Clay Mathematics Proceedings

Compactifications and cohomology of modular varieties

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1. Overview

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over $\mathbb{Q}$. Denote by $\mathbf{G}(\mathbb{Q})$ (resp. $\mathbf{G}(\mathbb{R})$) the group of points in $\mathbf{G}$ with entries in $\mathbb{Q}$ (resp. $\mathbb{R}$). It is common to write $G = \mathbf{G}(\mathbb{R})$. Fix a maximal compact subgroup $K \subset G$ and let $A_G = \mathbf{A_G}(\mathbb{R})^+$ (see §4.1) be the (topologically) connected identity component of the group of real points of the greatest $\mathbb{Q}$-split torus $\mathbf{A_G}$ in the center of $\mathbf{G}$. (If $\mathbf{G}$ is semisimple then $A_G = \{1\}$.) We refer to $D = G/KA_G$ as the "symmetric space" for $\mathbf{G}$. We assume it is Hermitian, that is, it carries a G-invariant complex structure. Fix an arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ and let $X = \Gamma\backslash D$. We refer to X as a locally symmetric space. In general, X is a rational homology manifold: at worst, it has finite quotient singularities. If Γ is torsion-free then X is a smooth manifold. It is usually noncompact. (If $\mathbb{A}$ denotes the adèles of $\mathbb{Q}$ and $\mathbb{A}_f$ denotes the finite adèles, and if $K_f \subset \mathbf{G}(\mathbb{A}_f)$ is a compact open subgroup, then the topological space $Y = \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/A_GK \cdot K_f$ is a disjoint union of finitely many locally symmetric spaces for $\mathbf{G}$. To compactify Y it suffices to compactify each of these locally symmetric spaces.)

There are (at least) four important compactifications of X : the Borel-Serre compactification $\overline{X}^{\mathrm{BS}}$ (which is a manifold with corners), the reductive Borel-Serre compactification, $\overline{X}^{\mathrm{RBS}}$ (which is a stratified singular space), the Baily-Borel (Satake) compactification $\overline{X}^{\mathrm{BB}}$ (which is a complex projective algebraic variety, usually singular), and the toroidal compactification $\overline{X}^{\mathrm{tor}}_{\Sigma}$ (which is a resolution of singularities of

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$\overline{X}^{\text{BB}}$). (Actually there is a whole family of toroidal compactifications, depending on certain choices Σ.) The identity mapping $X \to X$ extends to unique continuous mappings

$$\overline{X}^{\text{BS}} \longrightarrow \overline{X}^{\text{RBS}} \xrightarrow{\tau} \overline{X}^{\text{BB}} \longleftarrow \overline{X}^{\text{tor}}_{\Sigma}.$$

The first three of these compactifications are obtained as the quotient under Γ of corresponding "partial compactifications"

$$\overline{D}^{\text{BS}} \longrightarrow \overline{D}^{\text{RBS}} \longrightarrow \overline{D}^{\text{BB}}$$

of the symmetric space D.

Besides its ordinary (singular) cohomology, the two singular compactifications $\overline{X}^{\text{BB}}$ and $\overline{X}^{\text{RBS}}$ also support various exotic sorts of cohomology, defined in terms of a complex of sheaves of differential forms with various sorts of restrictions near the singular strata. The L^2 cohomology of X may be realized as the cohomology of the sheaf of L^2 differential forms on $\overline{X}^{\text{BB}}$. The (middle) intersection cohomology of $\overline{X}^{\text{BB}}$ is obtained from differential forms which satisfy a condition (see §6.8) near each singular stratum, defined in terms of the dimension of the stratum. The Zucker conjecture [**Z1**], proven by E. Looijenga [**Lo**] and L. Saper and M. Stern [**SS**], says that the L^2 cohomology of $\overline{X}^{\text{BB}}$ coincides with its intersection cohomology, and that the same is true of any open subset $U \subset \overline{X}^{\text{BB}}$.

The (middle) weighted cohomology complex on $\overline{X}^{\text{RBS}}$ is defined in a manner similar to that of the intersection cohomology, however the restrictions on the chains (or on differential forms) are defined in terms of the weights of a certain torus action which exists near each singular stratum. Although the weighted cohomology and the intersection cohomology do not agree on every open subset of $\overline{X}^{\text{RBS}}$, it has recently been shown ([**S1**], [**S2**]) that they do agree on subsets of the form $\tau^{-1}(U)$ for any open set $U \subset \overline{X}^{\text{BB}}$.

This article is in some sense complementary to the survey articles [**Sch**] and [**B4**].

Notation. Throughout this article, algebraic groups over $\mathbb{Q}$ will be indicated in bold, and the corresponding group of real points in Roman, so $G = \mathbf{G}(\mathbb{R})$. The group of $n \times r$ matrices over a field k is denoted $M_{n\times r}(k)$. The rank n identity matrix is denoted I_n and the zero matrix is 0_n.

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2. The Baily Borel (Satake) compactification

2.1. The case $\mathbf{G} = \mathbf{SL}(2)$. Recall the fundamental domains for the action of $\Gamma = \mathbf{SL}(2, \mathbb{Z})$ on the upper half plane $\mathfrak{h}$.

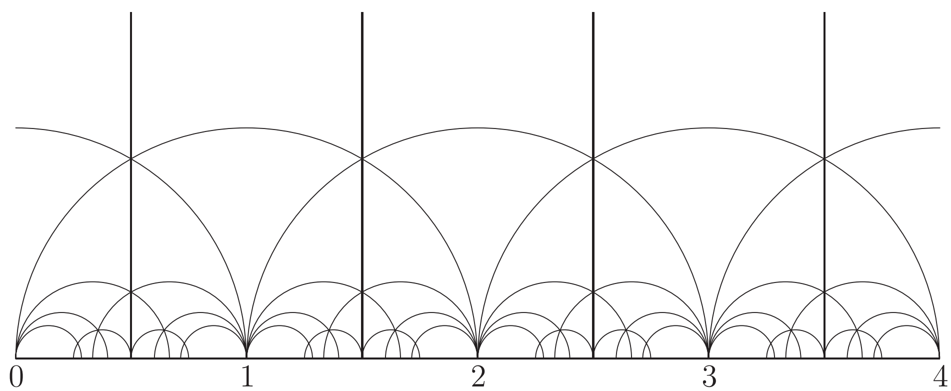

FIGURE 1. Fundamental domains for $\mathbf{SL}(2, \mathbb{Z})$

The quotient $X = \Gamma\backslash\mathfrak{h}$ may be compactified, $\overline{X} = X \cup \{\infty\}$ by adding a single cusp[1] at infinity. If we wish to realize this as the quotient under Γ of a partial compactification $\overline{\mathfrak{h}}^{\mathrm{BB}}$ of the upper half plane, then we must add to $\mathfrak{h}$ all the Γ-translates of $\{\infty\}$. This consists of all the rational points $x \in \mathbb{Q}$ on the real line (which also coincides with the $\mathbf{SL}(2, \mathbb{Q})$ orbit of the point at infinity). With this candidate for $\overline{\mathfrak{h}}^{\mathrm{BB}}$, the quotient under Γ will fail to be Hausdorff. The solution is to re-topologise this union so as to "separate" the added points $x \in \mathbb{Q}$.

A neighborhood basis for the point at infinity may be chosen to consist of the open sets $U_\tau = \{z \in \mathfrak{h} | \ \mathrm{Im}(z) > \tau\}$ for $\tau \geq 2$ (say). If we also throw in all the $\mathbf{SL}(2, \mathbb{Q})$-translates of these sets U_τ then we obtain a new topology, the *Satake topology*, on $\overline{\mathfrak{h}} = \mathfrak{h} \cup \mathbb{Q} \cup \{\infty\}$, in which each point $x \in \mathbb{Q}$ has a neighborhood homeomorphic to the neighborhood of the point at infinity. The group Γ still fails to act "properly" on $\overline{\mathfrak{h}}$

[1]Although they are called cusps, the points which are added to compactify a modular curve are in fact nonsingular points of the resulting compactifications.

because for each boundary point $x \in \mathbb{Q} \cup \infty$ there are infinitely many elements $\gamma \in \Gamma$ which fix x. However it does satisfy (cf. [**AMRT**] p. 258) the following conditions:

(S1) If $x, x' \in \overline{\mathfrak{h}}$ are not equivalent under Γ then there exist neighborhoods U, U' of x, x' respectively, such that $(\Gamma \cdot U) \cap U' = \phi$.

(S2) For every $x \in \overline{\mathfrak{h}}$ there exists a fundamental system of neighborhoods $\{U\}$, each of which is preserved by the stabilizer Γ_x, such that if $\gamma \notin \Gamma_x$ then $(\gamma \cdot U) \cap U = \phi$.

These properties guarantee that the quotient $\overline{X} = \Gamma \backslash \overline{\mathfrak{h}}$ is Hausdorff, and in fact it is compact. The same partial compactification $\overline{\mathfrak{h}}$ may be used for any arithmetic subgroup $\Gamma' \subset \mathbf{SL}(2, \mathbb{Q})$, giving a uniform method for compactifying all arithmetic quotients $\Gamma' \backslash \mathfrak{h}$. One would like to do the same sort of thing for symmetric spaces of higher rank.

2.2. A warmup problem. The following example, although not Hermitian, illustrates the phenomena which are encountered in the Baily Borel compactification of higher rank locally symmetric spaces. See [**AMRT**] Chapt. II for more details. The group $\mathbf{GL}(n, \mathbb{R})$ acts on the vector space $S_n(\mathbb{R})$ of real symmetric $n \times n$ matrices (through change of basis) by

$$g \cdot A = gA\,{}^t g. \tag{2.2.1}$$

The orbit of the identity matrix I_n is the open (homogeneous self adjoint) convex cone $\mathcal{P}_n$ of positive definite symmetric matrices. The stabilizer of I is the maximal compact subgroup $\mathbf{O}(n)$. The center A_G of $\mathbf{GL}(n, \mathbb{R})$ (which consists of the scalar matrices) acts by homotheties. The action of $\mathbf{GL}(n, \mathbb{R})$ preserves the closure $\overline{\mathcal{P}}_n$ of $\mathcal{P}_n$ in $S_n(\mathbb{R})$, whose boundary $\partial \mathcal{P}_n = \overline{\mathcal{P}_n} - \mathcal{P}_n$ decomposes into a disjoint union of (uncountably many) *boundary components* as follows. A *supporting* hyperplane $H \subset S_n(\mathbb{R})$ is a hyperplane such that $H \cap \mathcal{P}_n = \phi$ and $H \cap \partial \mathcal{P}_n$ contains nonzero elements. Let $\overline{F} = \partial \mathcal{P}_n \cap H$ where H is a supporting hyperplane. Then there is a unique smallest linear subspace $L \subset S_n(\mathbb{R})$ containing $\overline{F}$. The interior F of $\overline{F}$ in L is called a boundary component of $\mathcal{P}_n$ (much in the same way that the closure of each face of a convex polyhedron $P \subset \mathbb{R}^m$ is the intersection $P \cap H$ of P with a supporting affine hyperplane $H \subset \mathbb{R}^m$). Distinct boundary components do not intersect.

Let $\mathbf{B} \subset \mathbf{GL}(n)$ be the (rational) Borel subgroup of upper triangular matrices. Parabolic subgroups containing $\mathbf{B}$ will referred to as *standard*. Each boundary component is a $\mathbf{GL}(n, \mathbb{R})$ translate of exactly one of the following *standard boundary components* F_r $(1 \le r \le n-1)$ consisting of matrices $\left(\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right)$ such that $E \in \mathcal{P}_r$ is positive definite.

The normalizer $P \subset \mathbf{GL}(n,\mathbb{R})$ of this boundary component (meaning the set of elements which preserve F_r) is the standard maximal parabolic subgroup consisting of matrices $g = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ where $A \in \mathbf{GL}(r,\mathbb{R})$, $B \in \mathbf{Hom}(\mathbb{R}^{n-r},\mathbb{R}^r)$, and $D \in \mathbf{GL}(n-r,\mathbb{R})$. (It is the group of real points $P = \mathbf{P}(\mathbb{R})$ of the obvious maximal parabolic subgroup $\mathbf{P} \subset \mathbf{G}$.) The supporting subspace L of F_r is the set of all symmetric matrices $t = \left(\begin{smallmatrix} T & 0 \\ 0 & 0 \end{smallmatrix}\right)$ where $T \in S_r(\mathbb{R})$. The action of such an element $g \in P$ on the element t is given by $g \cdot t = gt\,{}^t g$, that is,

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} AT\,{}^t A & 0 \\ 0 & 0 \end{pmatrix} \tag{2.2.2}$$

for any $T \in S_r(\mathbb{R})$. In particular, the Levi component of P decomposes as a product $\mathbf{GL}(r,\mathbb{R}) \times \mathbf{GL}(n-r,\mathbb{R})$ where the first factor, A, acts transitively on the boundary component F_r and the second factor, D, acts trivially. The standard parabolic subgroups correspond to subsets of the Dynkin diagram of $\mathbf{G}$, and the maximal parabolic subgroup $\mathbf{P}$ corresponds to the deletion of a single node, α.

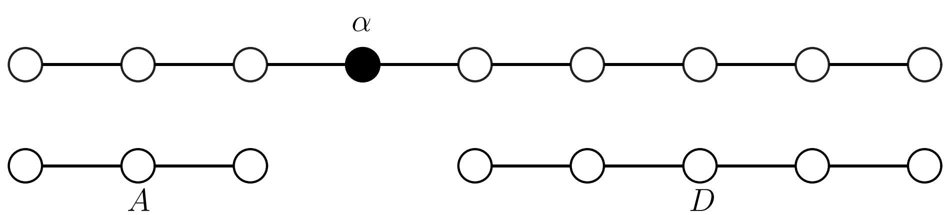

FIGURE 2. Dynkin diagrams for $\mathbf{G}$, $\mathbf{P}$, and its Levi factor.

A boundary component F is *rational* if the subspace L containing it is rational, or equivalently, if the normalizer P is a rational parabolic subgroup. Define the *standard* partial compactification, $\overline{\mathcal{P}}_n^{\mathrm{std}}$ to be the union of $\mathcal{P}_n$ with all its rational boundary components, with the Satake topology[2]. Then $\mathbf{GL}(n,\mathbb{Q})$ acts on $\overline{\mathcal{P}}_n^{\mathrm{std}}$. For any arithmetic group $\Gamma \subset \mathbf{GL}(n,\mathbb{Q})$ the quotient

$$\overline{X}^{\mathrm{std}} = \Gamma\backslash\overline{\mathcal{P}}_n^{\mathrm{std}}/A_G = \Gamma\backslash\overline{\mathcal{P}}_n^{\mathrm{std}}/\text{homotheties}$$

is a compact singular space which is stratified with finitely many strata of the form $X_F = \Gamma_F\backslash F/\text{homotheties}$ (where F is a rational boundary

[2]The Satake topology ([**AMRT**] p. 258, [**BB**] Thm. 4.9): is uniquely determined by requiring that conditions (S1) and (S2) (above) hold for any arithmetic group Γ, as well as the following: for any Siegel set $S \subset D$ its closure in $\overline{D}$ and its closure in $\overline{D}^{\mathrm{BB}}$ have the same topology.

component and Γ_F is an appropriate arithmetic group). The closure $\overline{X}_F$ in $\overline{X}^{\text{std}}$ is the standard compactification of $F/$homotheties.

A similar construction holds for any rational self adjoint homogeneous cone. These are very interesting spaces. Although they are not algebraic varieties, they have a certain rigid structure. For example, each Γ-invariant rational polyhedral simplicial cone decomposition of $\overline{\mathcal{P}}_n^{\text{std}}$ (in the sense of [**AMRT**]) passes to a "flat, rational" triangulation of $\overline{X}^{\text{std}}$. In some cases ([**GT2**]) there is an associated real algebraic variety.

