1}^{r} \chi_c(F'_j) \cdot n_j,$$

where χ_c denotes the Euler characteristic with compact supports. Furthermore if $\dim(F'_j)$ *is even, or if F'_j is a locally symmetric space then* $\chi_c(F'_j) = \chi(F'_j)$.

(10.4) Proof. The proof will appear in § 11.6.

(10.5) Remarks. Theorem 10.3 is also valid if we replace $\mathbf{A}^\bullet = \mathbf{A}_4^\bullet$ by the sheaf $\mathbf{A}_5^\bullet = i_R^! j_R^* \mathbf{S}^\bullet$ whose cohomology is the local group A_5. The Euler characteristic of the resulting constructible function $n(x)$ will be the same.

11 Remarks on the Euler characteristic

Throughout this section we assume that R is an integral domain. (Note as in § 2.13 that the trace of an endomorphism of R-modules is defined by tensoring with the field of fractions, so we may as well assume that R is a field.) We also assume that Y is a locally compact union of strata in a compact stratified space. This guarantees that the cohomology of Y is finite dimensional.

(11.1) It follows from the universal coefficient theorem that the Euler characteristic of Y may be computed using either homology or cohomology, and with coefficients in any field.

(11.2) If Y is a (not necessarily compact) manifold of dimension n, then by Poincaré duality, $H_c^i(Y; \mathbf{Z}/(2)) \cong H_{n-i}(Y; \mathbf{Z}/(2))$ and hence $\chi_c(Y) = (-1)^n \chi(Y)$ where χ_c denotes the Euler characteristic with compact supports. If Y can be compactified by adding a compact boundary manifold ∂Y then by the resulting long exact sequence on cohomology,

$$\chi(Y) = \chi_c(Y) + \chi(\partial Y).$$

For example, if Y is a locally symmetric space then the Euler characteristic of its Borel–Serre boundary is 0 (since it can be piecewise fibered by nilmanifolds [BS]), hence $\chi_c(Y) = \chi(Y)$. If Y is also odd dimensional then $\chi(Y) = -\chi_c(Y)$ so they both vanish.

(11.3) The Euler characteristic with compact supports is additive: Let Y be a (compactifiable) stratified space with strata $Y_1, Y_2, \ldots, Y_r$. Let $\mathbf{S}$ be a constructible sheaf (concentrated in degree 0) on Y, i.e. for each j, the restriction $\mathbf{S}|Y_j$ is locally constant with finite rank F_j. Then the Euler characteristic with compact supports is a sum over strata,

$$\chi_c(Y; S) = \sum_i (-1)^i \operatorname{rank} H_c^i(Y; \mathbf{S}) = \sum_j \chi_c(Y_j) F_j \in \mathbf{Z}.$$

This follows from induction and the long exact cohomology sequence for the pair

$$H^i(Y, Y - Y_1; \mathbf{S}) = H_c^i(Y_1; \mathbf{S}),$$

where Y_1 is a stratum in Y.

Remark. The Euler characteristic with compact supports $\chi_c(Y; \mathbf{S})$ depends only on the constructible function $F: Y \to \mathbf{Z}$ which is given by $F(y) = F_j$ for any $y \in Y_j$. It is therefore called the Euler characteristic (with compact supports) of the constructible function F, and is denoted $\chi_c(Y; F)$.

(11.4) If Y has only odd dimensional strata and if it has a compactification $\bar{Y}$ consisting of odd dimensional strata then $\chi(Y;F)=0$ for any constructible function F, and in particular $\chi(Y_j)=0$ for each j.

This may be seen as follows: We have $\chi(\bar{Y})=0$ [Su] as may be seen by calculating the Euler characteristic of the suspension of $\bar{Y}$, which has even dimensional strata. Now observe that the function F can be written as a linear combination of characteristic functions of closures of strata, and use induction.

