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# WU NUMBERS OF SINGULAR SPACES

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## §1. INTRODUCTION

IN THIS paper we consider the following question: for how singular a space is it possible to define cobordism invariant characteristic numbers? This question was one of the motivations behind the development of intersection homology: there are many singular spaces (complex algebraic varieties, for example) for which Whitney [40], Chern [28], and L[18] classes can be defined, but these are homology classes and cannot necessarily be multiplied so as to give characteristic numbers. It was hoped ([18]) that these classes could, in certain cases, be lifted canonically to intersection homology groups where their products could be formed. Thus, as a space is allowed to become more and more singular, it should become possible to multiply fewer and fewer characteristic classes, and so the corresponding cobordism groups would be determined by fewer and fewer characteristic numbers. This approach has (so far) failed completely, except in the "extreme" cases where there is a single characteristic number (for example, in Sullivan's theory of mod 2 Euler spaces ([40], [1]), where the Euler characteristic is the only cobordism invariant, or in P. Siegel's theory of mod 2 Witt spaces ([37], [15]), where the intersection homology Euler characteristic is the only cobordism invariant. We will exhibit four interesting classes of singular spaces with increasingly severe singularities:

(d) Orientable $\bar{s}$-duality spaces
(c) Orientable locally square-free spaces
(b) Locally orientable Witt spaces
(a) Locally orientable spaces

for which various (cobordism invariant) characteristic numbers can be constructed, and for which these characteristic numbers completely determine the cobordism groups. However, we construct characteristic numbers by lifting Wu classes to intersection homology, and multiplying them, rather than lifting and multiplying Stiefel Whitney classes. (Of course, if the space is a manifold then the Wu numbers are just linear combinations of Stiefel Whitney numbers, but for the spaces considered here, these corresponding Stiefel Whitney numbers cannot be defined.) The geometrical techniques needed to determine the cobordism groups are quite different in each of these four cases (which accounts for the extraordinary length of this paper), but they all use surgery on singular spaces and they all involve an understanding of the geometric consequences of the vanishing of a Wu number.

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(a) *Locally orientable spaces.* A normal pseudomanifold $X$ is locally orientable if the link of every simplex in some (and hence in any) triangulation of $X$ is orientable. If $X$ is locally orientable then the orientation cycle $v^1(X)$ is naturally a cohomology class. Thus the product $(v^1)^n \in \mathbb{Z}/(2)$ can be defined and it is a cobordism invariant (for locally orientable cobordisms) which we call the *orientation number.*

THEOREM 9.3. *A locally orientable pseudomanifold $X$ is the boundary of a locally orientable pseudomanifold $Y$ iff the orientation number $(v^1(X))^n = 0$.*

(Our study of locally orientable spaces uses only techniques from P.L. topology which were available in 1934—see, e.g. [36] §24 for a discussion of the orientation cycle.)

(b) *Witt spaces and Locally orientable Witt spaces.* A stratified pseudomanifold $X$ is a (mod 2) — Witt space ([37]) if, for each stratum of odd codimension $c = 2k + 1$, we have

$$IH_k^{\bar{m}}(L; \mathbb{Z}/(2)) = 0$$

where $L$ is the link of the stratum. For such spaces, the middle intersection homology groups satisfy Poincaré duality (over $\mathbb{Z}/(2)$), and the Wu classes can be constructed ([15]) as middle intersection homology classes. The intersection homology Euler characteristic

$$I\mathscr{X}(X; \mathbb{Z}/(2)) = v^k(X) . v^k(X)$$

is the only cobordism invariant ([37]). If an $n$-dimensional Witt space $X$ is also locally orientable, then it is also possible to construct the characteristic numbers $v^i(X)v^j(X)(v^1(X))^{n-i-j}$. In this way, Wu classes in cohomology are multiplied with Wu classes in intersection homology in order to produce characteristic numbers.

THEOREM 10.5. *A locally orientable Witt space $X$ of dimension $n$ is a boundary of a locally orientable Witt space $Y$ iff each of the characteristic numbers $v^i v^j (v^1)^{n-i-j}$ vanish.*

(c) *Locally square-free spaces.* The nontrivial cobordism invariant characteristic numbers considered above are somehow artificial since they are simple combinations of cohomology classes and the intersection homology Euler characteristic. The main goal of this paper is to find a class of spaces with another nontrivial characteristic number. We consider the largest class of spaces for which the operation $Sq^1$ may be defined as an operation on the middle intersection homology.

*Definition.* A normal connected pseudomanifold $X$ is a *locally square-free* (or LSF) space if it is a (mod 2) Witt space and if, for each stratum $S$ of even codimension $c = 2l$, the operation $Sq^1 : IH_l^{\bar{m}}(L; \mathbb{Z}/(2)) \to IH_{l-1}^{\bar{m}}(L; \mathbb{Z}/(2))$ vanishes, where $L$ is the link of the stratum $S$.

If $X$ is a $2k + 1$ dimensional LSF space then the Wu number $v^k Sq^1(v^k)$ can be formed and it is a cobordism invariant.

THEOREM 13.1. *An orientable $2k + 1$ dimensional LSF space $X$ is null cobordant iff $v^k Sq^1 v^k(X) = 0$.*

This result is particularly interesting because the proof reduces to a geometric and algebraic understanding (§14.2, §15.1) of the implications of the vanishing of the characteristic number $v^k Sq^1(v^k)$ for $2k + 1$ dimensional spaces. For orientable $4k$ dimensional LSF spaces, an interesting new phenomenon occurs: the Pontrjagin square

$\mathscr{P}$: $IH^{\bar{m}}_{2k}(X; \mathbb{Z}/(2)) \to \mathbb{Z}/(4)$ may be defined, and it gives rise to an additional invariant of $4k$ dimensional LSF spaces (which is analogous to the Arf and deRham invariants [8], [32]):

$$\pi(H) = \frac{1}{2\pi} \arg \sum_{v \in H} \exp(2\pi i \mathscr{P}(v)/2) \in \mathbb{Z}/(4)$$

where $H = IH^{\bar{m}}_{2k}(X; \mathbb{Z}/(2))$.

THEOREM. *The cobordism group of orientable $4k$-dimensional LSF spaces is isomorphic to $\mathbb{Z}/(4)$ and the invariant is given by $\pi$.*

(d) *$\bar{s}$-duality spaces.* Of course, LSF spaces may be combined with local orientability to produce a host of characteristic numbers involving $v^1$. Instead, we prefer to strengthen the LSF condition by demanding not just that $Sq^1$ vanish on the intersection homology of links of even codimension strata, but that the whole group vanish:

*Definition.* A normal pseudomanifold $X$ is an *$\bar{s}$-duality space* if, for each stratum of odd codimension $c = 2k + 1$, we have

$$IH^{\bar{m}}_k(L; \mathbb{Z}/(2)) = 0$$

and for each stratum of even codimension $c = 2k$ we have

$$IH^{\bar{m}}_k(L; \mathbb{Z}/(2)) = IH^{\bar{m}}_{k-1}(L; \mathbb{Z}/(2)) = 0$$

where $L$ is the link of the stratum. For such spaces the middle intersection homology groups admit a Bockstein operation $\beta$: $IH^{\bar{m}}_k(X; \mathbb{Z}/(2)) \to IH^{\bar{m}}_{k-1}(X; \mathbb{Z})$ whose mod 2 reduction is a Steenrod operation, $Sq^1: IH^{\bar{m}}_k(X; \mathbb{Z}/(2)) \to IH^{\bar{m}}_{k-1}(X; \mathbb{Z}/(2))$. It even turns out that there is a universal coefficient theorem for $\bar{s}$ — duality spaces, and that the intersection homology of $\bar{s}$-duality spaces satisfies Poincaré duality over the localization $\mathbb{Z}_{(2)}$.

THEOREM 16.5. *An orientable $n$ dimensional $\bar{s}$-duality space $X$ of odd dimension $n = 2k + 1$ is the boundary of an orientable $\bar{s}$-duality space $Y$ if and only if the characteristic number $v^k Sq^1 v^k(X)$ vanishes.*

In fact we show that the cobordism group $\Omega^{\bar{s}}_i$ of orientable $\bar{s}$-duality spaces is naturally isomorphic to the higher Mischenko–Ranicki–Witt symmetric L group ([31], [35]), $\Omega^{\bar{s}}_i \cong L_i(\mathbb{Z}_{(2)})$. (compare [33] for the analogous theory over $\mathbb{Z}$)

Our proof uses surgery on singular spaces, a procedure which was first carried out successfully by Siegel ([37]) and later generalized by Pardon ([33]). The technique consists of killing certain homology classes by removing the regular neighborhood $N$ of a representative cycle $\xi$ (not necessarily a sphere, not necessarily framed, and not necessarily below the middle dimension, which is nevertheless carefully chosen) and replacing it with an appropriate null cobordism of the boundary $\partial N$. (In Siegel's case this cobordism is always the cone, $c(\partial N)$). Once the appropriate homology groups and homology classes in $X$ have been killed by surgery, then $X$ can be realized as the boundary of a space $Y = \text{cone}(X)$.

One amazing consequence is that surgery on singular spaces is easier than surgery on manifolds: for the singular spaces considered here, the cobordism groups can be completely determined by geometric surgery techniques. Such a program does not work for manifolds, whose cobordism groups can only be calculated from the homotopy groups of the classifying spaces (although the work [10], [11] represents a major advance in the geometric techniques). For example, in the $\bar{s}$-duality theory, it is necessary to perform surgery on classes *above* the middle dimension, a procedure which cannot normally be applied to manifolds.

*Other Cobordism Theories.* In [41] D. Sullivan introduced the idea of allowing singularities to kill cobordism classes. Although his approach did not involve an analysis of characteristic numbers, a general theory was developed ([2]. [3], [1], [12]). Particular examples involving intersection homology were studied by J. Morgan (1975, unpublished), P. Siegel ([37]), and W. Pardon ([33]). The four theories considered here may be thought of as part of a chain of cobordism groups:

| | | | | |
|---|---|---|---|---|
| oriented $\mathbb{Z}/(2)$ homology manifolds | $\rightarrow$ oriented $\bar{s}$-duality spaces | $\rightarrow$ oriented LSF spaces | $\rightarrow$ locally orientable Witt spaces | $\rightarrow$ locally orientable spaces |

where each homomorphism kills certain cobordism classes because the corresponding products of characteristic classes can no longer be made. Obviously, there are many other interesting cobordism groups which could be inserted in this chain (see §21).