2.3. Hermitian symmetric domains. (Standard references for this section include [**AMRT**] Chapt. III and [**Sa1**] Chapt. II.) Assume that $\mathbf{G}$ is semisimple, defined over $\mathbb{Q}$, that $K \subset \mathbf{G}(\mathbb{Q})$ is a maximal compact subgroup and that $D = G/K$ is Hermitian. The symmetric space D may be holomorphically embedded in Euclidean space $\mathbb{C}^m$ as a bounded (open) domain, by the Harish Chandra embedding ([**AMRT**] p. 170, [**Sa1**] §II.4). The action of G extends to the closure $\overline{D}$. The boundary $\partial D = \overline{D} - D$ is a smooth manifold which decomposes into a (continuous) union of *boundary components*. Let us say that a real affine hyperplane $H \subset \mathbb{C}^m$ is a *supporting* hyperplane if $H \cap \overline{D}$ is nonempty but $H \cap D$ is empty. Let H be a supporting hyperplane and let $\overline{F} = H \cap \overline{D} = H \cap \partial D$. Let L be the smallest affine subspace of $\mathbb{C}^m$ which contains $\overline{F}$. Then $\overline{F}$ is the closure of a nonempty open subset $F \subset L$ which is then a single boundary component of D ([**Sa1**], III.8.11). The boundary component F turns out to be a bounded symmetric domain in L. Distinct boundary components have nonempty intersection, and the collection of boundary components decomposes ∂D. Alternatively, it is possible ([**Sa1**] III.8.13) to characterize each boundary component as a single holomorphic path component of ∂D : two points $x, y, \in \partial D$ lie in a single boundary component F iff they are both in the image of a holomorphic "path" $\alpha : \Delta \to \partial D$ (where Δ denotes the open unit disk). In this case $\alpha(\Delta)$ is completely contained in F.

Fix a boundary component F. The normalizer $N_G(F)$ (consisting of those group elements which preserve the boundary component F) turns out to be a (proper) parabolic subgroup of G. The boundary component F is *rational* if this subgroup is rationally defined in $\mathbf{G}$. There are countably many rational boundary components. If we decompose $\mathbf{G}$ into its $\mathbb{Q}$ simple factors, $\mathbf{G} = \mathbf{G_1} \times \ldots \times \mathbf{G_k}$ then the symmetric space D decomposes similarly, $D = D_1 \times \ldots \times D_k$. Each (rational) boundary component F of D is then the product $F = F_1 \times \ldots \times F_k$ where either

$F_i = D_i$ or F_i is a proper (rational) boundary component of D_i. The normalizer of F is the product $N_G(F) = N_{G_1}(F_1) \times \ldots \times N_{G_k}(F_k)$ (writing $N_{G_i}(D_i) = G_i$ whenever necessary). If $\mathbf{G}$ is $\mathbb{Q}$-simple then the normalizer $N_G(F)$ is a *maximal* (rational) proper parabolic subgroup of $\mathbf{G}$.

2.4. Definition. The Baily-Borel-Satake partial compactification $\overline{D}^{\mathrm{BB}}$ is the union of D together with all its rational boundary components, with the Satake topology.

2.5. Theorem. ([**BB**]) *The closure $\overline{F}$ of each rational boundary component $F \subset \overline{D}^{BB}$ is the Baily-Borel-Satake partial compactification $\overline{F}^{BB}$ of F. The group $\mathbf{G}(\mathbb{Q})$ acts continuously, by homeomorphisms on the partial compactification $\overline{D}^{BB}$. The action of any arithmetic group $\Gamma \subset \mathbf{G}(\mathbb{Q})$ on $\overline{D}^{BB}$ satisfies conditions (S1) and (S2) of §2.1 and the quotient $\overline{X}^{BB} = \Gamma\backslash\overline{D}^{BB}$ is compact. Moreover, it admits the structure of a complex projective algebraic variety.*

2.6. Remarks. Dividing by Γ has two effects: it identifies (rational) boundary components whose normalizers are Γ-conjugate, and it makes identifications within each (rational) boundary component. The locally symmetric space X is open and dense in $\overline{X}^{\mathrm{BB}}$. If $\kappa : \overline{D}^{\mathrm{BB}} \to \overline{X}^{\mathrm{BB}}$ denotes the quotient mapping, and if F is a rational boundary component then its image $X_F = \kappa(F)$ is the quotient $\Gamma_F\backslash F$ under the subgroup $\Gamma_F = \Gamma \cap N_G(F)$ which preserves F, and it is referred to as a *boundary stratum.* If Γ is *neat* (see §4.1) then the stratum X_F is a complex manifold.

2.7. Symplectic group. In this section we illustrate these concepts for the case of the symplectic group $G = \mathbf{G}(\mathbb{R}) = \mathbf{Sp}(2n, \mathbb{R})$, which may be realized as the group of $2n$ by $2n$ real matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ such that ${}^tAD - {}^tCB = I$; ${}^tAC$ and ${}^tBD$ are symmetric. These are the linear transformations which preserve the symplectic form $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$ on $\mathbb{R}^{2n}$. The symplectic group acts on the Siegel upper half space

$$\mathfrak{h}_n = \left\{ Z = X + iY \in M_{n\times n}(\mathbb{C}) \mid {}^tZ = Z, Y > 0 \right\}$$

(meaning that Y is positive definite) by fractional linear transformations:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

The stabilizer of the basepoint iI_n is the unitary group $K = \mathbf{U}(n)$, embedded in the symplectic group by $A+iB \mapsto \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$. It is a maximal compact subgroup, so $\mathfrak{h}_n = G/K$ is a Hermitian locally symmetric space.

The Harish-Chandra embedding $\phi : \mathfrak{h}_n \to D_n$ is given by the Cayley transformation. Here,

$$D_n = \left\{ w \in M_{n\times n}(\mathbb{C}) \mid {}^t w = w \text{ and } I_n - w\overline{w} > 0 \right\}$$

is a bounded domain, and

$$\phi(z) = (z - iI_n)(z + iI_n)^{-1}.$$

The closure $\overline{D}_n$ is given by relaxing the positive definite condition to positive semi-definite: $I_n - w\overline{w} \geq 0$. Each boundary component (resp. rational boundary component) is a $\mathbf{G}(\mathbb{R})$-translate (resp. $\mathbf{G}(\mathbb{Q})$-translate) of one of the n different *standard* boundary components $D_{n,r}$ (with $0 \leq r \leq n-1$) consisting of all complex $n\times n$ matrices of the form $\left(\begin{smallmatrix} w & 0 \\ 0 & I_{n-r} \end{smallmatrix}\right)$ such that $w \in D_r$. The normalizer $P_{n,r}$ in G of the boundary component $D_{n,r}$ is the maximal parabolic subgroup consisting of matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ such that (cf. [**Kl**] §5)

$$A = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix},\ C = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix},\ D = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

(The upper left block has size $r \times r$ in each of these.)

Each maximal parabolic subgroup P of $\mathbf{Sp}(2n, \mathbb{R})$ is the normalizer of an isotropic subspace $E \subset \mathbb{R}^n$ (meaning that the symplectic form vanishes on E). If a symplectic group element preserves E then it also preserves the symplectic orthogonal subspace $E^\perp \supseteq E$ (which is co-isotropic, meaning that the induced symplectic form vanishes on $\mathbb{R}^{2n}/E^\perp$). So P may also be described as the normalizer of the isotropic-co-isotropic flag $E \subset E^\perp$. In the case of $P_{n,r}$,

$$E = (\mathbb{R}^{n-r} \times 0_r) \times 0_n \subset \mathbb{R}^n \times \mathbb{R}^n, \text{ and } E^\perp = \mathbb{R}^n \times (\mathbb{R}^r \times 0_{n-r}) \subset \mathbb{R}^n \times \mathbb{R}^n.$$

To make these matrices look more familiar, reverse the numbering of the coordinates in the first copy of $\mathbb{R}^n$. Then the symplectic form becomes

$$J' = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$$

where α is the anti-diagonal matrix of ones. In these coordinates, the parabolic subgroup $P_{n,r}$ consists of all matrices preserving J' of the following form:

$$\left(\begin{array}{c|cc|c} *_{n-r} & * & * & * \\ \hline 0 & *_r & *_r & * \\ 0 & *_r & *_r & * \\ \hline 0 & 0 & 0 & *_{n-r} \end{array}\right)$$

(where $*_t$ denotes a square $t \times t$ matrix). As such, it has a Levi decomposition $P_{n,r} = LU$ with

$$L = \left(\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & B & 0 \\ \hline 0 & 0 & A' \end{array}\right), \quad U = \left(\begin{array}{c|c|c} I_{n-r} & * & * \\ \hline 0 & I_r & * \\ \hline 0 & 0 & I_{n-r} \end{array}\right)$$

where $A \in \mathbf{GL}(n-r, \mathbb{R})$, $A' = \alpha\, {}^t A^{-1} \alpha^{-1}$, where $B \in \mathbf{Sp}(2r, \mathbb{R})$. The center, Z_U of U is

$$Z_U = \left(\begin{array}{c|c|c} I_{n-r} & 0 & C \\ \hline 0 & I_r & 0 \\ \hline 0 & 0 & I_{n-r} \end{array}\right),$$

which is easily seen to be isomorphic to the vector space $S_{n-r}(\mathbb{R})$ of symmetric matrices. So L splits as a direct product of a Hermitian factor $L_{Ph} = \mathbf{Sp}(2r, \mathbb{R})$ and a "linear" factor $L_{P\ell} = \mathbf{GL}(n-r, \mathbb{R})$. The Dynkin diagrams for these factors are obtained from the Dynkin diagram for $\mathbf{G}$ by deleting the node α corresponding to the maximal parabolic subgroup P as illustrated in Figure 2.

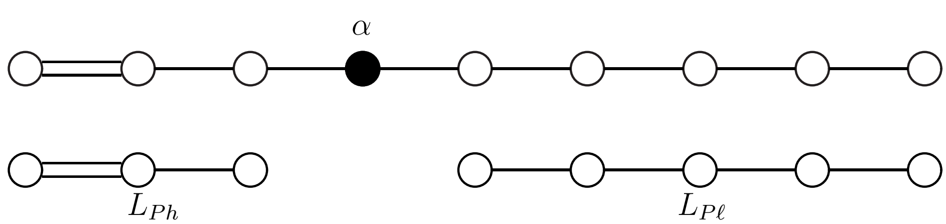

FIGURE 3. Dynkin diagrams for L_{Ph} and $L_{P\ell}$

It is not too difficult to verify that the standard parabolic group P acts on the boundary component $D_{n,r}$ via the first factor $\mathbf{Sp}(2r, \mathbb{R})$, in analogy with the situation in equation (2.2.2). Observe also that the second factor $\mathbf{GL}(n-r, \mathbb{R})$ acts linearly on the Lie algebra $\mathfrak{z} \cong S_{n-r}(\mathbb{R})$ of the center $Z(U_P)$ of the unipotent radical of P, by the action (2.2.1). This action preserves the self-adjoint homogeneous cone $\mathcal{P}_{n-r} \subset \mathfrak{z}$ described in §2.2. If $r' < r$ then $D_{n,r'} \subset \overline{D}_{n,r}$ and $\mathcal{P}_{n-r} \subset \overline{\mathcal{P}}_{n-r'}$.

2.8. Stratifications. A subset S of a locally compact Hausdorff space Y is locally closed iff it is the intersection of an open set and a closed set. A *manifold decomposition* of a locally compact Hausdorff space Y is a decomposition $Y = \coprod_\alpha S_\alpha$ of Y into locally finitely many locally closed smooth manifolds S_α (called strata), which satisfies the axiom of the frontier: the closure of each stratum is a union of strata. In this case the open cone

$$c^0(Y) = Y \times [0,1)/(y,0) \sim (y',0) \text{ for all } y, y' \in Y$$

may be decomposed with strata $S_\alpha \times (0,1)$ and the cone point $*$. Stratified sets are defined inductively. Every smooth manifold is stratified with a single stratum. Let B^s denote the open unit ball in $\mathbb{R}^s$.

2.9. Definition. A manifold decomposition $Y = \coprod_\alpha S_\alpha$ of a locally compact Hausdorff space Y is a *stratification* if for each stratum S_α there exists a compact stratified space L_α, and for each point $x \in S_\alpha$ there exists an open neighborhood $V_x \subset Y$ of x and a stratum preserving homeomorphism

$$V_x \cong B^s \times c^0(L_\alpha) \tag{2.9.1}$$

(where $s = \dim(S_\alpha)$ which is smooth on each stratum, which takes x to $0 \times \{*\}$ and which takes $V_x \cap S_\alpha$ to $B^s \times \{*\}$. Such a neighborhood V_x is a *distinguished* neighborhood of x.

The space L_α is called the *link* of the stratum S_α. A stratification $Y = \coprod_\alpha S_\alpha$ is *regular* if the local trivializations (2.9.1) fit together to make a bundle over S_α. (We omit the few paragraphs that it takes in order to make this precise since we will not have to make use of regularity.) There are many other possible "regularity" conditions on stratified sets, but all the useful ones (such as the Whitney conditions) imply the local triviality (2.9.1) of the stratification.

2.10. Singularities of the Baily-Borel compactification. Returning to the general case, suppose $\mathbf{G}$ is a semi-simple algebraic group defined over $\mathbb{Q}$, of Hermitian type, meaning that the symmetric space $D = G/K$ is Hermitian. Let F be a rational boundary component with normalizing parabolic subgroup P. Let U_P be the unipotent radical of P and $L_P = P/U_P$ the Levi quotient. There is ([**BS**]) a unique lift $L_P \to P$ of the Levi quotient which is stable under the Cartan involution corresponding to K. The group L_P decomposes as an almost direct product (meaning a commuting product with finite intersection), $L_P = L_{Ph}L_{P\ell}$ into factors of Hermitian and "linear" type[3]

[3] It is possible to absorb the compact factors of L_P, if there are any, into L_{Ph} and $L_{P\ell}$ in such a way that both $L_{Ph}, L_{P\ell}$ are defined over $\mathbb{Q}$.

with $A_P \subset L_{P\ell}$. Here, "linear" means that the symmetric space $L_{P\ell}/K_\ell$ for $L_{P\ell}$ is a self-adjoint homogeneous cone C_P, which is open in some real vector space V (in this case, $V = \mathrm{Lie}(Z(U_P))$) on which $L_{P\ell}$ acts by linear transformations which preserve C_P. The group P acts on F through L_{Ph}, identifying F with the symmetric space for L_{Ph}. There is a diffeomorphism $D = P/K_P \cong \mathcal{U}_P \times F \times C_P$.

2.11. Lemma. ([**AMRT**] *§4.4*) *Let* $\mathbf{P} \neq \mathbf{P}'$ *be standard rational parabolic subgroups, normalizing the standard boundary components* $F \neq F'$ *respectively. Then the following statements are equivalent, in which case we write* $\mathbf{P}' \prec \mathbf{P}$:

1. $L_{P'h} \subset L_{Ph}$
2. $L_{P\ell} \subset L_{P'\ell}$
3. $Z(U_P) \subset Z(U_{P'})$
4. F' *is a rational boundary component of* F
5. *The cone* C_P *is a rational boundary component of* $C_{P'}$.

Suppose $\mathbf{P}$ is a rational parabolic subgroup of $\mathbf{G}$ such that $P = \mathbf{P}(\mathbb{R})$ normalizes a rational boundary component F. Let $L_P = L_{Ph}L_{P\ell}$ be the almost direct product decomposition of its Levi component as discussed above. So we obtain identifications $D = P/K_P$, $F = L_{Ph}/K_h$, and $C_P = L_{P\ell}/K_\ell$ for appropriate maximal compact subgroups $K_P = K \cap P \subset L_P \subset P$, $K_h \subset L_{Ph}$, and $K_\ell \subset L_{P\ell}$.

Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a neat ([**B1**]) arithmetic subgroup. Set $\Gamma_P = \Gamma \cap P$ and $\Gamma_U = \Gamma \cap U_P$. Then

$$N_P = \Gamma_U \backslash U_P$$

is a compact "nilmanifold" whose fundamental group is the nilpotent group Γ_U. Let $\Gamma_L = \Gamma_P/\Gamma_U \subset L_P$ and set $\Gamma_\ell = \Gamma_L \cap L_{P\ell}$. Let $\Gamma_h \subset L_{Ph}$ be the projection of Γ_L to the Hermitian factor L_{Ph}. We obtain an identification between the boundary stratum X_F and the quotient $\Gamma_h \backslash F = \Gamma_h \backslash L_{Ph}/K_h$ (cf. §2.6).