(11.5) Suppose $\mathbf{S}$ is a sheaf of R-modules, concentrated in degree 0, on the (compactifiable) locally compact stratified space Y, which is constructible with respect to the stratification $Y=Y_1\cup Y_2\cup\ldots\cup Y_r$. Suppose $\mathbf{\Phi}:\mathbf{S}\to\mathbf{S}$ is a (constructible) sheaf map such that the trace on the stalks,

$$F(x)=\mathrm{Tr}(\mathbf{\Phi}_x:\mathbf{S}_x\to\mathbf{S}_x)\in R$$

is constant on each stratum. Then the Lefschetz number (with compact supports) of $\mathbf{\Phi}$,

$$L_c(\mathbf{\Phi})=\sum_i(-1)^i\,\mathrm{Tr}(\mathbf{\Phi}^i_*:H^i_c(Y;\mathbf{S})\to H^i_c(Y;\mathbf{S}))\in R$$

is equal to the Euler characteristic (with compact supports) $\chi_c(Y;F)$ of the constructible function F:

$$L_c(\mathbf{\Phi})=\chi_c(Y;F)=\sum_j F_j\,\chi_c(Y_j)\in R.$$

The proof uses the long exact sequences of § 11.3 to reduce to the case that $\mathbf{S}$ is a locally constant sheaf and F is a constant function, then uses the Mayer–Vietoris theorem as applied to an open covering of Y which trivializes $\mathbf{S}$.

(11.6) Suppose $\mathbf{A}^\bullet$ is a constructible complex of sheaves of R-modules on a compact stratified space $Y=Y_1\cup Y_2\cup\ldots\cup Y_r$, and that $\mathbf{\Phi}:\mathbf{A}^\bullet\to\mathbf{A}^\bullet$ is a self map (which covers the identity on Y). Suppose the local Lefschetz number,

$$F(x)=\sum_i(-1)^i\,\mathrm{Tr}(\mathbf{\Phi}^i_x:H^i_x(\mathbf{A}^\bullet)\to H^i_x(\mathbf{A}^\bullet))\in R$$

is constant along each stratum. Theorem 10.3 now reduces to a special case of the following result:

Proposition. *The Lefschetz number of $(\mathbf{\Phi},\mathbf{A}^\bullet)$ is equal to the Euler characteristic of Y with coefficients in the constructible function F, i.e.*

$$L(\mathbf{\Phi})=\chi(Y;F)=\sum\chi_c(Y_j)F_j\in R$$

(where F_j denotes the value of F on any point in Y_j). By § 11.2 the Euler characteristic with compact supports, $\chi_c(Y_j)$ may be replaced by the Euler characteristic $\chi(Y_j)$ if Y_j is even dimensional or if it is a locally symmetric space.

(11.7) Proof. Consider the E_2 term of the spectral sequence for hypercohomology,

$$E_2^{iq}=H^i(Y;H^q(\mathbf{A}^\bullet))\Rightarrow H^{i+q}(Y;\mathbf{A}^\bullet)$$

Since the alternating sum of traces behaves properly with respect to exact sequences and since $\mathbf{\Phi}$ induces sheaf automorphisms $\mathbf{\Phi}^q:H^q(\mathbf{A}^\bullet)\to H^q(\mathbf{A}^\bullet)$, the

Lefschetz number of Φ is equal to the Lefschetz number of its action on E_2, in other words,

$$(11.8)\quad L(\Phi) = \sum_q (-1)^q \sum_i (-1)^i \operatorname{Tr}(H^i(\Phi^q): H^i(Y; H^q(\mathbf{A}^\bullet)) \to H^i(Y; H^q(\mathbf{A}^\bullet))).$$

By refining the stratification if necessary (and using the additivity of the Euler characteristic with compact supports), we may assume that the local cohomology sheaves $H^q(\mathbf{A}^\bullet)$ are locally constant on each stratum Y_j of Y and that the alternating sum of traces,

$$F^q(x) = \sum_i (-1)^i \operatorname{Tr}(\Phi_x^q: H_x^q(\mathbf{A}^\bullet) \to H_x^q(\mathbf{A}^\bullet))$$

is constant on the strata of Y. Thus F^q is a constructible function. The inner sum in (11.8) is the Lefschetz number of the sheaf morphism $\Phi^q: H^q(\mathbf{A}^\bullet) \to H^q(\mathbf{A}^\bullet)$ which may be evaluated by (11.5) so we obtain