## §2. PSEUDOMANIFOLDS AND STRATIFICATIONS

2.1. *Definitions.* An $n$-dimensional *pseudomanifold* $X$ is a (purely $n$-dimensional) piecewise linear pseudomanifold in the sense of [36], [23], [18], i.e. a polyhedron which admits a triangulation such that each $n-1$ dimensional simplex is a face of exactly two $n$-dimensional simplices. A pseudomanifold $X$ admits a piecewise linear *stratification,*

$$\phi \subset X_0 \subset X_1 \subset \ldots \subset X_{n-2} = \Sigma \subset X_n = X$$

such that each $X_i - X_{i-1}$ is a (possibly empty) union of $i$-dimensional P.L. manifolds, each of which has a neighborhood in $X$ which is a locally trivial mapping cylinder, i.e. for each $x \in X_i - X_{i-1}$ there is a neighborhood $U$ and a piecewise linear stratum preserving homeomorphism,

$$U \cong \mathbb{R}^i \times \operatorname{cone}(L)$$

where $L$ is the *link* of the stratum. We will say that $X^0 = X - \Sigma$ is the *nonsingular part* of $X$. A pseudomanifold $X$ will be called *orientable* if its nonsingular part is an orientable manifold.

2.2. *Remark on whitney stratifications.* It is possible to replace "piecewise linear pseudomanifold" by "Whitney stratified pseudomanifold" throughout this paper. This is because (a) every Whitney stratified pseudomanifold admits the structure of a P.L. pseudomanifold [16] with the same stratification, and (b) every piecewise linear pseudomanifold $X$ admits an embedding in Euclidean space as a Whitney stratified pseudomanifold $X'$, but with a refined stratification: just choose any triangulation of $X$ and embed it so that the simplices are flat.

2.3. *Surgery and Cobordism.* A *pseudomanifold $W$ with boundary $\partial W$* is a compact P.L. space such that $W - \partial W$ is a pseudomanifold, and $\partial W$ is a compact P.L. subspace of $W$ which has a collared neighborhood $U$, i.e. there is a P.L. homeomorphism $\phi$: $U \cong \partial W \times [0, 1]$ such that the restriction $\phi | \partial W: \partial W \rightarrow \partial W \times \{0\}$ is the identity. (It follows that $\partial W$ is a pseudomanifold). If $\partial W = X_1 \cup X_2$ is a disjoint union of two compact pseudomanifolds, then we say $W$ is an (unoriented) *cobordism* between $X_1$ and $X_2$. If $X_2 = \phi$ we say $W$ is a cobordism of $X_1$ to the empty set $\phi$.

A *surgery* on an $n$ dimensional pseudomanifold $X$ is the following data:

(1) An open subset $U \subset X$ whose closure $\bar{U}$ is a pseudomanifold with (bicollared) boundary $\partial U \subset X$ (so that the inclusion $\partial U \subset X$ is normally nonsingular with trivial normal bundle).

(2) An $n$-dimensional pseudomanifold $W$ with boundary $\partial W \cong \partial U$.

(3) An $n + 1$ dimensional cobordism $Z$ between $U \cup_{\partial U} W$ and $\phi$.

It follows that the pseudomanifold

$$X' = (X - U) \cup_{\partial U} W$$

is cobordant to $X$, and it will be called the result of the surgery.

2.4. *Normal Pseudomanifolds.* The pseudomanifold $X$ is normal if the link of each stratum is connected ([18]). Every pseudomanifold $X$ has a canonical normalization $\tilde{X} \to X$. Throughout this paper we will assume that all pseudomanifolds are normal and connected, although this is not an essential restriction because

(a) for any pseudomanifold $X$, the mapping cylinder of the normalization map $\pi$: $\tilde{X} \to X$ determines a cobordism between $X$ and $\tilde{X}$, and

(b) if $Y$ is a cobordism between two normal pseudomanifolds $X_1$ and $X_2$ then its normalization $\tilde{Y}$ is also a cobordism between $X_1$ and $X_2$.

## §3. INTERSECTION HOMOLOGY AND COHOMOLOGY

3.1. *Definitions.* Recall ([17], [18], [19]) that a perversity $\bar{p}$ is a sequence of integers $(p(2), p(3), p(4), \ldots)$ such that $p(2) = 0$, and $p(c) \leq p(c+1) \leq p(c) + 1$. We will be specifically interested in the following perversities:

| | |
|---|---|
| The zero perversity | $\bar{0} = (0, 0, 0, 0, 0, 0, \ldots)$ |
| The one perversity | $\bar{1} = (0, 1, 1, 1, 1, 1, \ldots)$ |
| The lower middle perversity | $\bar{m} = (0, 0, 1, 1, 2, 2, \ldots)$ |
| The upper middle perversity | $\bar{n} = (0, 1, 1, 2, 2, 3, \ldots)$ |
| The special perversity | $\bar{s} = (0, 0, 0, 1, 1, 2, \ldots)$ |
| The cospecial perversity | $\bar{r} = (0, 1, 2, 2, 3, 3, \ldots)$ |
| The top perversity | $\bar{t} = (0, 1, 2, 3, 4, 5, \ldots)$ |

Suppose $X$ is an $n$ dimensional pseudomanifold (which is not necessarily orientable or compact). Choose a stratification of $X$, and let $\Sigma$ denote the singular set of $X$. Fix a perversity $\bar{p}$. We will use the notation $IH^{n-i}_{\bar{p}}(X; \mathbb{Z})$ to denote the $(n-i)$th hypercohomology group $\mathbb{H}^{n-i}(\underline{\mathbb{P}}^{\cdot}(\underline{\mathbb{Z}}))$ where $\underline{\mathbb{P}}^{\cdot}(\underline{\mathbb{Z}})$ is Deligne's complex of sheaves,

$$\mathbb{P}^{\cdot} = \tau_{\leq p(n)} R i_{n*} \tau_{\leq p(n-1)} R i_{(n-1)*} \ldots \tau_{\leq p(2)} R i_{2*} \underline{\mathbb{Z}}$$

corresponding to the constant system $\underline{\mathbb{Z}}$ on $X - \Sigma$ (with $i_k: (X - X_{n-k}) \to (X - X_{n-k-1})$ denoting the inclusion). In other words, if we fix a P.L. structure on $X$, then $IH^{n-i}_{\bar{p}}(X; \mathbb{Z})$ is the cohomology group of the cochain complex

$$IC^{n-i}_{\bar{p}} = \{\xi \in C^{n-i}(X; \mathbb{Z}) | \left\{ \begin{array}{l} \dim(|\xi|) \cap X_{n-c} \leq i - c + p(c) \text{ and} \\ \dim(|\partial\xi|) \cap X_{n-c} \leq i - 1 - c + p(c) \end{array} \right\}$$

where $C^{n-i}(X;\mathbb{Z})$ denotes the abelian group of all normally oriented $i$-dimensional PL chains with closed (or "infinite") supports in $X$.

We will also study $IH_i^{\bar{p}}(X;\mathbb{Z})$ which is the $(n-i)$th hypercohomology group with compact supports, $\mathbb{H}_c^{n-i}(\underline{\mathbb{P}}^\cdot(\underline{\omega}))$ where $\underline{\omega}$ is the orientation sheaf on $X-\Sigma$. Since $X$ has a P.L. structure, this group is the $i$th homology group of the chain complex

$$IC_i^{\bar{p}}(X;\mathbb{Z}) = \{\xi \in C_i(X;\mathbb{Z}) | \left\{\begin{array}{l}\dim(|\xi|) \cap X_{n-c} \le i - c + p(c) \text{ and} \\ \dim(|\partial\xi|) \cap X_{n-c} \le i - 1 - c + p(c)\end{array}\right\}$$

where $C_i(X;\mathbb{Z})$ denotes the group of all tangentially oriented compact $i$-dimensional P.L. chains on $X$. We obtain natural homomorphisms

$$H^i(X:\mathbb{Z}) \to IH^i_{\bar{p}}(X;\mathbb{Z})$$

and
$$IH_i^{\bar{p}}(X;\mathbb{Z}) \to H_i(X;\mathbb{Z})$$

If $\bar{p}+\bar{q} \le \bar{t}$, (where $\bar{t}$-denotes the top perversity, $\bar{t}(c) = c-2$), then there are cup products,

$$IH^i_{\bar{p}}(X;\mathbb{Z}) \times IH^j_{\bar{q}}(X;\mathbb{Z}) \to IH^{i+j}_{\bar{p}+\bar{q}}(X;\mathbb{Z})$$

and cap products,

$$IH^i_{\bar{p}}(X;\mathbb{Z}) \times IH_j^{\bar{q}}(X;\mathbb{Z}) \to IH_{j-i}^{\bar{p}+\bar{q}}(X;\mathbb{Z})$$

such that, if $X$ is compact, then the product

$$IH^i_{\bar{p}}(X;\mathbb{Z}) \times IH_i^{\bar{t}-\bar{p}}(X;\mathbb{Z}) \to IH_0^{\bar{t}}(X;\mathbb{Z}) \to \mathbb{Z}$$

is nondegenerate over $\mathbb{Q}$. If $X$ is compact and orientable then the natural homomorphism

$$IH^i_{\bar{p}}(X;\mathbb{Z}) \to IH^{\bar{p}}_{n-i}(X;\mathbb{Z})$$

is an isomorphism. Similar constructions can be made using $\mathbb{Z}/(2)$ coefficients instead of $\mathbb{Z}$ coefficients; however, in this case $IH^i_{\bar{p}}(X;\mathbb{Z}/(2))$ is canonically isomorphic to $IH^{\bar{p}}_{n-i}\ (X;\mathbb{Z}/(2))$ for compact $X$, because $\underline{\omega} \otimes \underline{\underline{\mathbb{Z}/(2)}} \cong \underline{\underline{\mathbb{Z}/(2)}}$ on $X-\Sigma$.