It follows that $\Gamma_P \backslash D = \Gamma_P \backslash P/K_P$ fibers over the locally symmetric space $X_P = \Gamma_L \backslash L_P/K_P$ (cf. equation (5.1.2)) with fiber N_P; and that X_P in turn fibers over the boundary stratum X_F with fiber $\Gamma_\ell \backslash C_P$.

2.12. Theorem. ([**BB**]) *The boundary strata of the Baily-Borel compactification form a regular stratification of* $\overline{X}^{BB}$. *Let* X_F *be such a stratum, corresponding to the* Γ*-conjugacy class of the rational parabolic subgroup* $\mathbf{P} \subset \mathbf{G}$. *Then there exists a parabolic neighborhood (see §4.6)* V_F *of* X_F *whose intersection* $V_F \cap X$ *with* X *is diffeomorphic to the quotient* $\Gamma_P \backslash D$. *Hence the geodesic projection* $\pi_F : V_F \cap X \to X_F$

is a smooth fiber bundle with a fiber W, which is itself a fiber bundle, $W \to \Gamma_\ell \backslash C_P$ with fiber diffeomorphic to the compact nilmanifold $N_P = \Gamma_U \backslash U_P$. If $x \in X_F$ and if $B_x \subset X_F$ is a sufficiently small ball in X_F, containing x, then the pre-image $\pi_F^{-1}(B_x) \subset V_F$ is a distinguished neighborhood of x in $\overline{X}^{BB}$, whose intersection with X is therefore homeomorphic to the product $B_x \times W$. The closure of the stratum X_F is the Baily-Borel compactification of X_F. It consists of the union of all strata $X_{F'}$ such that the normalizing parabolic subgroup $\mathbf{P'}$ is Γ-conjugate to some $\mathbf{Q} \prec \mathbf{P}$.

Despite its precision, this result does not fully describe the topology of the neighborhood V_F; only that of $V_F \cap X$. Moreover it does not describe the manner in which such neighborhoods for different strata are glued together. A complete (but cumbersome) description of the local structure of the Baily-Borel compactification exists, but it is sometimes more useful to describe various sorts of "resolutions" of $\overline{X}^{\mathrm{BB}}$.

3. Toroidal compactifications and automorphic vector bundles

3.1. The toroidal compactification is quite complicated and we will not attempt to provide a complete description here. (The standard reference is [**AMRT**]. An excellent introduction appears in the book [**Nk**], but it takes many pages. Brief summaries are described in [**GT2**] §7.5 and [**GP**] §14.5.) Instead we will list some of its main features. As in the preceding section we suppose that $X = \Gamma \backslash G/K$ is a Hermitian locally symmetric space arising from a semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$. There are many toroidal compactifications. Each depends on a certain choice Σ of combinatorial data, and we sometimes indicate this by writing $\overline{X}_\Sigma^{\mathrm{tor}}$ for this compactification. Each $\overline{X}_\Sigma^{\mathrm{tor}}$ admits the structure of a complex algebraic space. For certain "good" choices of Σ the resulting variety is projective and nonsingular, and the complement $\overline{X}_\Sigma^{\mathrm{tor}} - X$ is a divisor with normal crossings, which is therefore stratified by the multi-intersections of the divisors. The identity extends to a unique continuous mapping $\kappa_\Sigma : \overline{X}^{\mathrm{tor}} \to \overline{X}^{\mathrm{BB}}$. It is a holomorphic morphism which takes strata to strata. If Σ is "good" in the above sense then κ_Σ is a resolution of singularities.

3.2. The cone again. In this section we describe the combinatorial data Σ which determines the choice of toroidal compactification $\overline{X}_\Sigma^{\mathrm{tor}}$ in the case that $\mathbf{G}$ is $\mathbb{Q}$-simple. In this case there is a natural ordering among the standard proper rational parabolic subgroups, with $\mathbf{P} \prec \mathbf{Q}$

if $L_{Ph} \subset L_{Qh}$, or equivalently (see Lemma 2.11), if $L_{Q\ell} \subset L_{P\ell}$. Let $\mathbf{P}$ be the standard maximal rational parabolic subgroup which comes first in this total ordering and let $C_P \cong L_{P\ell}/K_{P\ell}$ be the corresponding self adjoint homogeneous cone, with its partial compactification $\overline{C}_P^{\text{std}}$, as in §2.2. It is contained in $Z(U_P)$ which we identify with $\mathfrak{z} = \text{Lie}(Z(U_P))$ by the exponential map, and it is rational with respect to the lattice $\Lambda = \Gamma \cap Z(U_P)$. Choose a $\Gamma_{P\ell}$-invariant rational simplicial cone decomposition of $\overline{C}_P^{\text{std}}$, or equivalently, a rational flat triangulation of the compact (singular) space $\Gamma_{P\ell}\backslash\overline{C}_P^{\text{std}}$/homotheties, which is subordinate to the stratification by boundary strata. (This means that the closure of each stratum should be a subcomplex.)

Up to Γ conjugacy, there are finitely many maximal rational parabolic subgroups P that are minimal with respect to the ordering $\prec$. The data Σ refers to a choice of cone decomposition of $\overline{C}_P^{\text{std}}$ for each of these, which are compatible in the sense that if $C_Q \subset \overline{C}_P^{\text{std}} \cap \overline{C}_{P'}^{\text{std}}$ then the two resulting cone decompositions of C_Q coincide.

By the theory of torus embeddings, such a cone decomposition of C_P determines a $\Gamma_{P\ell}$ equivariant partial compactification of the algebraic torus $(\mathfrak{z} \otimes \mathbb{C})/\Lambda$, which is one of the key ingredients in the construction of the toroidal compactification. The rest of the construction, which is rather complicated, consists of "attaching" the resulting torus embedding to X.

Actually, this data only determines a resolution $\overline{X}_\Sigma^{\text{tor}}$ which is "rationally nonsingular" (has finite quotient singularities). A truly nonsingular compactification is obtained when we place a further integrality condition on the cone decomposition of $\overline{C}_P^{\text{std}}$, namely that the shortest vectors in the 1-dimensional cones in any top dimensional simplicial cone should form an integral basis of the lattice Λ. There is a further (convexity) criterion on the cone decompositions to guarantee that the resulting $\overline{X}_\Sigma^{\text{tor}}$ is projective. Cone decompositions satisfying these additional conditions exist, although the literature is a little sketchy on this point. A more difficult problem is to find (canonical) models for $\overline{X}_\Sigma^{\text{tor}}$ defined over a number field, or possibly over the reflex field, when X is a Shimura variety. See, for example [**FC**].

3.3. Automorphic vector bundles. Let $\lambda : K \to \mathbf{GL}(E)$ be a representation of K on some complex vector space E. Then we obtain a homogeneous vector bundle $\mathbf{E} = G \times_K E$ on D, meaning that we identify (h, e) with $(hk, \lambda(k)^{-1}e)$ whenever $k \in K$ and $h \in G$. Denote the equivalence class of such a pair by $[h, e]$. The action of G on D is covered by an action of G on $\mathbf{E}$ which is given by $g \cdot [h, e] = [gh, e]$. So dividing by Γ we obtain a *automorphic vector bundle* $\mathbf{E}_\Gamma = \Gamma \backslash \mathbf{E} \to X$, which may also be described as $\mathbf{E}_\Gamma = (\Gamma\backslash G) \times_K E$. Such a vector bundle carries a canonical connection. If the representation λ is the restriction to K of a representation of G, then E_Γ also carries a (different) flat connection (cf. [**GP**] §5).

Smooth sections of $\mathbf{E}_\Gamma$ may be identified (see also §3.5 below) with smooth mappings $f : G \to E$ such that $f(\gamma g k) = \lambda(k^{-1}) f(g)$ (for all $k \in K, \gamma \in \Gamma$). The *holomorphic* sections of $\mathbf{E}_\Gamma$ correspond to those functions that are killed by certain differential operators, as observed in [**B3**]. The complexified Lie algebra of G decomposes under the Cartan involution into $+1, +i$, and $-i$ eigenspaces,

$$\mathfrak{g}(\mathbb{C}) = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{k}(\mathbb{C}) \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

respectively (where $\mathfrak{k}(\mathbb{C})$ is the complexification of Lie(K)). Each $V = X + iY \in \mathfrak{g}(\mathbb{C})$ acts on functions $f : G \to \mathbb{C}$ by $V(f) = X(f) + iY(f)$. Then a smooth section of $\mathbf{E}_\Gamma$ is holomorphic if and only if the corresponding function $f : G \to \mathbb{E}$ satisfies the Cauchy-Riemann equations: $V(f) = 0$ for all $V \in \mathfrak{p}^-$. Let us further say ([**B3**]) that such a holomorphic section f is a *holomorphic automorphic form* if it has *polynomial growth*, that is, if there exists $C > 0$ and $n \geq 1$ such that $|f(g)| \leq C||g||_G^n$. (Here, $||g||_G$ is the norm $||g||_G = tr(Ad(\theta(g^{-1})) \cdot Ad(g))$, where θ is the Cartan involution.)

One might hope to interpret this condition in terms of the Baily Borel compactification $\overline{X}^{\text{BB}}$. However the automorphic vector bundle $\mathbf{E}_\Gamma \to X$ does not necessarily extend to the Baily-Borel compactification. It does extend to $\overline{X}^{\text{RBS}}$, but only as a topological vector bundle. However, in [**M**], Mumford constructs a canonical extension $\mathbf{E}_\Sigma \to \overline{X}_\Sigma^{\text{tor}}$ as a holomorphic vector bundle, and shows that the (global holomorphic) sections of $\mathbf{E}_\Sigma$ are precisely the holomorphic sections of $\mathbf{E}_\Gamma \to X$ with polynomial growth, that is, they are holomorphic automorphic forms.

3.4. Proportionality theorem. In [**M**], Mumford proved that the Chern classes $c^i(\mathbf{E}_\Sigma) \in H^{2i}(\overline{X}_\Sigma^{\text{tor}})$ of the bundle $\mathbf{E}_\Sigma$ satisfy Hirzebruch's proportionality theorem: there exists a single rational number $v(\Gamma)$ so that for any automorphic vector bundle $\mathbf{E}_\Gamma$ on X, for any toroidal compactification $\overline{X}_\Sigma^{\text{tor}}$, and for any partition $I : n_1 + n_2 + \cdots + n_k = 2n$ where $n = \dim_{\mathbb{C}}(X)$, the corresponding Chern number of the canonical extension $\mathbf{E}_\Sigma$

$$c^I(\mathbf{E}_\Sigma) = c^{n_1}(\mathbf{E}_\Sigma) \cup c^{n_2}(\mathbf{E}_\Sigma) \cup \cdots \cup c^{n_k}(\mathbf{E}_\Sigma) \cap [\overline{X}_\Sigma^{\text{tor}}] \in \mathbb{Q}$$

(where $[\overline{X}^{\text{tor}}_{\Sigma}] \in H_{2n}(\overline{X}^{\text{tor}}_{\Sigma})$ is the fundamental class) satisfies

$$c^I(\mathbf{E}_\Sigma) = v(\Gamma) c^I(\check{\mathbf{E}})$$

where $\check{\mathbf{E}}$ is the corresponding vector bundle on the compact dual symmetric space, $\check{D}$. The fact that these Chern numbers are independent of the resolution $\overline{X}^{\text{tor}}_{\Sigma}$ suggests that they might be related to the topology of $\overline{X}^{\text{BB}}$. This possibility was realized in [**GP**] where it was shown that for any automorphic vector bundle $\mathbf{E}_\Gamma \to X$, each Chern class $c^k(\mathbf{E}_\Gamma) \in H^{2k}(X;\mathbb{C})$ has a particular lift to cohomology $\overline{c}^k(\mathbf{E}_\Gamma) \in H^{2k}(\overline{X}^{\text{BB}};\mathbb{C})$ such that for any toroidal resolution $\kappa_\Sigma : \overline{X}^{\text{tor}}_{\Sigma} \to \overline{X}^{\text{BB}}$ the lift satisfies $\kappa_\Sigma^*(\overline{c}^k(\mathbf{E}_\Gamma)) = c^k(\mathbf{E}_\Sigma)$. Therefore the proportionality formula holds for these lifts $\overline{c}^k(\mathbf{E}_\Sigma)$ as well. In many cases this accounts for sufficiently many cohomology classes to prove that the cohomology $H^*(\check{D},\mathbb{C})$ of the compact dual symmetric space is contained in the cohomology $H^*(\overline{X}^{\text{BB}},\mathbb{C})$ of the Baily-Borel compactification.

3.5. Automorphy factors. There is a further (and more classical) description of the sections of an automorphic vector bundle $\mathbf{E}_{\boldsymbol{\Gamma}} = \Gamma\backslash(G \times_K E)$ corresponding to a representation $\lambda : K \to \mathbf{GL}(E)$. A (smooth) *automorphy factor* $J : G \times D \to \mathbf{GL}(E)$ for $\mathbf{E}$ is a (smooth) mapping such that

(1) $J(gg', x) = J(g, g'x)J(g', x)$ for all $g, g' \in G$ and $x \in D$
(2) $J(k, x_0) = \lambda(k)$ for all $k \in K$.

It follows (by taking $g = 1$) that $J(1, x) = I$. The automorphy factor J is determined by its values $J(g, x_0)$ at the basepoint: any smooth mapping $j : G \to \mathbf{GL}(E)$ such that $j(gk) = j(g)\lambda(k)$ (for all $k \in K$ and $g \in G$) extends in a unique way to an automorphy factor $J : G \times D \to \mathbf{GL}(E)$ by setting $J(g, hx_0) = j(gh)j(h)^{-1}$.

An automorphy factor J determines a (smooth) trivialization

$$\Phi_J : G \times_K E \to (G/K) \times E$$

by $[g, v] \mapsto (gK, J(g, x_0)v)$. With respect to this trivialization the action of $\gamma \in G$ is given by

$$\gamma \cdot (x, v) = (\gamma x, J(\gamma, x)v). \tag{3.5.1}$$

Conversely any smooth trivialization $\Phi : \mathbf{E} \cong (G/K) \times E$ of $\mathbf{E}$ determines a unique automorphy factor J such that $\Phi = \Phi_J$. Such a trivialization allows one to identify smooth sections s of $\mathbf{E}$ with smooth mappings $r : D \to E$. If the section s is given by a smooth mapping $s : G \to E$ such that $s(gk) = \lambda(k^{-1})s(g)$ then the corresponding mapping r is $r(gK) = J(g, x_0)s(g)$ (which is easily seen to be well defined).

By (3.5.1), sections s which are invariant under $\gamma \in \Gamma \subset \mathbf{G}(\mathbb{Q})$ then correspond to functions $r : D \to E$ which satisfy the familiar relation

$$r(\gamma x) = J(\gamma, x) r(x) \tag{3.5.2}$$

for all $x \in D$. Moreover, there exists a *canonical automorphy factor* ([**Sa1**] II §5),

$$J_0 : G \times D \to \mathbf{K}(\mathbb{C})$$

which determines an automorphy factor $J = \lambda_{\mathbb{C}} \circ J_0$ for every homogeneous vector bundle $\mathbf{E} = G \times_K E$, where $\lambda_{\mathbb{C}} : \mathbf{K}(\mathbb{C}) \to \mathbf{GL}(E)$ denotes the complexification of λ. With this choice for J, *holomorphic* sections s of $\mathbf{E}_{\boldsymbol{\Gamma}}$ correspond to *holomorphic* functions $r : D \to E$ which satisfy (3.5.2).

4. Borel-Serre compactification

4.1. About the center, and other messy issues. In this section and in the remainder of this article, $\mathbf{G}$ will be a connected reductive algebraic group defined over $\mathbb{Q}$; $K \subset G$ will be a chosen maximal compact subgroup and $\Gamma \in \mathbf{G}(\mathbb{Q})$ will be an arithmetic group.