$$L(\Phi) = \sum_q (-1)^q \chi(Y; F^q) = \chi(Y; F). \quad \square$$

12 Contracting fixed points

(12.1) Suppose $c: C \to X \times X$ is a correspondence with a fixed point component $F \subset C$. The component F is weakly contracting if there is a neighborhood $W' \subset X$ of $F' = c_1(F) = c_2(F)$ and a real valued function (to be thought of as the distance from F'), $t: W' \to \mathbb{R}_{\geq 0}$ such that $t^{-1}(0) = F'$ and so that for all $w \in C$, sufficiently close to F we have, $tc_1(w) \geq tc_2(w)$. Thus F is weakly hyperbolic and the indicator map t sends the neighborhood W' to the y-axis only. Let $\mathbf{S}^\bullet$ be a complex of sheaves on X and let

$$\Phi: c_2^*(\mathbf{S}^\bullet) \to c_1^!(\mathbf{S}^\bullet)$$

be a lift of c to the sheaf $\mathbf{S}^\bullet$.

(12.2) Proposition. *The lift Φ induces a self map of the restriction,*

$$\Phi': \mathbf{S}^\bullet | F' \to \mathbf{S}^\bullet | F'$$

and the local contribution to the Lefschetz number is

$$n(F) = \sum (-1)^i \operatorname{Tr}(\Phi'_*: H^i(F'; \mathbf{S}^\bullet) \to H^i(F'; \mathbf{S}^\bullet)).$$

If F' is stratified $F' = F'_1 \cup F'_2 \cup \ldots \cup F'_r$ so that the local trace $L_x(\Phi')$ is constant on each stratum, set $L_j = L_x(\Phi')$ for any $x \in F_j$. Then we have

$$n(F) = \chi(F'; L_x) = \sum_j L_j \chi_c(F'_j)\cdot$$

(12.3) Proof. The local sheaves $\mathbf{A}_3^\bullet$ and $\mathbf{A}_4^\bullet$ are both canonically isomorphic to $\mathbf{S}^\bullet | F'$. Apply Theorem 4.7 and Theorem 10.3.

(12.4) Remark. If F is an expanding fixed point component then the local contribution is the Euler characteristic of F' with coefficients in the constructible function M whose value M_x at a point $x \in F'$ is the alternating sum of the traces of Φ' on the compactly supported stalk cohomology of $\mathbf{S}^\bullet$ at x.

13 The case of a complex analytic endomorphism

(13.1) When the correspondence is the graph of a complex analytic function, much more explicit results can be obtained: under very general circumstances the local Lefschetz number is equal to the local trace [K, KS]. Unfortunately this pleasant state of affairs does not carry over to correspondences (see § 14.4).

(13.2) If $\mathbf{S}^\bullet$ is a complex of sheaves on a subanalytic space X and $\mathbf{T}^\bullet$ is a complex of sheaves on a subanalytic space Y, and if $f: X \to Y$ is a function, then the graph of f is a correspondence,

$$C = \{(X, f(x) | x \in X\} \to X \times Y$$

and a lift of C to the sheaf level,

$$\Phi : c_2^*(\mathbf{T}^\bullet) \to c_1^!(\mathbf{T}^\bullet)$$

is the same as a sheaf morphism $f^* \mathbf{T}^\bullet \to \mathbf{S}^\bullet$ (see [V]). Apply c_{2*} (or f_*) to obtain a map

$$\mathbf{T}^\bullet \to f_* \mathbf{S}^\bullet$$

which induces the pullback homomorphism on cohomology, $H^i(Y; \mathbf{T}^\bullet) \to H^i(X; \mathbf{S}^\bullet)$. Now suppose f is a self map, i.e. $X = Y$ and $\mathbf{S}^\bullet = \mathbf{T}^\bullet$. Then we obtain a self map on the stalk,

$$\Phi_x : \mathbf{S}_x^\bullet \to \mathbf{S}_x^\bullet$$

for each fixed point $x \in X$. The number

$$\sum_i (-1)^i \operatorname{Tr}(\Phi_x^i : H^i(\mathbf{S}_x^\bullet) \to H^i(\mathbf{S}_x^\bullet))$$

is called the *local trace.* (If $\mathbf{S}^\bullet$ is self dual or if $f^{-1}(x)$ consists of finitely many points, then it is also possible to define a local trace on the compactly supported stalk cohomology at a fixed point x.) The following beautiful result has been proven by Kashiwara and Schapira [KS]:

(13.3) Proposition. *Suppose X is a complex manifold, $f: X \to X$ is a holomorphic self map, $\mathbf{S}^\bullet$ is a complex analytically constructible complex of sheaves on $\mathbf{S}^\bullet$, and $\Phi : f^*(\mathbf{S}^\bullet) \to \mathbf{S}^\bullet$ is a lift of Φ to $\mathbf{S}^\bullet$. Let $x \in X$ be an isolated fixed point and suppose that the linear map $df(x)$ has no eigenvalue equal to 1. Then the local contribution at x to the Lefschetz number is equal to the alternating sum of the traces of Φ' on the stalk cohomology $H_x^i(\mathbf{S}^\bullet)$.*

(13.4) Remarks. This result appears to disagree with Theorem 4.7 which says that the local contribution is equal to the alternating sum of traces on the local groups A_1^i. These apparently conflicting results do, in fact, give the same number. This may be seen by first considering the following "conical" special case: Stratify X so that the complex of sheaves is constructible with respect to the stratification and suppose there exists a coordinate chart near the fixed point x such that this stratification is conical (i.e. invariant under $\mathbf{C}^*$) and the self map is linear. Then composing the self map with the time ε flow of the vectorfield $\sqrt{-1}\, dr$, (where r denotes the distance from x) one obtains a homotopic self map with no real eigenvalues. Thus if we further compose this with a contraction we will create no

new fixed points. However a contracting self map has local contribution given by the local trace. The general case can be reduced to this special case by deformation to the normal cone [BFM, F].

This argument also shows (by composing with an expansion instead of a contraction) that if the differential $df(x)$ has no kernel, then the local contribution may also obtained by taking the alternating sum of traces on the local compactly supported stalk cohomology.

14 Examples and counterexamples

(14.1) Cohomology. If $f:X \to Y$ is a map between subanalytic spaces then the graph of f is a correspondence $c:G(f) \to X \times Y$ which has a natural lift

$$c_2^*(\mathbf{Z}_Y) \to c_1^!(\mathbf{Z}_X) \cong \mathbf{Z}_{G(f)}.$$

Apply c_{2*} to obtain maps

$$\mathbf{Z}_Y \to c_{2*}\, c_2^*\, \mathbf{Z}_Y \to c_{2*}\, c_1^!\, \mathbf{Z}_X = f_*(\mathbf{Z}_X),$$

which gives the usual map on cohomology, $H^*(Y) \to H^*(X)$.

(14.2) Homology. If $f:X \to Y$ is a map between compact subanalytic spaces we may view the graph of f as a reverse correspondence, $d:G(f) \to Y \times X$. If $\mathbf{D}_Y^\bullet$ denotes the dualizing sheaf on Y, we obtain a canonical lift,

$$\mathbf{D}_{G(f)}^\bullet \cong d_2^*\, \mathbf{D}_X^\bullet \to d_1^!\, \mathbf{D}_Y^\bullet .$$

Apply $d_{2!}$ to obtain maps

$$Rf_!\, \mathbf{D}_X^\bullet = d_{2!}\, d_2^*\, \mathbf{D}_X^\bullet \to d_{2!}\, d_2^!\, \mathbf{D}_Y^\bullet \to \mathbf{D}_Y^\bullet ,$$

which gives the usual map on homology, $H_*(X) \to H_*(Y)$.

(14.3) Placid correspondences. A map $f: Y \to X$ between subanalytic pseudomanifolds is *placid* [GM2] if there exists a stratification of X such that for each stratum $S \subset X$ we have

$$\operatorname{codim}_Y(f^{-1}(S)) \leqq \operatorname{codim}_X(S).$$

A correspondence $c:C \to X \times X$ is placid if each component is placid. It is shown [GM2] that a placid correspondence has a canonical lift to the intersection cohomology sheaf $\mathbf{S}^\bullet = IC^\bullet$ (with any perversity, including homology and cohomology). This can be obtained as the composition

$$c_2^*(\mathbf{IC}_X^\bullet) \to \mathbf{IC}_C^\bullet \to c_1^!(\mathbf{IC}_Y^\bullet),$$

with the first map given by pullback of transverse cycles, and the second map given by pushforward of cycles. A special case is when each component of the correspondence is a finite map: this happens, for example, when C is a Hecke correspondence and X is a Satake compactification of a locally symmetric space.