## §4. STEENROD SQUARES

4.1. *Definitions.* In [15], mod 2 Steenrod operations were defined

$$Sq^i: IH^j_{\bar{p}}(X;\mathbb{Z}/(2)) \to IH^{i+j}_{\bar{p}}(X;\mathbb{Z}/(2))$$

provided $2\bar{p} \le \bar{t}$. These operations are 0 if $i > j$, and are given by the self intersection if $i = j$. They are "cohomology" (or "normal") operations, in the sense that the following diagram commutes:

$$\begin{array}{ccc} H^j(X;\mathbb{Z}/(2)) & \xrightarrow{Sq^i} & H^{i+j}(X;\mathbb{Z}/(2)) \\ \downarrow & & \downarrow \\ IH^j_{\bar{p}}(X:\mathbb{Z}/(2)) & \xrightarrow{Sq^i} & IH^{i+j}_{2\bar{p}}(X;\mathbb{Z}/(2)) \end{array}$$

The Steenrod operations satisfy the Cartan formula (whenever both sides of the equation make sense), are compatible with change of perversity, and may also be defined on the intersection homology with compact supports,

$$Sq^i: IH_k^{\bar{p}}(X;\mathbb{Z}/(2)) \to IH_{k-i}^{2\bar{p}}(X;\mathbb{Z}/(2))$$

(provided $2\bar{p} \leq \bar{t}$), so as to agree with the previous operations $Sq^i$ when $X$ is compact. If $j\colon U \to X$ is the inclusion of an open subset, then

$$j_*\colon IH_*^{\bar{p}}(U;\mathbb{Z}/(2)) \to IH_*^{\bar{p}}(X;\mathbb{Z}/(2))$$

commutes with $Sq^i$.

More generally, suppose $j\colon Y \to X$ is a normally nonsingular inclusion ([13], [14], [18]), with normal bundle $\nu$. Then $j$ induces homomorphisms

$$j_*\colon IH_*^{\bar{p}}(Y;\mathbb{Z}/(2)) \to IH_*^{\bar{p}}(X;\mathbb{Z}/(2))$$
$$j^*\colon IH_{\bar{p}}^*(X;\mathbb{Z}/(2)) \to IH_{\bar{p}}^*(Y;\mathbb{Z}/(2))$$

4.2. Proposition. *For any $\xi \in IH_k^{\bar{p}}(Y;\mathbb{Z}/(2))$ and for any $\eta \in IH_{\bar{p}}^j(X;\mathbb{Z}/(2))$, we have*

$$Sq^i j_*(\xi) = \sum_{a+b=i} j_*(w^a(\nu)Sq^b(\xi))$$
$$Sq^i j^*(\eta) = j^* Sq^i(\eta)$$

*where $w^a(\nu) \in H^a(Y;\mathbb{Z}/(2))$ is the Whitney class of the vector bundle $\nu$.*

*Proof.* The second formula is proven in [15], so we will concentrate on proving the first formula. Since $j$ is normally nonsingular, there is a homeomorphism $h\colon \nu \to N \subset X$ between the total space of the normal bundle $\nu$ and some open neighborhood $N$ of $Y$, which restricts to $j$ on the zero section of $\nu$. Since $Sq$ commutes with the pushforward for open inclusions, we may replace $X$ by $N$. If $[Y]$ denotes the fundamental cohomology class of $Y$, then

$$U := j_*[Y] \in H_*(N)$$

is the Thom class of $N$, and the formula reduces to the classical result of Thom [43],

$$Sq(U) = j_*(w(\nu)).$$

Since $N$ is homeomorphic to a vectorbundle over $Y$, we have canonical (Thom) isomorphisms,

$$IH_{k+c}^{\bar{p}}(N) \xrightarrow{j^*} IH_k^{\bar{p}}(Y) \xrightarrow{j_*} IH_k^{\bar{p}}(N)$$

where $c = \dim(X) - \dim(Y)$. Therefore, for arbitrary $\xi \in IH_k^{\bar{p}}(Y)$ there is a class $\tilde{\xi} \in IH_{k+c}^{\bar{p}}(N)$ such that $\xi = j^*(\tilde{\xi})$. Thus

$$Sqj_*(\xi) = Sqj_*j^*(\tilde{\xi}) = Sqj_*(j^*(\tilde{\xi})\cdot[X]) = Sq(\tilde{\xi}\cdot U) = Sq(\tilde{\xi})Sq(U)$$
$$= Sq(\tilde{\xi})j_*(w(\nu)) = j_*(j^*Sq(\tilde{\xi})\cdot w(\nu)) = j_*(Sq(\xi)\cdot w(\nu))$$

as desired.

4.3. *Remark.* If $X$ is an $n$-dimensional pseudomanifold and if $\xi \in IH_i^{\bar{m}}(X;\mathbb{Z}/(2))$ then the number $Sq^i(\xi) \in \mathbb{Z}/(2)$ has the following interpretation: If $2i = n$ then $Sq^i(\xi) = \xi\,.\,\xi$. Otherwise, it is possible to find a normally nonsingular subpseudomanifold $j\colon Y \to X$ such that $Y$ has a trivial normal bundle in $X$, $\dim(Y) = 2i$, and $\xi = j_*(\tilde{\xi})$ for some class $\tilde{\xi} \in IH_i^{\bar{m}}(Y)$. Thus, $Sq^i(\xi) = Sq^i(\tilde{\xi}) = \tilde{\xi}\,.\,\tilde{\xi}$ (where the intersection takes place inside $Y$). Such a space $Y$ can be found as follows: Let $\partial U$ denote the boundary of a regular neighborhood $U$ of $\xi$. It has a trivial normal bundle in $X$. By the long exact sequence for intersection homology, $\xi$ comes from a class in $IH_i^{\bar{m}}(\partial U)$. If $2i = n - 1$ we can take $Y = \partial U$, otherwise let $\partial U'$ be the boundary of a regular neighborhood $U'$ of $\xi$ in $\partial U$. Then $\xi$ pulls back to a class $\xi'$ in $\partial U'$ which has a trivial normal bundle in $X$. Continuing in this way, we will eventually lift $\xi$ to some class $\tilde{\xi} = \xi^{(n-2i)}$ in $Y = \partial U^{(n-2i)}$ which has a trivial normal bundle in $X$.

## §5. WU CLASSES

5.1. *Definitions.* Let $X$ be an $n$-dimensional (not necessarily compact) pseudomanifold. Fix a perversity $\bar{a}$ such that $2\bar{a} \leq \bar{t}$ (where $\bar{t}(c) = c - 2$). Let $\bar{b} = \bar{t} - \bar{a}$ be the complementary perversity. The operation

$$Sq^i : IH^{\bar{a}}_i(X; \mathbb{Z}/(2)) \to IH^{2\bar{a}}_0(X; \mathbb{Z}/(2)) \to \mathbb{Z}/(2)$$

is given (via Poincaré duality) by multiplication with some class

$$v^i = v^i_{\bar{b}} \in IH^i_{\bar{b}}(X; \mathbb{Z}/(2))$$

which is called the $i$th Wu class. As usual, $v^i = 0$ for $i > [n/2]$.

5.2. *Remark.* For a general pseudomanifold $X$, the largest such perversity $\bar{a}$ is the "lower middle" perversity

$$\bar{m}(c) = \left[\frac{c-2}{2}\right]$$

It follows that we may always take the Wu classes to live in the intersection homology with "upper middle" perversity,

$$\bar{n}(c) = \left[\frac{c-1}{2}\right].$$

One goal of this paper is to find spaces for which the Wu classes can be lifted to lower perversities so that top dimensional products of Wu classes can be formed, for these products will be cobordism invariant characteristic numbers.

## §6. BOCKSTEIN HOMOMORPHISM

6.1. *Definitions.* Let $X$ be an $n$-dimensional pseudomanifold. The short exact sequence of sheaves on $X - \Sigma$

$$0 \to \underline{\mathbb{Z}} \xrightarrow{\times 2} \underline{\mathbb{Z}} \to \underline{\underline{\mathbb{Z}/(2)}} \to 0$$

gives rise to an exact triangle in the derived category $D^b(X - \Sigma)$

$$\begin{array}{ccc} \underline{I}^\cdot & \longrightarrow & \underline{I}^\cdot \\ & {}^{\beta}\nwarrow \quad \swarrow & \\ & \underline{J}^\cdot & \end{array}$$

where $\underline{I}^\cdot$ denotes an injective resolution of $\underline{\mathbb{Z}}$ and where $\underline{J}^\cdot$ denotes an injective resolution of $\underline{\underline{\mathbb{Z}/(2)}}$. Fix a perversity $\bar{p}$. Let $\underline{\mathbb{P}}^\cdot$ denote Deligne's sheaf ([19]) as in §3.1. We want to ask whether $\beta$ extends to a morphism on $\underline{\mathbb{P}}^\cdot$.

6.2. *Definition.* The space $X$ admits a $\bar{p}$ Bockstein iff the map $\beta$ has an extension (in the derived category $D^b(X)$) to a morphism

$$\beta : \underline{\mathbb{P}}^\cdot(\mathbb{Z}/(2)) \to \underline{\mathbb{P}}^\cdot(\mathbb{Z})[1]$$

6.3. PROPOSITION. *The following statements are equivalent:*

(a) *$X$ admits a $\bar p$ Bockstein*

(b) *For every stratum $S$ of $X$,*

$$\operatorname{Tor}_{\mathbb{Z}}(IH_{\bar p}^{\bar p(c)+1}(L;\mathbb{Z}), \mathbb{Z}/(2)) = 0$$

*where $L$ is the link of the stratum $S$, and $c$ is the codimension of $S$.*

(c) *For every stratum $S$, the link of $S$ admits a $\bar p$ Bockstein and the operation*

$$\beta\colon IH_{\bar p}^{\bar p(c)}(L;\mathbb{Z}/(2)) \to IH_{\bar p}^{\bar p(c)+1}(L;\mathbb{Z})$$

*vanishes.*

Furthermore, in this case the extension $\beta$ is unique (in the derived category) and the triangle above extends to an exact triangle

$$\begin{array}{ccc} \underline{\mathbb{P}}^{\cdot}(\mathbb{Z}) & \longrightarrow & \underline{\mathbb{P}}^{\cdot}(\mathbb{Z}) \\ {\scriptstyle\beta}\nwarrow & & \swarrow \\ & \underline{\mathbb{P}}^{\cdot}(\mathbb{Z}/(2)) & \end{array}$$

and in particular, there is a long exact Bockstein sequence

$$\ldots \to IH_{\bar p}^{i}(X;\mathbb{Z}) \to IH_{\bar p}^{i}(X;\mathbb{Z}) \to IH_{\bar p}^{i}(X;\mathbb{Z}/(2)) \xrightarrow{\beta} IH_{\bar p}^{i+1}(X;\mathbb{Z}) \to \ldots$$

6.4. *Proof.* This follows immediately from Deligne's construction of $\mathbb{P}^{\cdot}$ ([19]) and induction.