The identity component (in the sense of algebraic groups) of the center of $\mathbf{G}$ is an algebraic torus defined over $\mathbb{Q}$. It has three parts: a greatest $\mathbb{Q}$-split subtorus $\mathbf{A_G}$, an $\mathbb{R}$-split but $\mathbb{Q}$-anisotropic part $\mathbf{A_G^1}$, and an $\mathbb{R}$-anisotropic (i.e. compact) part, $\mathbf{A_G^2}$. Unfortunately it is not simply the direct product of these three parts, however we can at least isolate the group $A_G = \mathbf{A_G}(\mathbb{R})^+$, the topologically connected identity component of the group of real points of $\mathbf{A_G}$. Define ([**BS**] §1.1)

$$^0\mathbf{G} = \bigcap_{\chi} \ker(\chi^2)$$

to be the intersection of the kernels of the squares of the rationally defined characters $\chi : \mathbf{G} \to \mathbf{G_m}$. It is a connected reductive linear algebraic group defined over $\mathbb{Q}$ which contains every compact subgroup and every arithmetic subgroup of G ([**BS**] §1.2). The group of real points $G = \mathbf{G}(\mathbb{R})$ decomposes as a direct product, $G = {}^0\mathbf{G}(\mathbb{R}) \times A_G$. Then $D = G/KA_G = {}^0G/K$. So, to study the topology and geometry of D (and its arithmetic quotients) one may assume that the group $\mathbf{G}$ contains no nontrivial $\mathbb{Q}$-split torus in its center. This is not a good assumption to make from the point of view of representation theory or from the point of view of Shimura varieties since in these cases the center $\mathbf{A_G}$ plays an important role. Nevertheless we will occasionally make this assumption when it simplifies the exposition.

The part A_G^1 of the center contributes a Euclidean factor to the symmetric space D. However, after dividing by Γ this Euclidean space will get rolled up into circles, which explains why it does not interfere with our efforts to compactify $\Gamma\backslash D$. Even if $\mathbf{A_G}$ and $\mathbf{A^1_G}$ are trivial, the group G may still contain a compact torus in its center, but this will be contained in any maximal compact subgroup $K \subset G$ so it will not appear in the symmetric space G/K.

We will also assume, for simplicity, that Γ is torsion free, which implies that Γ acts freely on D and that the quotient X is a smooth manifold. It is often convenient to make the slightly stronger assumption that Γ is *neat* ([**B1**]), which implies ([**AMRT**] p. 276) that $(\Gamma \cap H_2(\mathbb{C}))/(\Gamma \cap H_1(\mathbb{C})$ is torsion-free whenever $\mathbf{H_1} \lhd \mathbf{H_2} \subset \mathbf{G}$ are rationally defined algebraic subgroups. This guarantees that all the boundary strata are smooth manifolds also. Every arithmetic group contains neat arithmetic subgroups of finite index, however much of what follows will continue to hold even when Γ has torsion.

4.2. The Borel-Serre compactification $\overline{X}^{\text{BS}}$ is (topologically) a smooth manifold (of some dimension m) with boundary. However the boundary has the differentiable structure of "corners": it is decomposed into a collection of smooth manifolds of various dimensions, and a point on one of these boundary manifolds of dimension d has a neighborhood which is diffeomorphic to the product $B^d \times [0, 1)^{m-d}$ where B^d is the open unit ball in $\mathbb{R}^d$. This compactification is obtained as the quotient under Γ of a "partial" compactification $\overline{D}^{\text{BS}}$ which is obtained from D by attaching a "boundary component" for each proper rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$.

4.3. Geodesic action. Let P be the group of real points of a rational parabolic subgroup $\mathbf{P}$. Let U_P be its unipotent radical and $\nu : P \to L_P$ be the projection to the Levi quotient. Then L_P is the group of real points of a rationally defined reductive group $\mathbf{L_P}$ and as such, we have $L_P = M_P A_P$ where $M_P = {}^0\mathbf{L_P}(\mathbb{R})$ as in §4.1. The choice of $K \subset G$ corresponds to a Cartan involution $\theta : G \to G$ and there is a unique θ stable lift ([**BS**]) of L_P to P. So we obtain the *Langlands decomposition*

$$P = U_P A_P M_P. \tag{4.3.1}$$

The intersection $K_P = K \cap P$ is completely contained in M_P. It follows from the Iwasawa decomposition that P acts transitively on D. Define the *right* action of A_P on $D = P/K_P$ by $(gK_P)\cdot a = gaK_P$ for $g \in P$ and $a \in A_P$. This action is well defined since A_P commutes with $K_P \subset M_P$, but it also turns out to be independent of the choice of basepoint. Moreover each orbit of this A_P action is a totally geodesic submanifold of D (with respect to any invariant Riemannian metric). Define the (Borel-Serre) boundary component $e_P = D/A_P$.

Intuitively, we want to "attach" e_P to D as the set of limit points of each of these geodesic orbits. For the upper half plane $\mathfrak{h}_1$ and the standard Borel subgroup $B \subset \mathbf{SL}(2,\mathbb{R})$, if $a = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right)$ then the geodesic action is $(x+iy)\cdot a = x + it^2 y$ for any $t \neq 0$, so the geodesic orbits are "vertical" half lines. Then e_P is a line at infinity, parallel to the real axis, which is glued onto the upper half plane so as to make a strip $\mathbb{R} \times (0,\infty]$. (See figure 3.)

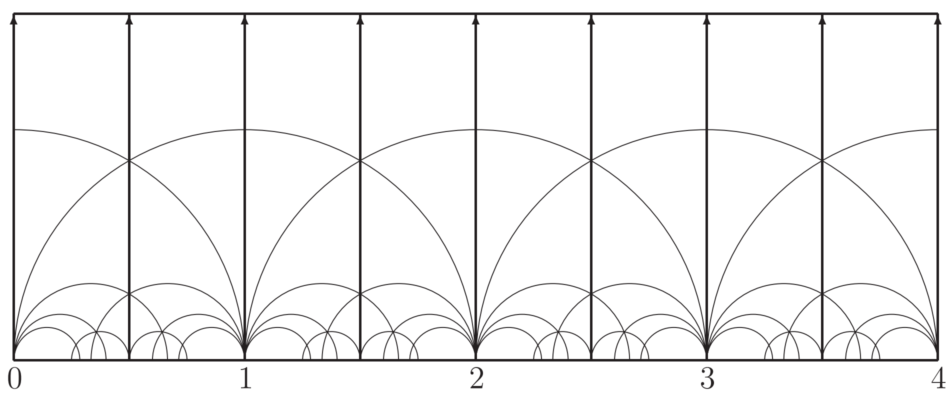

FIGURE 4. $\overline{\mathfrak{h}}^{\mathrm{BS}}$ and geodesic action

When Γ "acts" on this union $D \cup e_P$, only the translations $\Gamma_P = \Gamma \cap P$ act nontrivially on the boundary component e_P so the resulting circle $\Gamma_P \backslash e_P$ becomes glued to X where, previously in the Baily-Borel compactification, we had placed a cusp. Unfortunately the group Γ does not actually act on $D \cup e_P$, a difficulty which may be rectified by attaching additional boundary components e_Q for every rational parabolic subgroup $\mathbf{Q}$, using a "Satake topology" in which each e_Q has a neighborhood isomorphic to that of e_P. Although it is difficult to visualize the resulting space $\overline{D}^{\mathrm{BS}}$, it is nevertheless a (real) two dimensional manifold with boundary, whose boundary consists of countably many disjoint copies of $\mathbb{R}$.

4.4. In this section we return to the general case but we assume for simplicity that $\mathbf{A_G}$ is trivial. As a set, $\overline{D}^{\mathrm{BS}}$ is defined to be the disjoint union of D and all the Borel-Serre boundary components e_P

corresponding to rational proper parabolic subgroups $\mathbf{P}$. Let $\mathbf{P_0} \subset \mathbf{G}$ be a fixed minimal rational parabolic subgroup. The parabolic subgroups containing $\mathbf{P_0}$ are the *standard* parabolics. Denote by $\mathbf{A_0}$ the greatest $\mathbb{Q}$ split torus in the center of the (canonical lift of the) Levi component $L_0 = L_{P_0}$ and let $\Phi = \Phi(\mathbf{G}, \mathbf{A_0})$ be the corresponding system of rational roots, with simple rational roots Δ. For any standard parabolic subgroup $\mathbf{P}$ let Δ_P be the set of restrictions of the roots in Δ to $A_P \subset A_0$. If $\Delta_P = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ the isomorphism

$$A_P \cong (0, \infty)^r \quad \text{given by } t \mapsto (\alpha_1(t), \alpha_2(t), \dots, \alpha_r(t))$$

determines a partial compactification $\overline{A}_P \cong (0, \infty]^r$. Let

$$D(P) = D \times_{A_P} \overline{A}_P$$

where A_P acts on D by the geodesic action. Then $D(P)$ contains D and it also contains e_P as the set of points (or, rather, equivalence classes) $[x, (\infty, \infty, \dots, \infty)]$. The projection $D \to e_P$ extends continuously to $\pi_P : D(P) \to e_P$ which we refer to as the *geodesic projection*.

It is a bit easier to picture this construction in terms of coordinates. For each $\alpha \in \Delta_P$ define the *root function* $f_\alpha^P : D \to (0, \infty)$ by

$$f_\alpha^P(x) = f_\alpha^P(gK_P) = f_\alpha^P(uamK_P) = \alpha(a)$$

where $x = gK_P$ and $g = uam \in P$ has been decomposed according to the Langlands' decomposition (4.3.1). The root function is well defined because the mapping $P \to A_P$ given by $uam \mapsto a$ is a group homomorphism. If $g' = u'a'm' \in P$ and if $b \in A_P$ then

$$f_\alpha^P(g'x \cdot b) = \alpha(a'b) f_\alpha^P(x). \tag{4.4.1}$$

The root functions clearly extend to $D(P)$ and together with π_P they determine a diffeomorphism

$$D(P) \cong e_P \times (0, \infty]^r. \tag{4.4.2}$$

If $\mathbf{P} \subset \mathbf{Q}$ then $A_Q \subset A_P$ and the restriction of the geodesic action for A_P to A_Q coincides with the geodesic action for A_Q. Therefore there is a natural inclusion $\overline{A_Q} \subset \overline{A_P}$ as a coordinate subspace, and we see that $D(P)$ is the disjoint union of boundary components e_Q for $\mathbf{Q} \supseteq \mathbf{P}$ (including $e_G = D$.) We wish to declare this set $D(P)$ to be an open neighborhood of e_P in $\overline{D}^{\mathrm{BS}}$.

The following theorem says that it is possible to similarly attach boundary components e_P for any rational parabolic subgroup $\mathbf{P}$, so as to obtain a partial compactification $\overline{D}^{\mathrm{BS}}$ of D.

FIGURE 5. $D(P)$ and level curves of f_α^P for $\alpha_1, \alpha_2 \in \Delta_P$.

4.5. THEOREM. ([**BS**] §*7.1*) *There is a unique topology (the Satake topology) on the union* $\overline{D}^{BS}$ *of* D *with all its rational boundary components, so that the action of* $\mathbf{G}(\mathbb{Q})$ *on* D *extends continuously to an action by homeomorphisms on* $\overline{D}^{BS}$, *and so that each* $D(P) \subset \overline{D}^{BS}$ *is open. The parabolic subgroup* P *is the normalizer of the boundary component* e_P. *The closure* $\overline{e}_P$ *of* e_P *in* $\overline{D}^{BS}$ *is the Borel-Serre partial compactification of* e_P.

(An annoying problem arises because e_P is not a symmetric space, and in fact it is a homogeneous space under the non-reductive group P. In order to apply inductive arguments, Borel and Serre found it necessary to work within a wider class of groups and homogeneous spaces which include P and e_P. Fortunately the current context provides the author with a poetic license to ignore these further complications.)

4.6. Quotient under Γ**.** Fix a neat arithmetic group $\Gamma \subset \mathbf{G}(\mathbb{Q})$. Let $\kappa : \overline{D}^{\mathrm{BS}} \to \overline{X}^{\mathrm{BS}} = \Gamma\backslash\overline{D}^{\mathrm{BS}}$ be the quotient. The action of Γ on $\overline{D}^{\mathrm{BS}}$ will identify some boundary components and it will also make identifications within a single boundary component. There is a risk that this will completely destroy the local picture $D(P)$ of e_P which was developed above. It is a remarkable fact that this risk never materializes. To be precise, there is a neighborhood V of e_P in $\overline{D}^{\mathrm{BS}}$ so that

(P1) Two points in V are identified under Γ if and only if they are identified under $\Gamma_P = \Gamma \cap P$.
(P2) The neighborhood V is preserved by the geodesic action of $A_P(\geq 1)$.

Here, $A_P(\geq 1) = \{a \in A_P | \alpha(a) \geq 1 \text{ for all } \alpha \in \Delta_P\}$ is the part of A_P that moves points in D "towards the boundary." Such a neighborhood V is called a Γ-*parabolic neighborhood* and we will also refer to its image $\kappa(V) \subset \overline{X}^{\mathrm{BS}}$ as a parabolic neighborhood. The intersection $\kappa(V) \cap X$ is diffeomorphic to the quotient $\Gamma_P\backslash D$.

There is another way to say this. Let V' be the image of V in $\Gamma_P\backslash\overline{D}^{\mathrm{BS}}$. Since $\Gamma_P \subset \Gamma$ we have a covering $\beta : \Gamma_P\backslash\overline{D}^{\mathrm{BS}} \to \Gamma\backslash\overline{D}^{\mathrm{BS}}$. For points far away from e_P this is a nontrivial covering. However for points in $V' \subset \overline{D}^{\mathrm{BS}}$ the covering $V' \to \beta(V')$ is actually one to one. This fact (a consequence of reduction theory) allows us to study a neighborhood of e_P using the structure of the parabolic subgroup P. (In the case of the upper half plane, it is easy to see from Figure 1 that the set of points $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 2$ forms such a parabolic neighborhood of the point at infinity.)

Define the *Borel-Serre stratum* $Y_P = \kappa(e_P)$ to be the image of the boundary component e_P. A Γ-parabolic neighborhood of $Y_P \subset \overline{X}^{\mathrm{BS}}$ is diffeomorphic to a neighborhood of Y_P in $\Gamma_P\backslash D(P)$. If $\gamma \in \Gamma_P$ then for any $\alpha \in \Delta_P$ and any $x \in D(P)$ we have: $f^P_\alpha(\gamma x) = f^P_\alpha(x)$. This follows from (4.4.1) and the fact that the projection $P \to A_P$ kills Γ_P. Therefore the diffeomorphism (4.4.2) passes to a diffeomorphism

$$\Gamma_P\backslash D(P) \cong \Gamma_P\backslash e_P \times (0,\infty]^r$$

which says that the stratum Y_P has a neighborhood in $\overline{X}^{BS}$ which is a manifold with corners. As described above, these corners fit together: if $\mathbf{P} \subset \mathbf{Q}$ then the inclusion $e_Q \subset D(P)$ induces an mapping $Y_P \times (0,\infty)^s \to Y_Q$ (for an appropriate coordinate subspace $(0,\infty)^s$), which is one to one near Y_P. (Once we leave the parabolic neighborhood of Y_P this mapping is no longer one to one.) With a bit more work one concludes the following

4.7. Theorem. ([**BS**]) *The quotient $\overline{X}^{BS} = \Gamma\backslash\overline{D}^{BS}$ is compact. It is stratified with finitely many strata $Y_P = \Gamma_P\backslash e_P$, one for each Γ-conjugacy class of rational parabolic subgroups $\mathbf{P} \subset \mathbf{G}$. Each stratum Y_P has a parabolic neighborhood V diffeomorphic to $Y_P \times (0,\infty]^r$ (where r is the rank of A_P) whose faces $Y_P \times (0,\infty)^s$ are the intersections $Y_Q \cap V$ for appropriate $\mathbf{Q} \supset \mathbf{P}$.*

Two important applications are given in §6.2. For many purposes the Borel-Serre compactification is too big. For example, each stratum Y_P is the quotient of a non-reductive group P by an arithmetic subgroup Γ_P. The reductive Borel-Serre compactification (first studied in [**Z1**] §4.2, p. 190; see also [**GHM**] §8) is better behaved. It is obtained by replacing this stratum by an appropriate arithmetic quotient of the Levi component of P.