(14.4) The local trace. For a general correspondence $c:C \to X \times X$ and a lift to a complex of sheaves,

$$\mathbf{\Phi}: c_2^*(\mathbf{S}^\bullet) \to c_1^!(\mathbf{S}^\bullet)$$

we do not, in general, obtain an induced map on the stalk cohomology of $\mathbf{S}^\bullet$ at a fixed point. Even in the cases where there is a natural map induced on the stalk cohomology, and even when the correspondence is complex algebraic, the alternating sum of its traces may not be equal to the local contribution to the Lefschetz number.

Consider the graph of the "function" $y = x^{b/a}$, i.e. the correspondence $c: \mathbb{C} \rightrightarrows \mathbb{C}$ given by $c_1(x) = x^a$ and $c_2(x) = x^b$. Since c is a finite map we obtain canonical morphisms on the pullback of the constant sheaf $\mathbf{Z}$,

$$c_2^*(\mathbf{Z}) \to \mathbf{Z}_{G(f)} \to c_1^!(\mathbf{Z}).$$

This induces the "pullback" homomorphism on the stalk cohomology at the origin,

$$\mathbf{Z} = H_0^0(\mathbb{C}) \to H_0^0(\mathbb{C}) = \mathbf{Z},$$

which is multiplication by a. It also induces a (pullback) homomorphism on the stalk cohomology with compact support at the origin,

$$\mathbf{Z} = H^2_{\{0\}}(\mathbb{C}) \to H^2_{\{0\}}(\mathbb{C}) = \mathbf{Z},$$

which is multiplication by b. However the local contribution to the Lefschetz number is $\min(a, b)$ which can be seen in two interesting ways:

(1) It is the intersection number of the diagonal with the plane curve $y^a = x^b$.

(2) If $b \geqq a$ the correspondence is weakly contracting so by Theorem 4.7 or by § 12 the local Lefschetz number is equal to the local trace on the stalk cohomology, which is multiplication by a. If $b \leqq a$ the correspondence is weakly expanding so the local Lefschetz number is the local trace on the compactly supported cohomology, which is multiplication by b.

An even more striking example is given by the correspondence on $\mathbb{C}^2$ with $c_1(x_1, x_2) = (x_1^{a_1}, x_2^{a_2})$ and $c_2(x_1, x_2) = (x_1^{b_1}, x_2^{b_2})$. The trace on the stalk cohomology at the origin is $a_1 a_2$ while the local Lefschetz number is $\min(a_1, b_1) \cdot \min(a_2, b_2)$.

Remark. The above example arises naturally in the study of modular curves. Suppose X is a modular curve and $\bar{X}$ is the compactification which is obtained by adding a point for each cusp. Such a point x_0 has a neighborhood which is locally parametrized by the complex numbers $\mathbb{C}$. If x_0 is an isolated fixed point of a Hecke correspondence, then the correspondence may be locally described by the equation $y = x^{b/a}$. As mentioned in the introduction, this case was previously analyzed by Langlands [L, Proposition 7.12], and Illusie [I, p. 144].

(14.5) Nonhyperbolic fixed points. The complex algebraic function $f(z) = z + 1$ extends to a self map of the Riemann sphere $\mathbb{C}\mathbf{P}^1$, which has a single fixed point which is not weakly hyperbolic at $z = \{\infty\}$ and Lefschetz number equal to $\chi(\mathbb{C}\mathbf{P}^1) = 2$. The differential $df(\infty)$ is 1 and the induced homomorphism on the stalk cohomology has trace equal to 1 at the fixed point.

(14.6) Noncompact spaces. The function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + 1$ has no fixed points, but the Lefschetz number

$$\sum_i (-1)^i \operatorname{Tr}(f^*: H^i(\mathbb{R}) \to H^i(\mathbb{R}))$$

is 1, while the Lefschetz number with compact supports,

$$\sum_i (-1)^i \operatorname{Tr}(f^*: H_c^i(\mathbb{R}) \to H_c^i(\mathbb{R}))$$

is -1. This simple example illustrates the principle that the Lefschetz number of an endomorphism on a noncompact space X depends on the behavior of the endomorphism at infinity, and is best studied by extending the endomorphism to some compactification $\bar{X}$ of the space X. In many naturally occurring examples, including those of Hecke correspondences, the space X may be a smooth manifold but the space $\bar{X}$ will have singularities, and it may even fail to have a natural embedding in a smooth manifold.

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