6.5. *Remarks.*

1. A homology Bockstein homomorphism

$$\beta\colon IH_i^{\bar p}(X;\mathbb{Z}/(2)) \to IH_{i-1}^{\bar p}(X;\mathbb{Z})$$

can be defined provided

$$\operatorname{Tor}_{\mathbb{Z}}(IH_{c-2-p(c)}^{\bar p}(L;\mathbb{Z}), \mathbb{Z}/(2)) = 0$$

in a completely analogous way, by replacing the sheaf $\underline{\mathbb{Z}}$ by the orientation sheaf $\underline{\omega}$.

2. If the links $L$ fail to satisfy the condition of the proposition, it is still possible to define a (cohomology or homology) Bockstein homomorphism, but it will increase perversity by 1.

3. The Bockstein has the usual geometric interpretation: Let $\xi$ be an $n-i$ dimensional P.L. chain with no mod 2 boundary. Choose a triangulation of $|\xi|$ and a normal orientation of each $n-i$ dimensional simplex. These choices induce a normal orientation on the $n-i-1$ dimensional simplices of $|\xi|$ (some of which may cancel) and it turns out that the resulting $n-i-1$ dimensional chain $\beta(\xi)$ is an integral cycle whose homology class was independent of the choices. Furthermore, $|\beta(\xi)| \subset |\xi|$, so for each stratum $S$ of $X$ we have

$$\dim(|\beta(\xi)| \cap S) \leq \dim(|\xi| \cap S)$$

which shows that if $\xi \in IC_{\bar p}^{i}(X;\mathbb{Z}/(2))$ then $\beta(\xi) \in IH_{\bar p+\bar 1}^{i+1}(X;\mathbb{Z})$.

6.6. *The operation $Sq^1$.* By considering the exact triangle corresponding to the exact sequence of sheaves on $X - \Sigma$,

$$0 \to \mathbb{Z}/(2) \xrightarrow{\times 2} \mathbb{Z}/(4) \to \mathbb{Z}/(2) \to 0$$

we obtain the following analogous result:

PROPOSITION. *The space X admits an operation*

$$Sq^1: \underline{\mathbb{P}}^{\cdot}(\mathbb{Z}/(2)) \to \underline{\mathbb{P}}^{\cdot}(\mathbb{Z}/(2))[1]$$

*if and only if for each stratum S of X the (inductively defined) operation*

$$Sq^1: IH_{\bar{p}}^{\bar{p}(c)}(L; \mathbb{Z}/(2)) \to IH_{\bar{p}}^{\bar{p}(c)+1}(L; \mathbb{Z}/(2))$$

*vanishes (where L denotes the link of the stratum S, and c denotes the codimension of S in X). In this case we obtain a long exact sequence*

$$\to IH_{\bar{p}}^i(X; \mathbb{Z}/(2)) \to IH_{\bar{p}}^i(X; \mathbb{Z}/(4)) \to IH_{\bar{p}}^i(X; \mathbb{Z}/(2)) \xrightarrow{Sq^1} IH_{\bar{p}}^{i+1}(X; \mathbb{Z}/(2)) \to$$

*and if X satisfies the conditions of* 6.3, *then* $Sq^1$ *is the mod* 2 *reduction of* $\beta$.

## §7. THE ORIENTATION CYCLE

7.1. *Definitions.* For a compact $n$ dimensional pseudomanifold $X$ it is possible to define an orientation class $v^1(X) \in H_{n-1}(X; \mathbb{Z}/(2))$ as follows: choose any orientation of each of the $n$-dimensional simplices $\Delta_i$ of $X$ and take

$$\tfrac{1}{2}\partial \sum_i \Delta_i$$

7.2. PROPOSITION. *The following statements are equivalent*:

(a) $X$ is orientable (i.e $X - \Sigma$ is an orientable manifold)
(b) $H^n(X; \mathbb{Z})$ has no 2-torsion
(c) The Bockstein homomorphism

$$\beta: H^{n-1}(X; \mathbb{Z}/(2)) \to H^n(X; \mathbb{Z})$$

vanishes
(d) $v^1(X) = 0$

7.3. *Proof.* The proof is standard but we note that $(c) \Rightarrow (d)$ because

$$Sq^1: H^{n-1}(X; \mathbb{Z}/(2)) \to H^n(X; \mathbb{Z}/(2))$$

is the reduction (mod 2) of $\beta$.

7.4. *Remark.* It is easy to see that the orientation cycle is the image in homology of the Wu class $v^1(X) \in IH_{\bar{n}}^1(X; \mathbb{Z}/(2))$, but even more is true. For a compact pseudomanifold $X$, the orientation cycle $v^1(X) \in H_{n-1}(X; \mathbb{Z}(2))$, has a canonical lift to $IH_{\bar{1}}^1(X; \mathbb{Z}/(2))$ where $\bar{1}$ is the perversity

$$\bar{1}(c) = \min(c-2, 1)$$

for each $c$. This is because the mod 2 reduction of the Bockstein gives a lift of the Steenrod operation $Sq^1$ to intersection homology,

$$Sq^1: IH_{\bar{t}-\bar{1}}^{n-1}(X; \mathbb{Z}/(2)) \to IH_{\bar{t}}^n(X; \mathbb{Z}(2)) \to \mathbb{Z}(2)$$

where $\bar{t} - \bar{1}$ is the perversity

$$(\bar{t} - \bar{1})(c) = \max(0, c-3)$$

By Poincaré duality, this homomorphism is given by multiplication with some class

$$v^1 \in IH^1_{\bar{1}}(X; \mathbb{Z}/(2))$$

and it is easy to verify that the image of $v^1$ in $H_{n-1}(X; \mathbb{Z}/(2))$ is the orientation class.

We conjecture that similar lifts exist for the other Wu classes:

7.5. *Conjecture.* For any pseudomanifold $X$ and for any perversity $\bar{p}$, there is a canonical extension of the Steenrod operation $Sq^i$ to a homomorphism

$$Sq^i: IH^j_{\bar{p}}(X; \mathbb{Z}/(2)) \to IH^{i+j}_{\bar{q}}(X; \mathbb{Z}/(2))$$

where $\bar{q}(c) = \min(2\bar{p}(c), \bar{p}(c) + i)$. Applying this extension to $IH^{n-i}_{\bar{s}}(X; \mathbb{Z}/(2))$, where

$$\bar{s}(c) = \max\left(\left[\frac{c-2}{2}\right], c-2-i\right)$$

would give rise to a canonical lift of $v^i$ to a class

$$v^i \in IH^i_{\bar{r}}(X; \mathbb{Z}(2)$$

where

$$\bar{r}(c) = \min\left(\left[\frac{c-1}{2}\right], i\right)$$

## §8. LOCALLY ORIENTABLE SPACES

8.1. *Definitions.* A pseudomanifold $X$ is called locally orientable if, for each stratum $S$, the link $L$ of that stratum is an orientable pseudomanifold.

8.2. PROPOSITION. *If $X$ is a locally orientable $n$ dimensional pseudomanifold then the orientation (Wu) class $v^1(X) \in IH^1_{\bar{n}}(X; \mathbb{Z}/(2))$ (where $\bar{n}$ is the upper middle perversity, $\bar{n}(c) = [(c-1)/2]$) has a canonical lift to the intersection homology group $IH^1_{\bar{0}}(X; \mathbb{Z}/(2))$ with perversity 0. If $X$ is also normal then $v^1(X)$ lifts to cohomology.*

8.3. *Proof.* By Proposition 6.3 the space $X$ admits a $\bar{t}$ Bockstein homomorphism (where $\bar{t}(c) = c - 2$), and so

$$Sq^1: IH^{n-1}_{\bar{t}}(X; \mathbb{Z}/(2)) \to IH^n_{\bar{t}}(X; \mathbb{Z}/(2)) \to \mathbb{Z}/(2)$$

is given by multiplication with $v^1(X) \in IH^1_{\bar{0}}(X; \mathbb{Z}/(2))$.

*Remarks.* This result can be seen by a simple geometric argument which R. MacPherson told us: Suppose $X$ is normal and locally orientable. It is then possible to orient each of the $n$-dimensional dual cones in a triangulation of $X$. This induces an orientation on the $n-1$ dimensional dual cones, so the orientation class is naturally an element of $H^1(X; \mathbb{Z}/(2))$. ([27, 14]).

COROLLARY. *If an $n$-dimensional pseudomanifold $X$ is orientable (i.e. if there exists an orientation of each $n$-simplex such that the induced orientations on each $n-1$-simplex cancel, or equivalently, if $v^1(X) = 0$), then $X$ is also locally orientable.*

*Proof.* The restriction $v^1(X)|L$ to any link $L$ is the orientation class of $L$.