5. Reductive Borel-Serre Compactification

5.1. As in the previous section we suppose $\mathbf{G}$ is a reductive algebraic group defined over $\mathbb{Q}$ with associated symmetric space $D = G/KA_G$. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup and set $X = \Gamma\backslash D$. Let $\mathbf{P}$ be a proper rational parabolic subgroup with Langlands' decomposition (4.3.1), let $\Gamma_P = \Gamma \cap P$, let $e_P = D/A_P$ be the Borel-Serre boundary component, and let $Y_P = \Gamma_P\backslash e_P$ be the Borel-Serre stratum. Let us first examine the structure of $e_P \to Y_P$.

Using the Levi decomposition $P = U_P L_P$ we may write $D = U_P L_P/K_P A_G$. The group K_P and the geodesic action of the group A_P act (from the right) only on the factor L_P. So we obtain a diffeomorphism

$$e_P \cong U_P \times (L_P/K_P A_P) = U_P \times D_P \tag{5.1.1}$$

where D_P is the *reductive Borel-Serre boundary component* $L_P/K_P A_P$. In these coordinates, the action of $g \in P$ is given by

$$g.(u, zK_P A_P) = (gui\nu_P(g)^{-1}, \nu_P(g)zK_P A_P)$$

where $\nu_P : P \to L_P$ is the projection to the Levi quotient, and where $i : L_P \to P$ is its canonical splitting from §2.10. So the unipotent radical of P acts only on the U_P factor, while P acts on the D_P factor through its Levi quotient.

Define the *reductive Borel-Serre stratum*

$$X_P = \Gamma_L\backslash D_P \ = \Gamma_L\backslash L_P/K_P A_P \tag{5.1.2}$$

where $\Gamma_L = \nu_P(\Gamma_P) \subset L_P = M_P A_P$. Then the Borel-Serre stratum Y_P is a fiber bundle over the reductive Borel-Serre stratum X_P,

$$Y_P = \Gamma_P\backslash e_P = \Gamma_P\backslash P/K_P A_P \to X_P = \Gamma_L\backslash D_P$$

whose fiber is the compact nilmanifold $N_P = \Gamma_U\backslash U_P$.

5.2. Definition. The reductive Borel-Serre partial compactification $\overline{D}^{\mathrm{RBS}}$ (resp. $\overline{X}^{\mathrm{RBS}}$) is the quotient of $\overline{D}^{\mathrm{BS}}$ (resp. $\overline{X}^{\mathrm{BS}}$) which is obtained by collapsing each e_P to D_P (resp. Y_P to X_P).

5.3. Theorem. ([**Z1**], [**GHM**] §*8.10*) *The group* Γ *acts on* $\overline{D}^{RBS}$ *with compact quotient* $\Gamma\backslash\overline{D}^{RBS} = \overline{X}^{RBS}$. *The boundary strata form a regular stratification of* $\overline{X}^{RBS}$ *and the stratum*

$$X_P = \Gamma_L\backslash M_P/K_P = \Gamma_L\backslash L_P/K_P A_P$$

is a locally symmetric space corresponding to the reductive group L_P. *Its closure* $\overline{X}_P$ *in* $\overline{X}^{RBS}$ *is the reductive Borel-Serre compactification of* X_P. *The geodesic projection* $\pi_P : D \to e_P \to D_P$ *passes to a geodesic* *projection,* $\pi_P : V \to X_P$ *defined on any parabolic neighborhood* $V \subset \overline{X}^{RBS}$ *of* X_P. *The pre-image* $\pi_P^{-1}(B^r)$ *of an open ball* $B^r \subset X_P$ *is a distinguished neighborhood of any* $x \in B^r$.

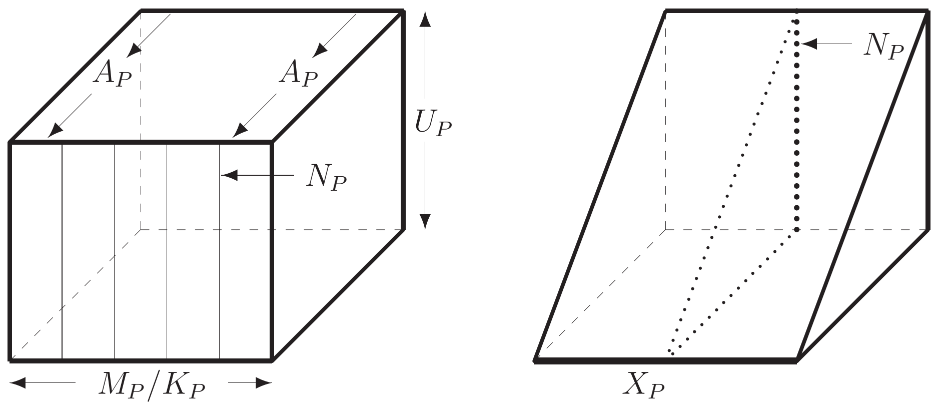

FIGURE 6. Borel-Serre and reductive Borel-Serre compactifications

The diagram on the left of Figure 5 represents the Borel-Serre compactification. This may be thought of as a "local" picture, but one may also imagine a "global" picture by identifying the top and bottom of the box, and identifying the left and right sides of the box. The box is a manifold with boundary: the front face is the boundary stratum Y_P. It is foliated by nilmanifolds isomorphic to N_P, and in general the vertical lines are the (images of) orbits of U_P. The geodesic action of A_P moves points towards the front face. The (images of) orbits of M_P are horizontal. On the right hand side, the nilmanifold fibers in Y_P have been collapsed to points, leaving the stratum X_P. Nothing else has changed. However we now see that a normal slice through X_P (indicated by a dotted triangle) is diffeomorphic to the cone over N_P, that is, the nilmanifold N_P is the *link* of X_P (see next section).

5.4. Singularities of $\overline{X}^{\mathbf{RBS}}$. The reductive Borel-Serre compactification of X is a highly singular, non-algebraic space. Although the singularities are complicated, they can be precisely described, and as a consequence it is possible to compute the stalk cohomology of various sheaves on $\overline{X}^{\mathrm{RBS}}$. Here is a description of the link (cf. §2.8) of the stratum X_P in $\overline{X}^{\mathrm{RBS}}$.

If $\mathbf{P}$ is a (proper) maximal rational parabolic subgroup of $\mathbf{G}$ then the link of the stratum X_P is the compact nilmanifold $N_P = \Gamma_{U_P} \backslash U_P$ where U_P is the unipotent radical of P and $\Gamma_{U_P} = \Gamma \cap U_P$.

If $\mathbf{P} \subset \mathbf{Q}$ then $\mathbf{P}$ determines a parabolic subgroup $P/U_Q \subset L_Q$ with unipotent radical $U_P^Q = U_P/U_Q$ and discrete group $\Gamma_P^Q = \Gamma_{U_P}/\Gamma_{U_Q}$. Let $N_P^Q = \Gamma_P^Q \backslash U_P^Q$ be the associated nilmanifold. It is the quotient of N_P under the action of U_Q so there is a surjection $T_{PQ} : N_P \to N_P^Q$. Similarly, if $\mathbf{P} \subset \mathbf{R} \subset \mathbf{Q}$ we obtain a canonical surjection

$$N_P^Q \to N_P^R. \tag{5.4.1}$$

To make the notation more symmetric, let us also write $N_P = N_P^G$.

If $\mathbf{P} = \mathbf{Q}_1 \cap \mathbf{Q}_2$ is the intersection of two maximal rational (proper) parabolic subgroups then the link of the stratum X_P in $\overline{X}^{\mathrm{RBS}}$ is the double mapping cylinder of the diagram

$$N_P^{Q_1} \xleftarrow[T_{PQ_1}]{} N_P^G \xrightarrow[T_{PQ_2}]{} N_P^{Q_2}.$$

In other words, it is the disjoint union $N_P^{Q_1} \cup (N_P \times [-1,1]) \cup N_P^{Q_2}$ modulo relations $(x,-1) \sim T_{PQ_1}(x)$ and $(x,1) \sim T_{PQ_2}(x)$ for all $x \in N_P$.

In the general case, suppose $\mathbf{P}$ is a rational parabolic subgroup of $\mathbf{G}$ with $\dim(A_P) = r$. The rational parabolic subgroups containing $\mathbf{P}$ (including $\mathbf{G}$) are in one to one correspondence with the faces of the $r-1$ dimensional simplex Δ^{r-1}, in an inclusion-preserving manner, with the interior face corresponding to $\mathbf{G}$.

5.5. Theorem. ([**GHM**] §*8*) *The link of the stratum X_P in $\overline{X}^{RBS}$ is homeomorphic (by a stratum preserving homeomorphism which is smooth on each stratum) to the geometric realization of the contravariant functor $N : \Delta^{r-1} \to \{manifolds\}$ defined on the category whose objects are faces of the $r-1$ simplex (and whose morphisms are inclusions of faces), which associates to each face $\mathbf{Q}$ the nilmanifold N_P^Q and to each inclusion of faces $\mathbf{R} \subset \mathbf{Q}$ the morphism (5.4.1).*

5.6. Theorem. ([**Z2**]) *Suppose the symmetric space $D = G/K$ is Hermitian and let $\overline{X}^{BB}$ be the Baily-Borel compactification of $X = \Gamma \backslash D$. Then there exist unique continuous mappings*

$$\overline{X}^{BS} \longrightarrow \overline{X}^{RBS} \xrightarrow{\ \tau\ } \overline{X}^{BB}.$$

which restrict to the identity on X.

5.7. The first map is part of the definition of the reductive Borel-Serre compactification. The mapping τ, if it exists, is determined by the fact that it is the identity on X. However at first glance it appears unlikely to exist since, when $\mathbf{G}$ is $\mathbb{Q}$ simple, the strata of $\overline{X}^{\mathrm{BB}}$ are indexed by (Γ conjugacy classes of) *maximal* rational parabolic subgroups, while the strata of $\overline{X}^{\mathrm{RBS}}$ are indexed by (Γ conjugacy classes of) all rational parabolic subgroups. Suppose for the moment that $\mathbf{G}$ is $\mathbb{Q}$ simple. (The general case follows from this.) Then the rational Dynkin diagram for G is of type C_n or BC_n, as in Figure 2. A rational parabolic subgroup corresponds to a subset of the Dynkin diagram, so its Levi quotient decomposes as an almost direct product (commuting product with finite intersections):

$$L_P = L_{Ph} \times L_{\ell 1} \times L_{\ell 2} \times \ldots \times L_{\ell m} \times H \tag{5.7.1}$$

of a (semisimple) Hermitian factor L_{Ph} with a number of "linear factors" $L_{\ell i}$, (each of which acts as a group of automorphisms of a self adjoint homogeneous cone in some real vector space) and a compact group H. (In what follows we will assume the compact factor H, if it exists, has been absorbed into the other factors. It is possible to arrange this so that each of the resulting factors is defined over the rational numbers.)

So there is a projection $D_P \to F$ from the reductive Borel-Serre boundary component $D_P = L_P/K_PA_P$ to the Borel-Serre boundary component $F = L_{Ph}/K_{Ph}$ (for appropriate maximal compact subgroup K_{Ph}). This boundary component F was associated (in §2.3) to the maximal (proper, rational) parabolic subgroup $\mathbf{Q}$ whose Levi component L_Q decomposes as $L_Q = L_{Qh} \times L_{Q\ell}$ with $L_{Qh} = L_{Ph}$. In other words, L_P and L_Q have the same Hermitian factor.

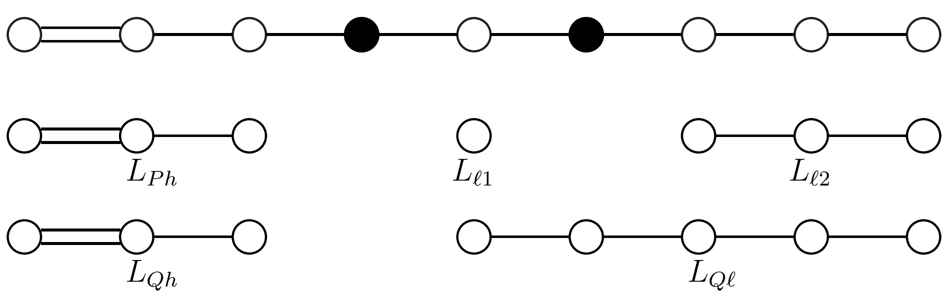

FIGURE 7. Dynkin diagrams for G, L_P, and L_Q

Moreover, the closure $\overline{D_P} = \overline{D_P}^{\mathrm{RBS}}$ decomposes as the product of reductive Borel-Serre partial compactifications of the locally symmetric spaces corresponding to the factors in (5.7.1). The symmetric spaces for the linear factors $L_{\ell 1} \times \ldots \times L_{\ell m}$ show up as boundary components of the symmetric space for $L_{Q\ell}$. With a bit more work it can be shown that

5.8. THEOREM. *The mapping* $\tau : \overline{X}^{RBS} \to \overline{X}^{BB}$ *takes strata to strata and it is a submersion on each stratum. If* $X_F \subset \overline{X}^{BB}$ *is the stratum corresponding to a maximal parabolic subgroup* $\mathbf{Q} \subset \mathbf{G}$ *then* $\tau^{-1}(X_F) \to X_F$ *is a fiber bundle whose fiber is isomorphic, by a stratum preserving isomorphism, to the reductive Borel-Serre compactification of the arithmetic quotient* $\Gamma_{Q\ell}\backslash L_{Q\ell}/K_{Q\ell}$ *of the self adjoint homogeneous cone* $L_{Q\ell}/K_{Q\ell}$.

(Here, $\Gamma_{Q\ell} \subset L_{Q\ell}(\mathbb{Q})$ is the arithmetic group which is obtained by first projecting $\Gamma \cap Q$ to L_Q and then intersecting with $L_{Q\ell}$.)

In summary we have a diagram of partial compactifications and compactifications,

$$\begin{array}{ccccc} \overline{D}^{\mathrm{BS}} & \longrightarrow & \overline{D}^{\mathrm{RBS}} & \longrightarrow & \overline{D}^{\mathrm{BB}} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{X}^{\mathrm{BS}} & \longrightarrow & \overline{X}^{\mathrm{RBS}} & \longrightarrow & \overline{X}^{\mathrm{BB}} \end{array}$$

with corresponding boundary components and boundary strata

$$\begin{array}{ccccc} e_P = P/K_PA_P & \longrightarrow & D_P = L_P/K_PA_P & \longrightarrow & F = L_{Ph}/K_{Ph} \\ \downarrow & & \downarrow & & \downarrow \\ Y_P = \Gamma_P\backslash e_P & \longrightarrow & X_P = \Gamma_L\backslash M_P/K_P & \longrightarrow & X_F = \Gamma_{Ph}\backslash F \end{array}$$

5.9. A very similar picture applies when $X = \Gamma\backslash\mathcal{P}_n$ is an arithmetic quotient of the symmetric cone of positive definite real matrices (or, more generally, when X is an arithmetic quotient of any rationally defined $\mathbb{Q}$ irreducible self adjoint homogeneous cone). The identity mapping $X \to X$ has unique continuous extensions

$$\overline{X}^{\mathrm{BS}} \longrightarrow \overline{X}^{\mathrm{RBS}} \xrightarrow{\ \tau\ } \overline{X}^{\mathrm{std}}$$

which take strata to strata. A stratum X_F of $\overline{X}^{\mathrm{std}}$ corresponds to a maximal rational parabolic subgroup $\mathbf{Q}$ whose Levi component factors, $L_Q = L_1L_2$ as a product of two "linear" factors. The stratum X_F is an arithmetic quotient of the self adjoint homogeneous cone for L_1. The pre-image of $\tau^{-1}(X_F)$ is a fiber bundle over X_F whose fiber over a point $x \in X_F$ is isomorphic to the reductive Borel-Serre compactification of an arithmetic quotient of the self adjoint homogeneous cone for L_2.

6. Cohomology

6.1. Group cohomology. (see [**Bro**] Chapt. I or [**W**] Chapt. 6) As in the preceding section we assume that $\mathbf{G}$ is reductive, defined over $\mathbb{Q}$, that $\mathbf{A_G}$ is trivial, that $K \subset \mathbf{G}(\mathbb{R})$ is a maximal compact subgroup and that the symmetric space $D = G/K$ is Hermitian with basepoint x_0. Fix a neat arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ and set $X = \Gamma\backslash D$. Let $\lambda : G \to \mathbf{GL}(E)$ be a finite dimensional representation of G on some complex vector space, and let $\mathbf{E} = E \times_\Gamma D$ be the resulting local system (flat vector bundle) on X. Since Γ acts freely on the contractible manifold D we see that the cohomology $H^i(X, \mathbf{E})$ is naturally isomorphic to the group cohomology $H^i(\Gamma, E)$ of the representation $\lambda|\Gamma$.

6.2. Theorem. *The cohomology $H^*(\Gamma, E)$ is finite dimensional. The group Γ is finitely presented.*

If Γ is neat, the proof follows just from the existence of the Borel-Serre compactification: the inclusion $X \to \overline{X}^{\mathrm{BS}}$ is a homotopy equivalence. Since Γ is the fundamental group of X, it is finitely presented. Moreover any compact manifold with boundary (or compact manifold with corners) may be triangulated using finitely many simplices, so its cohomology is finite dimensional (and vanishes in dimensions greater than $\dim(X)$). In fact, these two consequences of the existence of the Borel-Serre compactification were first proven by M. S. Raghunathan [**R**], who showed that X was diffeomorphic to the interior of a smooth compact manifold with boundary.

It can also be shown, using the Borel-Serre compactification, that the Euler characteristic $\chi(X)$ and the Euler characteristic with compact supports $\chi_c(X) = \sum_{i\geq 0}(-1)^i \dim(H^i_c(X))$ are equal. This follows from the fact that their difference is the Euler characteristic of the boundary $\partial\overline{X} = \overline{X} - X$ for any compactification $\overline{X}$ of X. One checks by induction that the Euler characteristic of the Borel-Serre boundary vanishes, since each "corner" Y_P is fibered over X_P with fiber a compact nilmanifold N_P, whose Euler characteristic $\chi(N_P) = 0$ vanishes.

6.3. L^2 cohomology. A choice of K-invariant inner product on the tangent space $T_{x_0}D$ determines a complete G-invariant Riemannian metric on D which then passes to a complete Riemannian metric (with negative curvature) on X. Let $\Omega^i(X)$ be the vector space of smooth complex valued differential i-forms and let

$$\Omega^i_{(2)}(X) = \left\{ \omega \in \Omega^i(X) \,\middle|\, \int \omega \wedge *\omega < \infty, \int d\omega \wedge *d\omega < \infty \right\}$$

be the vector space of L^2 differential i-forms on X. These form a complex whose cohomology

$$H^i_{(2)}(X) = \ker(d)/\mathrm{im(d)}$$

is called the L^2 cohomology of X. It is finite dimensional (when D is Hermitian symmetric, which we are currently assuming). We may similarly define the L^2 cohomology $H^i(X, \mathbf{E})$ with coefficients in a local system $\mathbf{E}$ arising from a finite dimensional irreducible representation $\lambda : \mathbf{G}(\mathbb{R}) \to \mathbf{GL}(E)$ on some complex vector space E.

The L^2 cohomology is a representation-theoretic object, and it may be identified ([**BW**]) with the relative Lie algebra cohomology

$$H^i_{(2)}(X, \mathbf{E}) \cong H^i(\mathfrak{g}, K; L^2(\Gamma\backslash G, E)) \tag{6.3.1}$$

of the module of L^2 functions on $\Gamma\backslash G$ with values in E. One would like to understand the decomposition of this module under the regular representation of G. (See lectures of J. Arthur in this volume.) This decomposition of $L^2(\Gamma\backslash G)$ is reflected in the resulting decomposition of its cohomology (6.3.1), which is somewhat easier to understand, but the information flows in both directions. For example, it is known (when $\mathbf{E}$ is trivial) that the trivial representation occurs exactly once in $L^2(\Gamma\backslash G)$, and that its $(\mathfrak{g}, K)$ cohomology coincides with the ordinary cohomology of the compact dual symmetric space $\check{D}$. Hence, $H^*(\check{D})$ occurs in $H^*_{(2)}(X, \mathbb{C})$.

6.4. Zucker conjecture. In [**Z1**], S. Zucker conjectured there is an isomorphism

$$H^i_{(2)}(X, \mathbf{E}) \cong IH^i(\overline{X}^{\mathrm{BB}}, \mathbf{E})$$

between the L^2 cohomology and the intersection cohomology of the Baily-Borel compactification. This beautiful conjecture relates an analytic and representation theoretic object, the L^2 cohomology, with a topological invariant, the intersection cohomology. (An analogous result, for "metrically conical" singular spaces, had been previously discovered [**Ch0, Ch1, Ch2**] by J. Cheeger. A relatively simple, piecewise linear analog is developed in [**BGM**].) Moreover, if X is (replaced by) a Shimura variety, then it has a canonical model defined over a number field and there is an associated variety defined over various finite fields. In this case the intersection cohomology of $\overline{X}^{\mathrm{BB}}$ has an étale version, on which a certain Galois group acts. So the Zucker conjecture provides a "path" from automorphic representations to Galois representations, the understanding of which constitutes one of the goals of Langlands' program.

Zucker proved the conjecture in the case of $\mathbb{Q}$-rank one. Further special cases were proven by Borel, Casselman, and Zucker ([**B2**], [**BC1**], [**BC2**], [**Z3**]). Finally, in [**Lo**] and [**SS**] the conjecture was proven in full generality. Looijenga's proof uses the decomposition theorem ([**BBD**] thm. 6.2.5) and the toroidal compactification, while the proof of Saper and Stern uses analysis (essentially on the reductive Borel-Serre compactification). Among the many survey articles on this material we mention [**B2**], [**CGM**], [**Go**], and [**S4**].

Both the L^2 cohomology and the intersection cohomology are the (hyper) cohomology groups of complexes of sheaves, $\mathbf{\Omega}^\bullet_{(2)}(\overline{X}^{\mathrm{BB}}, \mathbf{E})$ and $\mathbf{I\Omega}^\bullet(\overline{X}^{\mathrm{BB}}, \mathbf{E})$ respectively. The proofs of Looijenga and Saper and Stern construct a quasi-isomorphism between these complexes of sheaves. This implies, for example, the existence of an isomorphism between the L^2 cohomology and the intersection cohomology of any open set $V \subset \overline{X}^{\mathrm{BB}}$, and these isomorphisms are compatible with the maps induced by inclusion of open sets, exact sequences of pairs, and Mayer Vietoris sequences.

6.5. Review of sheaf theory. Let Z be a stratified space with a regular stratification and let $\mathbf{S}$ be a sheaf (of finite dimensional vector spaces over some field) on Z. The stalk of the sheaf $\mathbf{S}$ at the point $x \in Z$ is denoted $\mathbf{S}_x$. A *local system* on Z is a locally trivial sheaf. Denote by $\Gamma(U, \mathbf{S})$ the sections of $\mathbf{S}$ over an open set $U \subset Z$. The sheaf $\mathbf{S}$ is *fine* if it admits partitions of unity. (That is, for any locally finite cover $\{U_\alpha\}$ of Z, and for any open $V \subset Z$, every section $\omega \in \Gamma(V, \mathbf{S})$ can be written as a sum of sections ω_α supported in $U_\alpha \cap V$.) If $f : Z \to W$ is a continuous mapping and if $\mathbf{S}$ is a sheaf on Z then its push forward $f_*(\mathbf{S})$ is the sheaf on W whose sections over an open set are

$$\Gamma(U, f_*(\mathbf{S})) = \Gamma(f^{-1}(U), \mathbf{S}).$$

Let

$$\mathbf{S}^0 \xrightarrow{d_0} \mathbf{S}^1 \xrightarrow{d_1} \cdots$$

be a complex of sheaves (of vector spaces) on Z which is bounded from below. Such a complex is denoted $\mathbf{S}^\bullet$ rather than $\mathbf{S}^*$ to indicate that it comes with a differential. It is common to write $\mathbf{S}[k]^\bullet$ or $\mathbf{S}^\bullet[k]$ for the shifted complex, $\mathbf{S}[k]^i = \mathbf{S}^{k+i}$.

If $\mathbf{S}^\bullet$ is a complex of sheaves on Z its *stalk cohomology* $H^i_x(\mathbf{S}^\bullet)$ and *stalk cohomology with compact supports* $H^i_{c,x}(\mathbf{S}^\bullet) = H^i_{\{x\}}(\mathbf{S}^\bullet)$ at a point $x \in X_F \subset \overline{X}^{\mathrm{BB}}$ are the limits

$$\varinjlim H^i(U_x, \mathbf{S}^\bullet) \text{ and } \varprojlim H^i_c(U_x, \mathbf{S}^\bullet) \tag{6.5.1}$$

respectively, over a basis of neighborhoods $U_x \subset \overline{X}^{\rm BB}$ of x (ordered with respect to containment $U_x \supset U'_x \supset \cdots$). Sheaves form an abelian category so $\ker(d_i)$ and $\mathrm{Im}(d_{i-1})$ are sheaves, and we may form the *cohomology sheaf* $\mathbf{H}^i(\mathbf{S}) = \ker(d_i)/\mathrm{Im}(d_{i-1})$. Its stalk at a point $x \in Z$ is the stalk cohomology $H^i_x(\mathbf{S}^\bullet)$. The complex of sheaves $\mathbf{S}^\bullet$ is *cohomologically constructible* with respect to the stratification if, for each i, the restriction of the cohomology sheaf $\mathbf{H}^i(\mathbf{S}^\bullet)$ to each stratum is finite dimensional and locally trivial. This implies that $H^i(Z, \mathbf{S}^\bullet)$ is finite dimensional provided Z is compact.

A morphism $f : \mathbf{S}^\bullet \to \mathbf{T}^\bullet$ of complexes of sheaves is a *quasi-isomorphism* if it induces isomorphisms $H^i_x(\mathbf{S}^\bullet) \cong H^i_x(\mathbf{T}^\bullet)$ for every i and for every $x \in Z$. Such a quasi-isomorphism $\mathbf{S}^\bullet \to \mathbf{T}^\bullet$ induces isomorphisms $H^i(U, \mathbf{S}^\bullet) \cong H^i(U, \mathbf{T}^\bullet)$ for any open set $U \subseteq Z$ and these isomorphisms are compatible with the maps induced by inclusions and with Mayer Vietoris sequences.

If $\mathbf{S}^\bullet$ is a complex of fine sheaves, then for any open set $U \subseteq Z$ the cohomology $H^i(U, \mathbf{S}^\bullet)$ is the cohomology of the complex of sections over U,

$$\to \Gamma(U, \mathbf{S}^{i-1}) \to \Gamma(U, \mathbf{S}^i) \to \Gamma(U, \mathbf{S}^{i+1}) \to$$

However if $\mathbf{S}^\bullet$ is not fine, then this procedure gives the wrong answer. (Take, for example, the constant sheaf on a smooth manifold.) A *fine resolution* of $\mathbf{S}^\bullet$ is a quasi-isomorphism $\mathbf{S}^\bullet \to \mathbf{T}^\bullet$ where $\mathbf{T}^\bullet$ is fine. Then, in general, the cohomology $H^i(U, \mathbf{S}^\bullet)$ is defined to be the cohomology $H^i(U, \mathbf{T}^\bullet)$ for any fine (or flabby, or injective) resolution $\mathbf{T}^\bullet$ of $\mathbf{S}^\bullet$.

A similar problem arises when $f : Z \to W$ is a continuous mapping: if $\mathbf{S}^\bullet$ is a complex of fine sheaves on Z then the push forward $f_*(\mathbf{S}^\bullet)$ will satisfy

$$H^i(U, f_*(\mathbf{S}^\bullet)) \cong H^i(f^{-1}(U), \mathbf{S}^\bullet) \tag{6.5.2}$$

for any open set $U \subseteq W$. However if $\mathbf{S}^\bullet$ is not fine then (6.5.2) may fail, and $\mathbf{S}^\bullet$ should first be replaced by a fine (or flabby or injective) resolution before pushing forward. The resulting complex of sheaves (or rather, its quasi-isomorphism class) is denoted $Rf_*(\mathbf{S}^\bullet)$.

These apparently awkward constructions have their most natural expression in terms of the *derived category* of sheaves on Z, for which many excellent references exist. (See [**I**], [**GeM**], [**GeM2**]). Brief summaries are given in [**GM**], [**B5**].) However, the sheaves to be studied in the following sections will be fine, so no further resolutions are required.

Originally it was felt that the "dual" of a sheaf (or of a complex of sheaves) should be a co-sheaf (an object similar to a sheaf, but for which the restriction arrows are reversed). However, in [**BM**], Borel and Moore constructed the *dual sheaf* $\mathbf{T}^\bullet$ of a complex of sheaves $\mathbf{S}^\bullet$ on Z. They showed, for any open set $U \subset Z$, that $H^i_c(U, \mathbf{T}^\bullet)$ is the vector space dual of $H^i(U, \mathbf{S}^\bullet)$. In [**V**], Verdier showed there was a sheaf $\mathbf{D}^\bullet$ (called the dualizing sheaf) such that the Borel-Moore dual $\mathbf{T}^\bullet$ was quasi-isomorphic to the sheaf $\mathbf{Hom}^\bullet(\mathbf{S}^\bullet, \mathbf{D}^\bullet)$. In particular, the dual of the dual of $\mathbf{S}^\bullet$ is not equal to $\mathbf{S}^\bullet$, however it is quasi-isomorphic to $\mathbf{S}^\bullet$. There are many quasi-isomorphic models for the dualizing sheaf. Possible models include the sheaf of (singular) chains on Z (or piecewise linear chains, or subanalytic chains, if Z has a piecewise linear or subanalytic structure). If Z is compact, orientable (meaning that the top stratum of Z is orientable), and purely n-dimensional, then $H^i(Z, \mathbf{D}^\bullet)$ is the homology $H_{n-i}(Z)$.

6.6. The L^2 sheaf. Return to the situation of §6, with $X = \Gamma\backslash G/K$ a Hermitian locally symmetric space. The sheaf $\mathbf{\Omega}^i_{(2)}(\overline{X}^{\text{BB}}, \mathbf{E})$ of (smooth) L^2 differential forms on $\overline{X}^{\text{BB}}$ is defined to be the sheafification of the presheaf whose sections over an open set $U \subset \overline{X}^{\text{BB}}$ are

$$\left\{ \omega \in \Omega^i(U \cap X, E) \,\middle|\, \int_{U\cap X} \omega \wedge *\omega < \infty \text{ and } \int_{U \cap X} d\omega \wedge *d\omega < \infty \right\}$$

A common mistake is to confuse this with the direct image

$$j_* \mathbf{\Omega}^i_{(2)}(X, \mathbf{E})$$

of the sheaf of (smooth) L^2 $\mathbf{E}$-valued differential forms on X, where $j : X \to \overline{X}^{\text{BB}}$ is the inclusion. In fact the sheafification of the presheaf of smooth L^2 ($\mathbf{E}$-valued) differential forms on X is the sheaf of all smooth ($\mathbf{E}$-valued) differential forms on X. Its cohomology is the ordinary cohomology $H^*(X, \mathbf{E})$ and so the same is true of $j_*\mathbf{\Omega}^i_{(2)}(X, \mathbf{E})$.