8.4. *Geometric Representative of the Orientation Cocycle.* Suppose $X$ is a normal, locally orientable pseudomanifold. Then it has an orientation bundle, which is a $\mathbb{Z}/(2)$ fibre bundle whose fibre at each point $x$ is the set of generators of $H_n(X, X-x; \mathbb{Z})$. The associated line bundle $\omega_X$ is classified by a map $f: X \to \mathbb{RP}^N$ (for sufficiently large $N$). The characteristic class $v^1(X)$ is $f^*(y)$, where $y$ is the generator of $H^1(\mathbb{RP}^N; \mathbb{Z}/(2))$. Since the generator of $H^1(\mathbb{RP}^N)$ is carried by the Thom class of the normal bundle of $\mathbb{RP}^{N-1}$ in $\mathbb{RP}^N$, making the map $f$ transverse to $\mathbb{RP}^{N-1}$ ([14], (13)] gives a normally nonsingularly embedded sub-pseudomanifold

$$Y = f^{-1}(\mathbb{RP}^{N-1}) \subset X$$

such that the Thom class of the normal bundle of $Y$ in $X$ represents $v^1(X)$. We also define higher orientation cocycles,

$$Y^{(i)} = f^{-1}(\mathbb{RP}^{N-i})$$

(for transverse choice of $f$) which represent the characteristic classes $(v^1)^i$, such that each inclusion $Y^{(i)} \subset Y^{(i-1)}$ is normally nonsingular. Let $\omega_Y$ denote the orientation line bundle of $Y$.

PROPOSITION.

(A) *Let $\nu$ denote the normal bundle of the inclusion $j_1: Y \to X$. Then*

$$j_1^*(v^i(X)) = v^i(Y) + \sum_{a=0}^{\infty} v^{i-2^a}(Y)(w^1(\nu))^{2^a}$$

(B) *Let $j_2: Y^{(2)} \to X$ denote the inclusion. Then $j_2^*(v^1(X)) = v^1(Y^{(2)})$.*

(See also Proposition 8.7)

*Proof.* By Poincaré duality, it suffices to verify that the product (of each side of the above equation) with any $\xi \in IH_i^{\bar{m}}(Y; \mathbb{Z}/(2))$ is equal. Since this is a number (in $\mathbb{Z}/(2)$) we may compute the product in $H_0(X; \mathbb{Z}/(2))$. Thus,

$$j_*(j^*v^i(X).\xi) = v^i(X).j_*(\xi) = Sq^i j_*(\xi) = j_* Sq^i(\xi) + j_* w^1(\nu) Sq^{i-1}(\xi) \quad \text{(by 4.2).}$$

By the Cartan formula, for any one-dimensional class $w$, we have

$$w.Sq^{i-1}(\xi) = Sq^{i-1}(w.\xi) + Sq^{i-2}(w^2.\xi) + Sq^{i-4}(w^4.\xi) + \ldots$$

which gives

$$j_*(j^*v^i(X).\xi) = j_*[(v^i(Y) + v^{i-1}(Y)w^1(\nu) + v^{i-2}(Y)w^2(\nu) + \ldots).\xi]$$

and this proves part (A). The proof of part (B) is similar:

$$j_{2*}(j_2^* v^1(X).\xi) = Sq^1 j_{2*}(\xi) = j_{2*}Sq^1(\xi) + j_{2*}w^1(\nu)Sq^0(\xi) \quad \text{(by 4.2).}$$

But the normal bundle $\nu(\mathbb{RP}^{N-2} \subset \mathbb{RP}^N)$ is orientable, so the second term vanishes.

8.5. LEMMA.

(a) *$\omega_{X-Y}$ is trivial (i.e. $X - Y$ is orientable)*
(b) *$\omega_Y$ is trivial (i.e. $Y$ is orientable)*
(c) *$\omega_{Y^{(2)}} \cong \nu(Y^{(2)} \subset Y) \cong \omega_X | Y^{(2)}$ where $\nu$ denotes the normal bundle*
(d) *$\nu(Y^{(2)} \subset X) \cong \omega_{Y^{(2)}} \oplus \omega_{Y^{(2)}}$*

8.6. *Proof of Lemma.* (a) is obvious. To prove (b), note that by 8.4(A), we have

$$v^1(Y) + w^1(v) = j^*(v^1(X))$$

Since $\omega_Y$ and $v$ are line bundles on $Y$, they are determined by their first Whitney classes, so we conclude that

$$\omega_Y \otimes v(Y \subset X) \cong \omega_X | Y$$

On the other hand, $\omega_X | Y \cong v(Y \subset X)$ because these bundles are pulled back from the tautological bundle $\xi$ on $\mathbb{RP}^N$, and the corresponding isomorphism is true for the corresponding bundles on $\mathbb{RP}^{N-1}$ and $\mathbb{RP}^N$. Therefore, $\omega_Y$ is trivial (which proves (b)).

The second isomorphism of part (c) is the pullback to $Y^{(2)}$ of the canonical isomorphism

$$v(\mathbb{RP}^{N-2} \subset \mathbb{RP}^{N-1}) \cong \xi | \mathbb{RP}^{N-2}$$

and by 8.4(B), we have $\omega_X | Y^{(2)} \cong \omega_{Y^{(2)}}$ since these line bundles are classified by their Whitney classes. This proves (c).

Finally,

$$v(Y^{(2)} \subset X) = v(Y^{(2)} \subset Y) \oplus v(Y \subset X) \cong \omega_{Y^{(2)}} \oplus \omega_X | Y^{(2)} \cong \omega_{Y^{(2)}} \oplus \omega_{Y^{(2)}}$$

by (c).

8.7. Corollary. *Let $j_2 \colon Y^{(2)} \to X$ denote the inclusion. Then, for any $i$ we have*

$$j_2^*(v^i(X)) = v^i(Y^{(2)}) + \sum_{a=1}^{\infty} v^{i-2^a}(Y^{(2)}) v^1(Y^{(2)})^{2^a}$$

*Proof.* Let $\eta$ denote the normal bundle of the inclusion $j_2$. By Lemma 8.5(d) we have $w^1(\eta) = 0$ and $w^2(\eta) = v^1(Y^{(2)})^2$. Using exactly the same procedure as in the proof of 8.4(A), these substitutions allow us to replace all the characteristic classes of $\eta$ with characteristic classes of $Y^{(2)}$.

## §9. COBORDISM OF LOCALLY ORIENTABLE PSEUDOMANIFOLDS

9.1. *Definition.* If $X$ is a compact $n$-dimensional locally orientable pseudomanifold, then the orientation number $\sigma(X) \in \mathbb{Z}/(2)$ is the number

$$\sigma(X) = \varepsilon(v^1(X))^n$$

where $\varepsilon \colon H_0(X; \mathbb{Z}/(2)) \to \mathbb{Z}/(2)$ denotes the augmentation.

9.2. *Remark.* This number is 0 if $n = 2k + 1$ is odd because

$$\sigma(X) = Sq^1((v^1)^k(v^1)^k) = 2(v^1)^k Sq^1(v^1)^k = 0 \pmod 2$$

9.3. Theorem. *The cobordism class of a locally orientable n-dimensional pseudomanifold is determined by its orientation number.*

Corollary. *If $\mathcal{N}_i$ denotes the cobordism group of i dimensional locally orientable pseudomanifolds, then we have*

$$\mathcal{N}_i = \begin{cases} \mathbb{Z}/(2) & \text{for } i \text{ even} \\ 0 & \text{for } i \text{ odd} \end{cases}$$

9.4. *Outline of Proof.* S. Buoncristiano pointed out to us that there is an elementary geometric proof of this result, along the lines of [10], [11], but we will give a "surgery" proof, using the method of Stong ([39]) because the same technique will be needed later in this paper. Here is the outline of proof: If $X$ is orientable then $Z = \text{cone}(X)$ is a cobordism to $\phi$. If $X$ is not orientable, then we want to make it orientable by doing surgery on (a neighborhood of) the orientation cycle $Y \subset X$. If $Y$ has a trivial normal bundle in $X$, then this surgery is easily accomplished because $Y$ itself is orientable. Otherwise, we must first do a surgery on the orientation cycle of $Y$ to trivialize the normal bundle of $Y$ in $X$. These arguments are most clearly organized in the exact sequences of Stong:

9.5. *$\mathbb{Z}/(2)$-pseudomanifolds and Stong's exact sequences.* A $\mathbb{Z}/(2)$-structure on a pseudomanifold $X$ is a choice of normally nonsingular codimension 1 submanifold $Y$ (possibly empty) such that $[Y]$ represents the orientation cycle, and such that $Y$ has trivial normal bundle in $X$. If $X$ admits a $\mathbb{Z}/(2)$-structure, then $X$ can be obtained from an oriented pseudomanifold with boundary $(W, \partial W)$ with the property that $\partial W = Y \cup -Y$, by identifying the two components of $\partial W$ (just cut $X$ along $Y$ to obtain $W$). If $v^1(X)^2 = 0$, then $X$ admits a $\mathbb{Z}/(2)$-structure because (Proposition 8.4) there exists a choice of orientation cycle $Y$ with trivial normal bundle. The analogous notion of a $\mathbb{Z}/(2)$-structure on a cobordism between two pseudomanifolds with $\mathbb{Z}/(2)$-structures gives rise to the cobordism group $\Omega_i(\mathbb{Z}/(2))$ of locally orientable pseudomanifolds which admit a $\mathbb{Z}/(2)$-structure. For any pseudomanifold $X \in \Omega_i(\mathbb{Z}/(2))$ the orientation cycle $Y$ is orientable and determines a homomorphism $\beta: \Omega_i(\mathbb{Z}/(2)) \to \Omega_{i-1}$ (where $\Omega$ denotes the cobordism group of orientable pseudomanifolds). Let $\mathcal{N}_i$ denote the cobordism group of $i$-dimensional locally orientable pseudomanifolds.

PROPOSITION. ([39]) *For any $i$ dimensional locally orientable pseudomanifold $X$, the association $X \to Y^{(2)}$ induces a surjective homomorphism*

$$(v^1)^2 : \mathcal{N}_i \to \mathcal{N}_{i-2}$$

*with kernel $\Omega_i(\mathbb{Z}/(2))$, and the resulting short exact sequence*

$$0 \to \Omega_i(\mathbb{Z}/(2)) \to \mathcal{N}_i \to \mathcal{N}_{i-2} \to 0$$

*is naturally split. Furthermore there is a natural long exact Bockstein sequence*

$$\Omega_i \xrightarrow{\times 2} \Omega_i \to \Omega_i(\mathbb{Z}/(2)) \xrightarrow{\beta} \Omega_{i-1} \xrightarrow{\times 2} \Omega_{i-1} \to \Omega_{i-1}(\mathbb{Z}/(2)) \to$$

*where $\beta$ associates to any $\mathbb{Z}/(2)$ pseudomanifold $X$ the pseudomanifold $Y = \beta(X)$ which represents the orientation class of $X$.*

COROLLARY. *If $\Omega_i$ consists of elements of order 2, then*

$$\mathcal{N}_i \cong \mathcal{N}_{i-2} \oplus \Omega_i \oplus \Omega_{i-1}$$

9.6. *Proof of Theorem* 9.5. This is exactly the same as in [39] p. 156.