6.7. Theorem ([**Z1**]). *The sheaf* $\mathbf{\Omega}^\bullet_{(2)}(\mathbf{E}) = \mathbf{\Omega}^\bullet_{(2)}(\overline{X}^{BB}, \mathbf{E})$ *of smooth* L^2 *differential forms on* $\overline{X}^{BB}$ *is fine.*

This implies that one may calculate the (hyper) cohomology of this complex of sheaves simply by taking the cohomology of the global sections (that is, globally defined L^2 differential forms), so we do indeed get the L^2 cohomology, that is,

$$H^i_{(2)}(X, \mathbf{E}) \cong H^i(\overline{X}^{\text{BB}}, \mathbf{\Omega}^\bullet_{(2)}(\mathbf{E})).$$

6.8. Middle Intersection cohomology. There is a construction of intersection cohomology using differential forms, which R. MacPherson and I worked out some years ago (see [**Bry**] and [**P**]). Let $\pi_F : V_F \to F$ be the geodesic projection of a parabolic neighborhood of a boundary stratum $X_F \subset \overline{X}^{\mathrm{BB}}$ of (complex) codimension c. Let us say that a smooth differential form $\omega \in \Omega^i(X, \mathbf{E})$ is *allowable near* X_F if there exists a neighborhood $V_\omega \subset V_F$ of X_F in $\overline{X}^{\mathrm{BB}}$ such that for any choice of c smooth vector fields $A_1, A_2, \ldots, A_c$ in $V_\omega \cap X$, each tangent to the fibers of π, the contractions

$$i(A_1)i(A_2)\cdots i(A_c)(\omega) = 0 \quad \text{and} \quad i(A_1)i(A_2)\cdots i(A_c)(d\omega) = 0$$

vanish in $V_\omega \cap X$. We say a smooth differential form $\omega \in \Omega^i(X, E)$ is *allowable* if it is allowable near X_F, for every stratum X_F of $\overline{X}^{\mathrm{BB}}$.

6.9. Definition. The sheaf $\mathbf{I\Omega^i}(\mathbf{E})$ on $\overline{X}^{\mathrm{BB}}$ is the sheafification of the presheaf whose sections over an open set $U \subset \overline{X}^{\mathrm{BB}}$ are

$$\left\{\omega \in \Omega^i(U \cap X, \mathbf{E}) \mid \omega \text{ is the restriction of an allowable form on } X\right\}.$$

This sheaf is fine, so its cohomology $IH^*(\overline{X}^{\mathrm{BB}}, \mathbf{E})$ coincides with the cohomology of the complex of allowable differential forms on X. Moreover its stalk cohomology and stalk cohomology with compact supports (6.5.1) are achieved in any *distinguished* neighborhood $V_x \subset \overline{X}^{\mathrm{BB}}$ (see §2.10 and §4.6), that is,

$$IH^i_x(\mathbf{E}) \cong IH^i(V_x, \mathbf{E}) \text{ and } IH^i_{\{x\}}(\mathbf{E}) \cong IH^i_c(V_x, \mathbf{E}).$$

The (stalk) cohomology is even the cohomology of the complex of allowable differential forms in V_x which satisfy the allowability condition with respect to X_F throughout the neighborhood V_x. (The corresponding statement for the stalk cohomology with compact supports is false.)

The complex of sheaves $\mathbf{I\Omega^\bullet}(\mathbf{E})$ has the following properties.

(1) It is constructible: its stalk cohomology (at any point) is finite dimensional, and its cohomology sheaves are locally trivial when restricted to any stratum.
(2) The restriction $\mathbf{I\Omega^\bullet}(\mathbf{E})|X$ is a fine resolution of the sheaf (of sections of) $\mathbf{E}$.
(3) If F is a stratum of complex codimension c then for any $x \in F$,

$$H^i_x(\mathbf{I\Omega^\bullet}(\mathbf{E})) = 0 \text{ for all } i \geq c$$
$$H^i_{c,x}(\mathbf{I\Omega^\bullet}(\mathbf{E})) = 0 \text{ for all } i \leq c.$$

Condition (3) says that the sheaf of differential forms has been *truncated by degree* at the stratum X_F, that is, the allowability condition has killed all the stalk cohomology of degree $\geq c$. In [**GM**] it is shown that any complex of sheaves $\mathbf{S}^\bullet$ satisfying these three conditions is quasi-isomorphic to the intersection complex, meaning that in the appropriate bounded constructible derived category $D^b_c(\overline{X}^{\mathrm{BB}})$ there is an isomorphism $\mathbf{S}^\bullet \cong \mathbf{I\Omega}^\bullet(\mathbf{E})$. So the proof of the Zucker conjecture amounts to checking that the sheaf of L^2 differential forms satisfies these conditions. Conditions (1) and (2) are easy, however checking condition (3), which is local in $\overline{X}^{\mathrm{BB}}$, involves a detailed understanding both of the local topology of $\overline{X}^{\mathrm{BB}}$ and of its metric structure.

The intersection cohomology sheaf is (Borel-Moore-Verdier) self dual. In particular, if E_1 and E_2 are dual finite dimensional representations of G then for each open set $U \subset \overline{X}^{\mathrm{BB}}$ the intersection cohomology vector spaces

$$IH^i(U, \mathbf{E_1}) \text{ and } IH_c^{2n-i}(U, \mathbf{E_2})$$

are dual, where $n = \dim_{\mathbb{C}}(X)$.

6.10. Remark. Condition (3) above says that $\mathbf{I\Omega}^\bullet$ is a perverse sheaf ([**BBD**]) on $\overline{X}^{\mathrm{BB}}$. In fact the simple objects in the category $\mathrm{Perv}_c(\overline{X}^{\mathrm{BB}})$ of (constructible) perverse sheaves are the just the intersection complexes $j_*(\mathbf{I\Omega}^\bullet(\overline{X}_F, \mathbf{E_F}))[c_F]$ of closures of strata, where $j : \overline{X}_F \to \overline{X}^{\mathrm{BB}}$ is the inclusion of the closure of a stratum X_F of codimension c_F and where $\mathbf{E_F}$ is a local coefficient system on X_F.

6.11. Weighted cohomology. If $f : Y \to Z$ is a morphism and if $\mathbf{S}^\bullet$ is a complex of fine sheaves on Y then $f_*(\mathbf{S}^\bullet)$ is a complex of fine sheaves on Z whose cohomology is the same: $H^i(Z, f_*(\mathbf{S}^\bullet)) \cong H^i(Y, \mathbf{S}^\bullet)$. So we can study the cohomology of $\mathbf{S}^\bullet$ locally on Z. However the converse is not always true: if $\mathbf{T}^\bullet$ is a complex of sheaves on Z, there does not necessarily exist a complex of sheaves $\mathbf{S}^\bullet$ on Y so that $f_*(\mathbf{S}^\bullet) \cong \mathbf{T}^\bullet$.

One would like to study the intersection cohomology $IH^*(\overline{X}^{\mathrm{BB}}, \mathbf{E})$ locally on the reductive Borel-Serre compactification, which is in many ways a simpler space than $\overline{X}^{\mathrm{BB}}$. One might hope to use the sheaf of L^2 differential forms on $\overline{X}^{\mathrm{RBS}}$, which again makes sense on the reductive Borel-Serre compactification. It is again a fine sheaf [**Z1**], and its cohomology is $H^*_{(2)}(X, \mathbf{E})$. However, the L^2 sheaf on $\overline{X}^{\mathrm{RBS}}$ is not constructible: its stalk cohomology at a boundary point x may be infinite dimensional. The weighted cohomology sheaf $\mathbf{WC}^\bullet(\overline{X}^{\mathrm{RBS}}, \mathbf{E})$ is designed to be a good replacement; see Theorem 6.14 below. The idea is the following. For any stratum $X_P \subset \overline{X}^{\mathrm{RBS}}$ the torus A_P

acts (by geodesic action) on any parabolic neighborhood V_P. This action should give rise to a decomposition of the stalk cohomology (of the sheaf $\mathbf{\Omega}^\bullet(\overline{X}^{\mathrm{RBS}}, \mathbf{E})$ of all smooth differential forms) at any point $x \in X_P$ into weight spaces. We would like to kill all the cohomology with weights greater than or equal to some fixed value, that is, we would like a *weight truncation* of the sheaf of smooth differential forms. Unfortunately, the complex of smooth differential forms is infinite dimensional, and the torus A_P does not act semi-simply (near X_P) on this complex. So it is first necessary to find an appropriate collection of differential forms with the same cohomology, which decomposes under the action of A_P. In [**GHM**] a subsheaf $\mathbf{\Omega}^\bullet_{\mathrm{sp}}(\overline{X}^{\mathrm{RBS}}, \mathbf{E})$ of smooth "special" differential forms is constructed with this property.

Assume $\mathbf{G}$ is reductive and $\mathbf{A_G}$ is trivial. Fix a standard minimal rational parabolic subgroup $\mathbf{P_0} \subset \mathbf{G}$. Let $\mathbf{A_0}$ be the greatest $\mathbb{Q}$ split torus in the center of $\mathbf{P_0}$. Fix a "weight profile" $\nu \in X^*_{\mathbb{Q}}(\mathbf{A_0})$, that is, a rational character of $\mathbf{A_0}$. This will be used to determine weight cutoffs for each stratum. Suppose P is a standard rational parabolic subgroup. The choice of basepoint $x_0 \in D$ determines a lift $L_P \subset P$ (see §2.10), so the action of P on its unipotent radical restricts to an action of $L_P \subset P$ on the complex

$$C^\bullet(\mathfrak{N}_P, E) = \mathrm{Hom}_{\mathbb{R}}(\wedge^\bullet(\mathfrak{N}_P), E)$$

(where $\mathfrak{N}_P = \mathrm{Lie}(U_P)$) and hence determines a local system

$$\mathbf{C}^\bullet(\mathfrak{N}_P, E) = C^\bullet(\mathfrak{N}_P, E) \times_{\Gamma_L} (L_P \backslash K_P A_P)$$

over the reductive Borel-Serre stratum $X_P = \Gamma_L \backslash L_P / K_P A_P$, cf. (5.1.2). The torus A_P acts on $C^\bullet(\mathfrak{N}_P, E)$ so we obtain a decomposition into weight submodules

$$C^\bullet(\mathfrak{N}_P, E) \cong \bigoplus_{\mu \in X(A_P)} C^\bullet(\mathfrak{N}_P, E)_\mu.$$

Using the weight profile ν, define the submodule

$$C^\bullet(\mathfrak{N}_P, E)_{\geq \nu} = \bigoplus_{\mu \geq \nu} C^\bullet(\mathfrak{N}_P, E)_\nu$$

where $\mu \geq \nu$ means that $\mu - (\nu | A_P)$ lies in the positive cone spanned by the simple rational roots $\alpha \in \Delta_P$. This definition also makes sense when $\mathbf{P}$ is an arbitrary rational parabolic subgroup, by conjugation.

Suppose $V \subset \overline{X}^{\mathrm{RBS}}$ is a parabolic neighborhood of X_P. Then it turns out that the complex of differential forms which are special throughout V may be identified with the complex $\Omega^\bullet_{sp}(X_P, \mathbf{C}^\bullet(\mathfrak{N}_P, E))$

of special differential forms on X_P with coefficients in the (finite dimensional) local system $\mathbf{C}^\bullet(\mathfrak{N}_P, E)$. Define $\Omega^\bullet_{sp}(V)_{\geq\nu}$ to be the subcomplex of special differential forms on X_P with coefficients in the subbundle $\mathbf{C}^\bullet(\mathfrak{N}_P, E)_{\geq\nu}$. The subcomplex of $\Omega^\bullet_{sp}(V)$ is independent of the choice of basepoint.

6.12. Definition. The weighted cohomology $\mathbf{W}^{\geq\nu}\mathbf{C}^\bullet(\overline{X}^{\mathrm{RBS}}, \mathbf{E})$ is the sheafification of the complex of sheaves whose sections over an open set $U \subset \overline{X}^{\mathrm{RBS}}$ consist of smooth differential forms ω on $U \cap X$ such that for each stratum X_P there exists a parabolic neighborhood $V = V(\omega, X_P) \subset \overline{X}^{\mathrm{RBS}}$ with $\omega | V \in \Omega^\bullet_{sp}(V)_{\geq\nu}$.

It is possible to similarly define $\mathbf{W}^{>\nu}\mathbf{C}^\bullet(\overline{X}^{\mathrm{RBS}}, \mathbf{E})$. It will coincide with the sheaf $\mathbf{W}^{\geq\nu}\mathbf{C}^\bullet(\overline{X}^{\mathrm{RBS}}, E)$ if, for each rational parabolic subgroup $\mathbf{P}$, the weight $\nu | A_P$ does not occur in any of the cohomology groups $H^i(\mathfrak{N}_P, E)$.

6.13. Theorem. *The complex $\mathbf{W}^{\geq\nu}\mathbf{C}^\bullet$ is constructible with respect to the canonical stratification of $\overline{X}^{RBS}$, so its cohomology is finite dimensional. Its restriction to X is a fine resolution of the sheaf (of sections of) $\mathbf{E}$. The stalk cohomology, and compactly supported stalk cohomology at a point $x \in X_P$ are given by*

$$WH^j_x \cong H^j(\mathfrak{N}_P, E)_{\geq\nu}$$
$$WH^j_{c,x} \cong H^{j-d-s}(\mathfrak{N}_P, E)_{<\nu}$$

where $s = \dim(A_P)$ and $d = \dim(X_P)$.

It is possible that these conditions uniquely determine the weighted cohomology sheaf in the bounded constructible derived category of $\overline{X}^{\mathrm{RBS}}$. In any case this theorem is considerably more complete than the corresponding result in §6.8 for intersection cohomology (which only specifies the region in which these stalk cohomology groups vanish). This illustrates the fact that the reductive Borel-Serre compactification is easier to understand than the Baily-Borel compactification.

There are many possible weight truncations. The two extreme truncations ($\nu = -\infty$ and $\nu = \infty$) give rise to a weighted cohomology sheaf on $\overline{X}^{\mathrm{RBS}}$ whose cohomology is the ordinary cohomology $H^*(X, \mathbf{E})$ and the ordinary cohomology with compact supports $H^*_c(X, \mathbf{E})$ of X, respectively. Another weight truncation $\nu = 0$ (and $\mathbf{E}$ trivial) gives the ordinary cohomology $H^*(\overline{X}^{\mathrm{RBS}}, \mathbb{C})$.

If E_1 and E_2 are dual (finite dimensional) irreducible representations of G and if $\mu + \nu = -2\rho$ then the weighted cohomology complexes

$\mathbf{W}^{\geq\nu}\mathbf{C}^{\bullet}(\overline{X}^{\mathrm{RBS}}, \mathbf{E_1})$ and $\mathbf{W}^{>\mu}\mathbf{C}^{\bullet}(\overline{X}^{\mathrm{RBS}}, \mathbf{E_2})$ are (Verdier) dual sheaves. (Here ρ is one-half the sum of the positive roots.) In particular, for any open set $U \subseteq \overline{X}^{\mathrm{RBS}}$ the cohomology groups

$$W^{\geq\nu}H^i(U, \mathbf{E_1}) \text{ and } W^{>\mu}H_c^{n-i}(U, \mathbf{E_2})$$

are dual vector spaces (where $n = \dim(X)$). Thus, taking $m = -\rho$ there are two "middle" weighted cohomology sheaves (which may coincide),

$$\mathbf{W}^{\geq m}\mathbf{C}^{\bullet}(\overline{E}^{\mathrm{RBS}}) \text{ and } \mathbf{W}^{>m}\mathbf{C}^{\bullet}(\overline{X}^{\mathrm{RBS}}, \mathbf{E}).$$

The weighted cohomology construction makes sense whether or not D is Hermitian. But in the Hermitian case we also have the mapping $\tau : \overline{X}^{\mathrm{RBS}} \to \overline{X}^{\mathrm{BB}}$ of §5.6. Let E be an irreducible finite dimensional representation of G.

6.14. Theorem. ([**GHM**] *Theorem 23.2*) *The above mapping τ induces quasi-isomorphisms*

$$\tau_*(\mathbf{W}^{\geq m}\mathbf{C}^{\bullet}(\overline{X}^{RBS}, \mathbf{E})) \cong \tau_*(\mathbf{W}^{>m}\mathbf{C}^{\bullet}(\overline{X}^{RBS}, \mathbf{E})) \cong \mathbf{I\Omega}^{\bullet}(\overline{X}^{BB}, \mathbf{E})$$

and in particular the weighted cohomology of $\overline{X}^{RBS}$ is canonically isomorphic to the intersection cohomology of $\overline{X}^{BB}$.