9.7. *Proof of Theorem* 9.3. First note that $\Omega_i = 0$ (if $i \geq 1$): if $X$ is orientable, then $Z = \text{cone}(X)$ is an orientable null cobordism. According to the second Stong exact sequence, this implies that the homomorphism

$$(v^1)^2 : \mathcal{N}_i \to \mathcal{N}_{i-2}$$

is an isomorphism. Now consider the general case. Suppose the orientation number of $X$ vanishes. Let $Y^{(2)}$ denote the second orientation cycle (8.4). By Corollary 8.7, the orientation number of $Y^{(2)}$ vanishes. By induction, this means that $Y^{(2)}$ is null-cobordant, so $[X]$ is also.

## §10. LOCALLY ORIENTABLE WITT SPACES

10.1. *Definition.* An $n$-dimensional pseudomanifold $X$ is a ($\mathbb{Z}/(2)$-) Witt space iff (for some and hence for any stratification of $X$) for each stratum of odd codimension $c = 2k + 1$, we have

$$IH^{k}_{\bar{m}}(L; \mathbb{Z}/(2)) = 0$$

where $L$ is the link of that stratum.

These spaces have been studied in ([37], [15]) where it was shown that if $X$ is a $\mathbb{Z}/(2)$-Witt space, then

(a) the natural homomorphism $IH^{i}_{\bar{m}}(X; \mathbb{Z}/(2)) \to IH^{i}_{\bar{n}}(X; \mathbb{Z}/(2))$ is an isomorphism

(b) The Wu classes $v^i$ lift canonically to $IH^{i}_{\bar{m}}(X; \mathbb{Z}/(2))$, and so one can define Whitney classes

$$IW_{n-j} = \sum_{a+b=j} Sq^a v^b(X) \in H_{n-j}(X; \mathbb{Z}/(2))$$

(c) The cobordism groups of $\mathbb{Z}/(2)$-Witt spaces of dimension $n$ are 0 for $n$ odd, and are isomorphic to $\mathbb{Z}/(2)$ for $n$ even, with the only invariant being given by the intersection homology Euler characteristic $IW_0(X) = v^j v^j = I\mathscr{X}(X; \mathbb{Z}/(2))$ where $j = n/2$.

10.2. *Definition.* An $n$-dimensional pseudomanifold $X$ is a locally orientable Witt space, if it is both locally orientable and is $\mathbb{Z}/(2)$-Witt space.

LEMMA. *If $X$ is a locally orientable Witt space then $Sq^1 Sq^{2i} = Sq^{2i+1}$ as homomorphisms*

$$IH^{\bar{m}}_{j}(X; \mathbb{Z}/(2)) \to IH^{\bar{m}}_{j-2i-1}(X; \mathbb{Z}/(2))$$

*Remarks.* This formula is the first Wu relation for Steenrod squares. It is conjectured that the Wu relations between the Steenrod squares hold in general, whenever both sides of the relation make sense.

COROLLARY. *If $X$ is orientable then the operation*

$$Sq^{2k+1}: IH^{\bar{m}}_{2k+1}(X; \mathbb{Z}/(2)) \to \mathbb{Z}/(2)$$

*vanishes.*

This is because $Sq^{2k+1}(\xi) = Sq^1 Sq^{2k}(\xi) = v^1(X) Sq^{2k}(\xi) = 0$.

*Proof of Lemma.* (See [38] p. 281, exercise # 5) As in [5], [15], the Steenrod squares may be constructed from certain sheaf maps

$$J_j: \underline{\underline{IC}}_{\bar{m}}(\mathbb{Z}/(2)) \otimes \underline{\underline{IC}}_{\bar{m}}(\mathbb{Z}/(2)) \to \underline{\underline{IC}}_{\bar{m}}(\mathbb{Z}/(2))[-j]$$

of degree $j$, by

$$Sq^i(c) = J_{q-i}(c \otimes c)$$

for any $c \in \Gamma(\underline{\underline{IC}}^q_{\bar{m}})$. As in [38] it is possible to find lifts of these sheaf maps to the complex of integral cochains,

$$D_j: \underline{\underline{IC}}_{\bar{m}}(\mathbb{Z}) \otimes \underline{\underline{IC}}_{\bar{m}}(\mathbb{Z}) \to \underline{\underline{IC}}_{\bar{t}}(\mathbb{Z})[-j]$$

such that

$$D_{2j+1}d + dD_{2j+1} = D_{2j} - D_{2j}T$$

$$D_{2j}d - dD_{2j} = D_{2j-1} + D_{2j-1}T$$

where

$$T(\sigma_1 \otimes \sigma_2) = (-1)^{\deg(\sigma_1)\deg(\sigma_2)}(\sigma_2 \otimes \sigma_1)$$

Now suppose that $c \in \Gamma(\underline{\underline{IC}}^q_{\bar{m}}(\mathbb{Z}/(2))$ is a (mod 2) cycle. Choose a lift to a chain $\tilde{c} \in \Gamma(\underline{\underline{IC}}^q_{\bar{m}}(\mathbb{Z}))$. Then $dc = 2y$ for some $y \in \Gamma(\underline{\underline{IC}}^{q+1}_{\bar{m}}(\mathbb{Z}))$ and if $q = 2j + 2k + 1$ is odd, we have

$$\begin{aligned}
Sq^1 Sq^{2k}(c) &= \tfrac{1}{2} dD_{2j+1}(\tilde{c} \otimes \tilde{c}) \pmod 2\\
&= \tfrac{1}{2} D_{2j}(2\tilde{c} \otimes \tilde{c}) - \tfrac{1}{2} D_{2j+1}(2y \otimes c - c \otimes 2y) \pmod 2\\
&= D_{2j}(\tilde{c} \otimes \tilde{c}) - D_{2j+1}(y \otimes c - c \otimes y) \pmod 2\\
&= Sq^{2k+1}(c) - D_{2j+1}(y \otimes c + c \otimes y) - 2D_{2j+1}(c \otimes y) \pmod 2\\
&= Sq^{2k+1}(c) - D_{2j+2}d(y \otimes c) + dD_{2j+2}(y \otimes c) \pmod 2\\
&= Sq^{2k+1}(c) - D_{2j+2}(dy \otimes c + y \otimes dc) + dD_{2j+2}(y \otimes c) \pmod 2\\
&= Sq^{2k+1}(c) - D_{2j+1}(2y \otimes y) + dD_{2j+2}(y \otimes c) \pmod 2
\end{aligned}$$

The results now follows by reducing mod 2 and reducing mod boundaries

10.3. *Characteristic numbers.* Suppose $X$ is an $n$-dimensional locally orientable Witt space. If $i + j \leqq n$ are nonnegative integers, we define the characteristic number

$$v^{ij}(X) = \varepsilon(v^i(X) v^j(X) v^1(X)^{n-i-j}) \in \mathbb{Z}/(2)$$

where $\varepsilon$: $H_0(X; \mathbb{Z}/(2)) \to \mathbb{Z}/(2)$ denotes the augmentation. Since $v^1$ is a cohomology class and $v^i v^j$ is a homology class, this product is a well defined cobordism invariant.

We will see (10.4) that the cobordism class of a locally orientable Witt space $X$ is determined by the characteristic numbers $v^{ij}(X)$. Since the relations between these characteristic numbers are quite complicated, we will give another basis for the space of characteristic numbers involving two new sets of invariants.

*Definition.* If $n = 2k = \dim(X)$ is even, we define

$$I\mathscr{X}(X) = \varepsilon(v^k v^k)$$

$$\mathscr{X}^i(X) = I\mathscr{X}(Y^{(i)})$$

when $I\mathscr{X}$ denotes the (mod 2) intersection homology Euler characteristic. If $n = 2k + 1$ is odd, we define similarly

$$\Phi(X) = \varepsilon(v^k v^k v^1)$$

$$\Phi^i(X) = \Phi(Y^{(i)})$$

It follows from 8.5(c) that $\mathscr{X}^i(X) = \mathscr{X}^{i-2}(Y^{(2)})$ and $\Phi^i(X) = \Phi^{i-2}(Y^{(2)})$. The numbers $\mathscr{X}^i$ and $\Phi^i$ are well defined and are cobordism invariants.

*Remark.* If $n = 2k + 1$ is odd, the number $\Phi(X) = v^{kk} = v^k v^k v^1$ vanishes when $X$ is a compact manifold because it equals $Sq^1(v^k v^k) = 2v^k Sq^1(v^k)$ by the Cartan formula. However, in our situation, $Sq^1$ cannot be defined as an operation on middle intersection homology, so the product $2v^k Sq^1(v^k)$ does not make sense, and the above argument fails (cf. 11). In fact, the number $\Phi(X)$ is nonzero on the following (singular) locally orientable Witt space $X$: The connected sum $\mathbb{CP}^2 \# \mathbb{CP}^2$ bounds, as an *orientable* Witt space: it is possible to kill the class represented by $\mathbb{CP}^1 \# \mathbb{CP}^1$ by surgery as in [37]. Let us say $\partial Y = \mathbb{CP}^2 \# \mathbb{CP}^2$. The natural identification of these boundary components is therefore an orientation reversing diffeomorphism, and if we glue two boundary components by this diffeomorphism we obtain a 5-dimensional space $X$ with the property that its orientation cycle $Y = \mathbb{CP}^2$ has odd Euler characteristic, and has trivial normal bundle in $X$. Thus, $v^1(X)v^k(X)^2 = \mathscr{X}(\mathbb{CP}^2) = 1 \pmod 2$.