6.15. Hecke correspondences. Any $g \in \mathbf{G}(\mathbb{Q})$ gives rise to a *Hecke correspondence* $X' \rightrightarrows X$, meaning that we have two finite surjective mappings c_1, c_2 from X' to X. It is defined as follows. Let $\Gamma' = \Gamma \cap g^{-1}\Gamma g$, $X' = \Gamma' \backslash D$. The two mappings are: $\Gamma' hK \mapsto (\Gamma hK, \Gamma ghK)$. They give an immersion $X' \to X \times X$ whose image may be thought of as a multi-valued mapping $X \to X$. The Hecke correspondence defined by any $g' \in \Gamma g \Gamma$ is the same as that defined by g (cf. [**GM2**] §6.6). The composition of Hecke correspondences defined by $g, g' \in \mathbf{G}(\mathbb{Q})$ is not the Hecke correspondence defined by gg', but rather, it is a finite linear combination of Hecke correspondences (cf. [**Sh**] §3.1). So the set of finite formal linear combinations of Hecke correspondences form a ring, the *Hecke ring* or Hecke algebra of Γ.

Fix a Hecke correspondence $(c_1, c_2) : X' \rightrightarrows X$. Differential forms on X may be pulled back by c_2 then pushed forward by c_1, and L^2 forms are taken to L^2 forms by this procedure. The induced mapping on $H^i_{(2)}(X, \mathbf{E})$ is called a Hecke operator. Using the trace formula, J. Arthur ([**A**]) gave an expression for the Lefschetz number of this operator, that is, the alternating sum of the traces of the induced mapping on the L^2 cohomology.

Both mappings $(c_1, c_2) : X' \rightrightarrows X$ extend to finite mappings

$$\overline{X'}^{\mathrm{RBS}} \rightrightarrows \overline{X}^{\mathrm{RBS}} \text{ and } \overline{X'}^{\mathrm{BB}} \rightrightarrows \overline{X}^{\mathrm{BB}}.$$

A *fixed point* $x \in \overline{X'}^{\mathrm{RBS}}$ is a point such that $c_1(x) = c_2(x)$. In [**GM2**] the Lefschetz fixed point formula for the action of this Hecke correspondence on the weighted cohomology of $\overline{X}^{\mathrm{RBS}}$ was computed. In [**GKM**] it was shown how the contributions from individual fixed point components in $\overline{X}^{\mathrm{RBS}}$ may be grouped together so as to make the Lefschetz formula (for the middle weighted cohomology) agree, term by term, with the L^2 Lefschetz formula of Arthur. This gives a purely topological interpretation (and re-proof) of Arthur's formula, as well as similar formulas for other weighted cohomology groups.

7. A selection of further developments

Throughout this section we assume that $X = \Gamma\backslash G/K$ is a Hermitian locally symmetric space, with $\mathbf{G}$ semi-simple, as in the preceding section.

7.1. Let $\widehat{X}$ be the closure of X in $\overline{X}_\Sigma^{\mathrm{tor}} \times \overline{X}^{\mathrm{RBS}}$. In [**GT1**] it is shown that the fibers of the mapping $\pi_1 : \widehat{X} \to \overline{X}_\Sigma^{\mathrm{tor}}$ are contractible, so there exists a homotopy inverse $\mu : \overline{X}_\Sigma^{\mathrm{tor}} \to \widehat{X}$, that is, $\pi_1\mu$ and $\mu\pi_1$ are homotopic to the identity. The composition $\overline{X}_\Sigma^{\mathrm{tor}} \to \widehat{X} \to \overline{X}^{\mathrm{RBS}}$ allows one to compare the cohomology of $\overline{X}^{\mathrm{RBS}}$ and $\overline{X}_\Sigma^{\mathrm{tor}}$.

7.2. In [**GHMN**] it is shown that the restriction of the weighted cohomology sheaf to the closure of any stratum of $\overline{X}^{\mathrm{RBS}}$ decomposes as a direct sum of weighted cohomology sheaves for that stratum. (The analogous statement for intersection cohomology is false.)

7.3. In [**Z4**], S. Zucker showed that for large p, the L^p cohomology of $X = \Gamma\backslash G/K$ is naturally isomorphic to the ordinary cohomology $H^*(\overline{X}^{\mathrm{RBS}})$ of the reductive Borel-Serre compactification. Although this result is much easier to prove than the original Zucker conjecture, it went surprisingly unnoticed for twenty years. In [**Nr**], A. Nair showed that the weighted cohomology $WH^*(\overline{X}^{\mathrm{RBS}}, \mathbf{E})$ is canonically isomorphic to the weighted L^2 cohomology of J. Franke [**Fr**]. In [**S1**], [**S2**], L. Saper showed that the push forward $\tau_*(\mathbf{I\Omega}^\bullet(\overline{X}^{\mathrm{RBS}}, \mathbf{E}))$ is canonically isomorphic to $\mathbf{I\Omega}^\bullet(\overline{X}^{\mathrm{BB}}, \mathbf{E})$. This gives the surprising result that

$$IH^i(\overline{X}^{\mathrm{RBS}}, \mathbf{E}) \cong IH^i(\overline{X}^{\mathrm{BB}}, \mathbf{E}) \cong W^{\geq m}H^i(\overline{X}^{\mathrm{RBS}}, \mathbf{E}).$$

However, on $\overline{X}^{\mathrm{RBS}}$, the weighted cohomology sheaf and sheaf of intersection forms are definitely not quasi-isomorphic: Saper's theorem depends on delicate global vanishing results for the weighted cohomology groups of various boundary strata.

7.4. Many other compactifications of $\Gamma\backslash G/K$ were constructed by Satake ([**Sa2**], [**Z2**]). Each Satake compactifications depend on a choice of (what is now called) a "geometrically rational" representation of G. If the representation is rational, then it is geometrically rational, however the Baily-Borel compactification arises from a geometrically rational representation that is not necessarily rational. So the issue of determining which representations are geometrically rational is quite subtle. See [**Ca**] and [**S3**] for more details.

7.5. The most successful method for computing the L^2 cohomology involves understanding relative Lie algebra cohomology and automorphic representations. See, for example, [**Ko, LS, Sch, BW**].

References

[A] J. Arthur, The L^2 Lefschetz numbers of Hecke operators, Invent. Math. **97** (1989), pp. 257–290.

[AMRT] A. Ash, D. Mumford, M. Rapoport, Y. -S. Tai, **Smooth Compactification of Locally Symmetric Varieties**, Math. Sci. Press, Brookline MA, 1975.

[BB] W. Baily and A. Borel, Compactification of arithmetic quotients of bounded symmetric domains. Ann. Math. **84** (1966), pp. 442-528.

[BBD] A. Beilinson, J. Bernstein and P. Deligne, **Faisceaux Pervers**, Astérisque **100**, Soc. Mat. de France, Paris, 1982.

[B1] A. Borel, **Introduction aux Groupes Arithmétiques**, Hermann, Paris, 1969.

[B2] A. Borel, L^2 cohomology and intersection cohomology of certain arithmetic varieties, in **Emmy Noether in Bryn Mawr**, Springer Verlag, New York, 1983, pp. 119–131.

[B3] A. Borel, Introduction to automorphic forms, in **Arithmetic Groups**, Proc. Symp. Pure Math. **IX**, Amer. Math. Soc., Providence R.I., 1966.

[B4] A. Borel, Cohomology and spectrum of an arithmetic group, **Proceedings of a Conference on Operator Algebras and Group Representations, Neptun, Romania**, Pitman, New York, 1983, p. 28–45.

[B5] A. Borel et al, **Intersection Cohomology**, Progress in Mathematics **50**, Birkhäuser Boston, Boston MA, 1984.

[BC1] A. Borel and W. Casselman, L^2 cohomology of locally symmetric manifolds of finite volume, Duke Math. J. **50** (1983), pp. 625–647.

[BC2] A. Borel and W. Casselman, Cohomologie d'intersection et L_2-cohomologie de variétés arithmétiques de rang rationnel 2, C. R. Acad. Sci. Paris **301** Ser. 1 no. 7 (1985), pp. 369–373.

[BM] A. Borel and J. C. Moore, Homology theory for locally compact spaces, Mich. Math. J. **7** (1960), pp. 137–159.

[BS] A. Borel and J. P. Serre, Corners and arithmetic groups, Comment. Math. Helv. **48** (1973), pp. 436-491.

[BW] A. Borel and N. Wallach, **Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups**, Princeton Univ. Press, Princeton N.J., 1980.

[BGM] J. P. Brasselet, M. Goresky, and R. MacPherson, Simplicial differential forms with poles, Amer. J. Math., **113** (1991), pp. 1019–1052.

[Bro] K. Brown, **Cohomology of Groups**, Graduate Texts in Mathematics **87**, Springer Verlag, New York, 1982.

[Bry] J. -L. Brylinski, Equivariant intersection cohomology, in **Kazhdan-Lusztig Theory and Related Topics**, Contemp. Math. **139** (1992), Amer. Math. Soc., Providence R.I., 1992.

[Ca] W. Casselman, Geometric rationality of Satake compactifications, in **Algebraic Groups and Lie Groups**, G. Lehrer ed., Cambridge University Press, Cambridge, UK, 1997.

[Ch0] J. Cheeger, On the spectral geometry of spaces with cone-like singularities, Proc. Nat. Acad. Sci. **76** (1979), pp. 2103–2106.

[Ch1] J. Cheeger, On the Hodge theory of Riemannian pseudomanifolds, in **Geometry of the Laplace Operator**, Proc. Symp. Pure Math **36**, Amer. Math. Soc., Providence R.I., 1980, pp. 91–146.

[Ch2] J. Cheeger, Spectral geometry of singular Riemannian spaces, J. Diff. Geom. **18** (1983), pp. 575–657.

[CGM] J. Cheeger, M. Goresky, and R. MacPherson, L^2 cohomology and intersection homology for singular varieties, in **Seminar in Differential Geometry** (S. T. Yau, ed.), Princeton University Press, Princeton N.J., 1982.

[FC] G. Faltings and C.-L. Chai, **Degeneration of Abelian Varieties**, Ergeb. Math. **22**, Springer Verlag, Berlin, 1990.

[Fr] J. Franke, Harmonic analysis in weighted L_2 spaces, Ann. Sci. École Norm. Sup. **31** (1998), pp. 181–279.

[GeM] S. Gelfand and Y. Manin, **Algebra V: Homological Algebra**, Encyclopedia of Mathematical Sciences **38**, Springer Verlag, Berlin, 1994.

[GeM2] S. Gelfand and Y. Manin **Methods of Homological Algebra**, Springer Verlag, Berlin, 2002.

[Go] M. Goresky, L^2 cohomology is intersection cohomology, in **The Zeta Functions of Picard Modular Surfaces**, R. P. Langlands and D. Ramakrishnan, ed., Centre de Recherches Mathématiques, Montréal, 1992.

[GHM] M. Goresky, G. Harder, and R. MacPherson, Weighted cohomology, Invent. Math. **116** (1994), pp. 139–213.

[GHMN] M. Goresky, G. Harder, R. MacPherson, and A. Nair, Local intersection cohomology of Baily Borel compactifications, Comp. Math. **134** (2002), pp. 243–268.

[GKM] M. Goresky, R. Kottwitz, and R. MacPherson, Discrete series characters and the Lefschetz formula for Hecke operators, Duke Math. J. **89** (1997), pp.477–554.

[GM] M. Goresky and R. MacPherson, Intersection homology II, Invent. Math. **72** (1983), pp. 77–129.

[GM2] M. Goresky and R. MacPherson, The topological trace formula, Crelle's Journal, **560** (2003), 77-150. .

[GP] M. Goresky and W. Pardon, Chern classes of automorphic vector bundles, Invent. Math. **147** (2002), pp. 561–612.

[GT1] M. Goresky and Y.-S. Tai, Toroidal and reductive Borel-Serre compactifications of locally symmetric spaces, Amer. J. Math. **121** (1999), pp. 1095–1151.

[GT2] M. Goresky and Y.-S. Tai, The moduli space of real abelian varieties with level structure, Comp. Math. (to appear).

[I] B. Iverson, **Cohomology of Sheaves**, Universitext, Springer Verlag, Berlin, 1986.

[Kl] H. Klingen, **Introductory Lectures on Siegel Modular Forms**, Cambridge University Press, Cambridge UK, 1990.

[Ko] R. Kottwitz, Shimura varieties and λ-adic representations, in **Automorphic Forms, Shimura varieties, and L-functions**, vol I, Academic Press, New York, 1990, p. 161–209.

[LS] J. -S. Li and J. Schwermer, On the Eisenstein cohomology of arithmetic groups. Preprint, Univ. Wien, Austria;

[Lo] E. Looijenga, L^2 cohomology of locally symmetric varieties. Comp. Math. **67** (1988), pp. 3-20.

[M] D. Mumford, Hirzebruch's proportionality theorem in the non-compact case, Inv. Math. **42** (1977), pp. 239–272.

[Nr] A. Nair, Weighted cohomology of arithmetic groups, Ann. Math. **150** (1999), pp. 1–31.

[Nk] Y. Namikawa, **Toroidal Compactification of Siegel Spaces**, Lecture Notes in Mathematics **812**, Springer Verlag, Berlin, 1980.

[P] G. Pollini, Modello di de Rham per la coomologia d'intersezione, Tesi di Laurea, University of Pisa, 2003.

[R] M. S. Raghunathan, A note on quotients of real algebraic groups by arithmetic subgroups, Invent. Math. **4** (1968), pp. 318–355.

[S1] L. Saper, L-modules and the conjecture of Rapoport and Goresky-MacPherson, `[arxiv.org/math.RT/0211138]`

[S2] L. Saper, L-modules and micro-support, `[arxiv.org/math.RT/0112251]`.

[S3] L. Saper, Geometric rationality of equal-rank Satake compactifications, preprint, Duke Univ., 2003.

[S4] L. Saper, On the Cohomology of Locally Symmetric Spaces and of their Compactifications, to appear in the proceedings of the Harvard/M.I.T. conference **Current Developments in Mathematics, 2002**, held in November 2002 [arxiv.org/math.RT/0306403] .

[SS] L. Saper and M. Stern, L_2 cohomology of arithmetic varieties, Ann. of Math. **132** (1990), pp. 1-69.

[Sa1] I. Satake, **Algebraic Structures Of Symmetric Domains**, Princeton University Press, Princeton N.J., 1980.

[Sa2] I. Satake, On compactifications of the quotient spaces for arithmetically defined discontinuous groups, Ann. Math. **72** (1960), pp. 555-580.

[Sch] J. Schwermer, Cohomology of arithmetic groups, automorphic forms and L-functions, in **Cohomology of Arithmetic Groups and Automorphic Forms**, Lecture Notes in Mathematics **1447**, Springer Verlag, Berlin, 1990, pp. 1–30.

[Sh] G. Shimura, **Introduction to the Arithmetic Theory of Automorphic Functions**, Princeton University Press, Princeton N.J., 1971.

[V] J. L. Verdier, Dualité dans la cohomologie des espaces localment compactes. Sém. Bourbaki **300** (1965).

[W] C. Weibel, **An Introduction to Homological Algebra**, Cambridge studies in advanced Mathematics **38**, Cambridge University Press, Cambridge UK, 1994.

[Z1] S. Zucker, L_2 cohomology of warped products and arithmetic groups. Invent. Math. **70** (1982), pp. 169–218.

[Z2] S. Zucker, Satake compactifications, Comment. Math. Helv. **58** (1983), pp. 312-343.

[Z3] S. Zucker, L_2 cohomology and intersection homology of locally symmetric varieties II, Comp. Math. **59** (1986), pp. 339–398.

[Z4] S. Zucker, On the reductive Borel-Serre compactification: L^p cohomology of arithmetic groups (for large p), Amer. J. Math. **123** (2001), pp. 951–984.