10.4. *Relations among the characteristic numbers.*

PROPOSITION A. *Every characteristic number $v^{ij}$ is a linear combination of the numbers*

$$v^{2i}v^{2i}(v^1)^{n-4i} \text{ where } \begin{cases} 0 \leqq i \leqq \left[\dfrac{n}{4}\right] & \text{if } n \text{ is even} \\ 1 \leqq i \leqq \left[\dfrac{n}{4}\right] & \text{if } n \text{ is odd} \end{cases}$$

*Proof.* We shall use induction on $\min(i, j)$, with the case $\min(i, j) = 1$ giving rise to some multiple of the orientation number, since $v^{1i} = v^i(v^1)^{n-i} = Sq^i(v^1)^{n-i} = \binom{n-i}{i}(v^1)^n$. For the inductive step, suppose that $j \leq i$. Notice first that, for any $p > 0$, $(Sq^{j-p}v^j)(v^1)^{n-2j-p}$ is (by induction) a linear combination of the numbers listed above because, by the Cartan formula,

$$\begin{aligned}(Sq^{j-p}v^j)(v^1)^{n-2j-p} &= Sq^{j-p}(v^j(v^1)^{n-2j-p}) + \text{terms involving } Sq^{j-p-q}v^j \\ &= v^{j-p}v^j(v^1)^{n-2j-p} + \text{terms involving } v^{j-p-q}v^j(v^1)^{n-2j-p-q}\end{aligned}$$

(where $q > 0$), to which the inductive hypothesis applies.

Now consider a general characteristic number $v^{ij}$ with $i = \min(i, j)$. There are two cases to eliminate: (a) $j = i$ is odd, and (b) $j > i$.

Case (a): We compute

$$\begin{aligned}v^{2i+1}v^{2i+1}(v^1)^{n-4i-2} &= Sq^{2i+1}(v^{2i+1}(v^1)^{n-4i-2}) \\ &= Sq^1 Sq^{2i}(v^{2i+1}(v^1)^{n-4i-2}) = v^1 Sq^{2i}(v^{2i+1})(v^1)^{n-4i-2} + \text{lower terms} \\ &= v^{2i}v^{2i+1}(v^1)^{n-4i-1} + \text{lower terms}\end{aligned}$$

(by Lemma 10.2) to which the inductive hypothesis applies.

Case (b): By the Cartan formula,

$$\begin{aligned}v^{ij} &= Sq^j(v^i(v^1)^{n-i-j}) = Sq^i v^i Sq^{j-i}(v^1)^{n-i-j} + \sum_{p>0} Sq^{i-p}v^i Sq^{j-i+p}(v^1)^{n-i-j} \\ &= v^i v^i \binom{n-i-j}{j-i}(v^1)^{n-i-j} + \text{terms involving } v^{ki} \text{ with } k < i\end{aligned}$$

which reduces us to case (a).

Finally, we observe that if $n=2k+1$ is odd, then there is an additional relation: the orientation number $v^{11}=(v^1)^n$ vanishes, because $(v^1)^{2k+1}=Sq^1((v^1)^k(v^1)^k)=2(v^1)^k Sq^1(v^1)^k=0$ (mod 2).

PROPOSITION B. *If $n=dim(X)$ is even, then every characteristic number $v^{ij}$ is a linear combination of the mod 2 invariants*

$$I\mathscr{X}(Y^{(n)}), I\mathscr{X}(Y^{(n-4)}), \ldots, I\mathscr{X}\left(Y^{n-4\left[\frac{n}{4}\right]}\right)$$

If $n$ is odd, then every characteristic number $v^{ij}$ is a linear combination of the invariants

$$\Phi(Y^{(n-5)}), \Phi(Y^{(n-9)}), \ldots, \Phi\left(Y^{(n-1-4)\left[\frac{n}{4}\right]}\right)$$

*Proof.* By the preceding proposition, it suffices to verify that the numbers $v^{2i}v^{2i}(v^1)^{n-4i}$ can be written as linear combinations of these invariants. Consider the case that $n$ is even. If $n-4i=0$, then $v^{2i}v^{2i}=I\mathscr{X}(X)=I\mathscr{X}(Y^{(0)})$. If $n-4i\geqq 2$, then

$$v^{2i}v^{2i}(v^1)^{n-4i}=v^{2i}v^{2i}(v^1)^2(v^1)^{n-4i-2}=j_2^*(v^{2i}v^{2i}(v^1)^{n-4i-2})$$

where $j_2$: $Y^{(2)}\to X$ is the inclusion of the second orientation cycle. According to proposition 8.4 and 8.7, this product becomes the following product of characteristic classes of $Y^{(2)}$:

$$(v^{2i}+v^{2i-2}(v^1)^2+v^{2i-4}(v^1)^4+\ldots)(v^{2i}+v^{2i-2}(v^1)^2+\ldots)(v^1)^{n-4i-2}$$

which is a linear combination of numbers $v^{pq}(Y^{(2)})$. By induction, each of these terms is a linear combination of the numbers $I\mathscr{X}(Y^{(n-4i)})$ as desired. The case $n$ is odd follows in an exactly parallel manner.

10.5. *Cobordism of locally orientable Witt Spaces.* Let $\Omega_i^W$ denote the cobordism group of $i$-dimensional orientable Witt spaces. (Recall from 8.3 that orientable $\Rightarrow$ locally orientable.)

THEOREM A. *The oriented cobordism group is*

$$\Omega_n=\begin{cases}\mathbb{Z}/(2) & \text{if } n\equiv 0 \pmod 4\\ 0 & \text{otherwise}\end{cases}$$

*and the single invariant is given by the (intersection homology) Euler characteristic.*

Let $\mathscr{N}_i^W$ denote the cobordism group of $i$-dimensional locally orientable Witt spaces. If $n$ is even, we have homomorphisms

$$\mathscr{X}^*: \mathscr{N}_n\to\mathbb{Z}/(2)^{\left[\frac{n}{4}\right]+1}$$

and

$$\Psi^*: \mathscr{N}_n\to\mathbb{Z}/(2)^{\left[\frac{n}{4}\right]+1}$$

given by

$$\mathscr{X}^*(X)=\left\{I\mathscr{X}(Y^{(n-4i)})\,\middle|\,0\leqq i\leqq\left[\frac{n}{4}\right]\right\}$$

$$\Psi^*(X)=\left\{v^{2i}v^{2i}(v^1)^{n-4i-1}\,\middle|\,0\leqq i\leqq\left[\frac{n}{4}\right]\right\}$$

If $n$ is odd, we have homomorphisms $\Psi^*$, $\Phi^*$: $\mathcal{N}_n \to \mathbb{Z}/(2)[\frac{n}{4}]$, given by

$$\Phi^*(X) = \left\{ \Phi(Y^{(n-3i)}) \middle| 0 \leqq i \leqq \left[\frac{n}{4}\right] - 1 \right\}$$

$$\Psi^*(X) = \left\{ v^{2i} v^{2i} (v^1)^{n-4i-1} \middle| 1 \leqq i \leqq \left[\frac{n}{4}\right] \right\}$$

THEOREM B. *The homomorphisms $\Psi^*$, $\mathscr{X}^*$, and $\Phi^*$ are isomorphisms.*

10.6. *Stong exact sequences.* We shall use the symbol $\Omega_i^W(\mathbb{Z}/(2))$ to denote the cobordism group of locally orientable Witt spaces with $(v^1)^2 = 0$; it can be identified with the cobordism group of $\mathbb{Z}/(2)$-locally orientable Witt spaces (see 9.2).

PROPOSITION. *The association $X \to Y^{(2)}$ induces a surjective homomorphism*

$$\mathcal{N}_i^W \to \mathcal{N}_{i-2}^W$$

*with kernel $\Omega_i^W(\mathbb{Z}/(2))$. The resulting short exact sequence*

$$0 \to \Omega_i^W(\mathbb{Z}/(2)) \to \mathcal{N}_i^W \to \mathcal{N}_{i-2}^W \to 0$$

*is naturally split. Furthermore, the long exact Bockstein sequence*

$$\Omega_i^W \xrightarrow{\times 2} \Omega_i^W \to \Omega_i^W(\mathbb{Z}/(2)) \xrightarrow{\beta} \Omega_{i-1}^W \xrightarrow{\times 2} \Omega_{i-1}^W \to$$

*splits into short exact sequences.*

COROLLARY. *There is an isomorphism*

$$\mathcal{N}_i^W \cong \mathcal{N}_{i-2}^W \oplus \Omega_i^W \oplus \Omega_{i-1}^W$$

*Proof.* The proof is exactly the same as that in 9.6. It is only necessary to observe that the Witt-space condition is preserved during each cobordism.

10.7. *Proof of Theorem* 10.5.

*Part A.* If $X$ is an $n$-dimensional orientable, locally orientable Witt space and $n$ is odd, then $X = \partial(\text{cone}(X))$ because the cone$(X)$ is a locally orientable Witt space. If $n$ is even, then $X$ is a boundary if and only if $I\mathscr{X}(X) = 0$ by the argument of [37]. However, if $\dim(X) = 4k + 2$, then the number $I\mathscr{X}(X)$ is 0, because

$$I\mathscr{X}(X) = Sq^{2k+1}(v^{2k+1}) = \beta Sq^{2k}(v^{2k+1}) = v^1 Sq^{2k}(v^{2k+1}) = 0$$

(by Lemma 10.2) since $X$ is orientable. This leaves nonzero Euler characteristics only in dimension $n = 4k$, and in this case $\mathbb{CP}^{2k}$ are nontrivial representatives of the cobordism groups.

*Part B.* Next we consider the injectivity of the homomorphisms $\mathscr{X}^*$, $\Psi^*$, and $\Phi^*$ in the non-orientable case. We shall use induction on the dimension of $X$, and consider only the case that $n = \dim(X)$ is even, the case of odd $n$ being entirely parallel. We have already seen (10.4) that each characteristic number $v^{2i} v^{2i} (v^1)^{n-4i}$ can be written as a linear combination of the numbers $I\mathscr{X}(Y^{(n-4i)})$, so it suffices to show that the homomorphism $\mathscr{X}^*$ is injective. Suppose that $X$ is a locally orientable $n$-dimensional Witt space and that $\mathscr{X}^*(X) = 0$. Then (see 8.5(c)), $\mathscr{X}^*(Y^{(2)})=0$, where $Y^{(2)}$ denotes the second orientation cycle (8.3). By induction, $Y^{(2)}$ is cobordant to 0, so by the first Stong exact sequence, $X$ is cobordant to a $\mathbb{Z}/(2)$-Witt space $X'$ which admits an orientation cycle $Y'$ with trivial normal bundle. Now consider the second Stong sequence. The orientation cycle $Y'$ is orientable, and its (intersection homology) Euler characteristic is

$$I\mathscr{X}(Y')=v^k(Y')v^k(Y')=i^*(v^k(X')v^k(X')v^1(X'))$$

by Proposition 8.4(A), and this vanishes by proposition 10.4.B. Therefore $X'$ is cobordant to an orientable Witt space $X''$, with the same (mod 2 intersection homology) Euler characteristic. Thus, by Theorem 10.5 (part A), $X''$ is null cobordant.

The case of $n$ odd is entirely parallel. Surjectivity follows from (10.6).

## §11. LOCALLY SQUARE FREE SPACES

11.1. *Definition.* A pseudomanifold $X$ is an LSF space $\Leftrightarrow$

(a) for each stratum $S$ of odd codimension $c=2l+1$ we have

$$IH_l^{\bar{m}}(L;\mathbb{Z}/(2))=0$$

(b) for each stratum of even codimension $c=2l$, the operation

$$Sq^1\colon IH_l^{\bar{m}}(L;\mathbb{Z}/(2))\to IH_{l-1}^{\bar{m}}(L;\mathbb{Z}/(2))$$

vanishes, where $L$ denotes the link of the stratum $S$.

11.2. PROPOSITION. *Let $X$ be an $n$ dimensional LSF space. Then*

(a) *the Wu classes $v^i(X)$ lift canonically to $IH_{n-i}^{\bar{m}}(X\colon\mathbb{Z}/(2))$*

(b) *the Steenrod operation $Sq^1$ is defined as a homomorphism*

$$Sq^1\colon IH_i^{\bar{m}}(X;\mathbb{Z}/(2))\to IH_{i-1}^{\bar{m}}(X;\mathbb{Z}/(2))$$

(c) *there exists a Pontrjagin square operation*

$$\mathscr{P}_2\colon IH_i^{\bar{m}}(X;\mathbb{Z}/(2))\to H_{2i-n}(X;\mathbb{Z}/(4))$$

*with the following properties:*

(i) $\mathscr{P}_2(x+y)=\mathscr{P}_2(x)+\mathscr{P}_2(y)+2x.y$

(ii) $\mathscr{P}_2(x)\equiv x.x\,(\text{mod } 2)$

(iii) *if $i\colon D\to X$ is the normally nonsingular inclusion of a submanifold with trivial normal bundle, then*

$$\mathscr{P}_2 i_*(x)=2i_*(x.Sq^1(x))$$

*Proof.* Parts (a) and (b) follow from §5 and §6. We will give an outline of two different approaches to the Pontrjagin square, each of which is quite complicated.

*Approach* 1. is to make sense of the usual formula

$$\mathscr{P}_2(x)=x.x+x\cup_1\partial x \quad (\text{mod } 4)$$

using Deligne's sheaf theoretic construction of intersection homology. The cup-1 product can be defined on $\mathbb{Z}/(4)$ chains (even on integral chains), following the procedure in [15]. In order to form the product $x \cup_1 \partial x$ it is necessary to find cochain complexes for the intersection chains such that the (mod 4) boundary of a (mod 2) chain is naturally a chain in $IC(X; \mathbb{Z}/(4))$. This can be done precisely if $X$ is an LSF space, and in fact it is possible to find a short exact sequence of chain complexes

$$0 \to IC^{\bar{m}}(X; \mathbb{Z}/(2)) \to IC^{\bar{m}}(X; \mathbb{Z}/(4)) \to IC^{\bar{m}}(X; \mathbb{Z}/(2)) \to 0$$

which compute the intersection homology. With this construction of $\mathscr{P}_2$, the proofs of (i) and (ii) are standard, while (iii) is a restatement of the fact ([44], [46], [47]) that the suspension of the Pontrjagin square is the Postnikov square. A complete proof of (iii) therefore consists of repeating the proof in [44], using the intersection chains instead of cellular chains, but the argument goes through without any changes.

*Approach* 2. is to define the operation $\mathscr{P}_2$ only on the group $IH_k(X; \mathbb{Z}/(2))$, when $X$ is an $n=2k$ dimensional pseudomanifold. (This is the only time we will ever use the Pontrjagin square). We first define the operation on $\ker(Sq^k)$ using property (iii), i.e. if $x^2=0$ then for any P. L. cycle representative $A$ of $x$, there is a class $x' \in IH^{\bar{m}}_{2k}(\partial N; \mathbb{Z}/(2))$ such that $i_*(x')=x$, where $i: \partial N \to X$ denotes the inclusion of the boundary of a regular neighborhood of $A$. Define $\mathscr{P}_2(x)=2x'Sq^1(x') \in \mathbb{Z}/(4)$. Since $X$ is a LSF space, $\partial N$ is also, and so the product $x'.Sq^1(x')$ may be formed in intersection homology. It is easy to see that this operation is well defined: any two cycle representatives $A_0$ and $A_1$ of $x$ are homologous by a chain $B \subset X \times [0, 1]$ such that $\partial B = A_1 \times \{1\} - A_0 \times \{0\}$. It is possible to choose a regular neighborhood $\tilde{N}$ of $B$ in $X \times [0, 1]$ which restricts to given neighborhoods $N_i$ of $A_i$ in $X \times \{i\}$ ($i=0, 1$). The relative homology class $[B] \in IH_{k+1}(X \times [0, 1])$ comes from a class $[B'] \in IH_{k+1}(\partial N, (\partial N_0) \cup (\partial N_1))$, so the chain $B'.Sq^1(B')$ defines a homology between the corresponding numbers $x'_0.Sq^1(x'_0)$ and $x'_1.Sq^1(x'_1)$. We omit the rather complicated geometric construction which is needed to show that $\mathscr{P}_2(x+y)=\mathscr{P}_2(x)+\mathscr{P}_2(y)+2x.y$. It is now possible to find a (noncanonical) extension of $\mathscr{P}_2$ to all of $IH_k(X)$ as follows: if $K=\ker(Sq^k) = IH^{\bar{m}}_k(X)$ then there is nothing left to do. Otherwise, choose a 1 dimensional complement $L \subset IH^{\bar{m}}_k(X; \mathbb{Z}/(2))$. The subspace $L$ contains a unique nonzero element $l \in L$, $l.l=1$ so we may choose a value of 1 or 3 for $\mathscr{P}_2(l)$. This gives projections $\pi_1: IH^{\bar{m}}_k(X) \to K$ and $\pi_2: IH^{\bar{m}}_k(X) \to L$, so we may define

$$\mathscr{P}_2(x)=\mathscr{P}_2(\pi_1(x))+\mathscr{P}_2(\pi_2(x))+\pi_1(x).\pi_2(x) \pmod 4.$$

It is easy to see that this function satisfies (i), and (ii) above. Property (iii) also follows from the argument above: if $x \in IH^{\bar{m}}_k(D)$ then $i_*(x) \in K$ and the regular neighborhood $N$ of a cycle representative of $i_*(x)$ may be chosen in the form $N=N' \times [0, 1]$, where $N'$ is a regular neighborhood in $D$ of a cycle representative of $x \in IH^{\bar{m}}_k(D)$.

*Remarks.* By making the other choice for $\mathscr{P}_2(l)$ we would obtain an operation $\mathscr{P}'_2$ which satisfies (i), (ii), and (iii) but which differs from $\mathscr{P}_2$ by an automorphism of $\mathbb{Z}/(4)$. The advantage of the more abstract sheaf approach is that no choices are involved in the construction of the Pontrjagin square.

If $X$ is an LSF space, then $Sq^1$ is defined as an operation on $IH^{\bar{m}}_*(X; \mathbb{Z}/(2))$. If $X$ is also locally orientable then $Sq^1$ is also defined as an operation on $H_*(X; \mathbb{Z}/(2))$. These two operations commute with the homomorphism $IH^{\bar{m}}_* \to H_*(X)$, but there may not exist $Sq^1$ operations which *preserve* the other intersection homology groups $IH^{\bar{p}}_*(X)$ for $\bar{m} < \bar{p} < \bar{t}$ (although, as remarked in §5, there always exists a $Sq^1$ operation $IH^{\bar{p}}(X) \to IH^{\bar{q}}(X)$ where $\bar{q}(c)=\min(\bar{p}(c)+1, 2\bar{p}(c))$.

## §12. CHARACTERISTIC NUMBERS AND INVARIANTS OF ORIENTABLE LSF SPACES

12.1. *The invariants.*

Suppose $X^n$ is an orientable LSF space. Then the following cobordism invariant characteristic numbers may be formed: $v^i v^{n-i}$, $v^i Sq^1(v^{n-i-1})$, $Sq^1(v^i)\, Sq^1(v^{n-i-2})$.

PROPOSITION. *If $X^n$ is an orientable* LSF*-space, then the following relations hold between the above characteristic numbers:*

(1) $$Sq^1(v^i)\, Sq^1(v^{n-i-2})=0$$

(2) $$v^i v^{n-i}=0 \text{ unless } n=2i$$

(3) $$v^i Sq^1(v^{n-i-1})=0 \text{ unless } n=4k+1 \text{ and } i=n-i-1=2k$$

*Proof.* Compute $Sq^1(v^i)Sq^1(v^{n-i-2})=Sq^1(v^i.Sq^1(v^{n-i-2}))=v^1.v^i.Sq^1(v^{n-i-2})=0$ (since $X$ is orientable). Since $v^i=0$ for $i>n/2$ we obtain (2). Similarly characteristic numbers of type (3) can only occur if $i$ and $n-i-1$ differ by at most one, which leaves the possibilities $v^i Sq^1(v^i)$ in a $2i+1$ dimensional space, and $v^{i-1}Sq^1(v^i)=v^i Sq^1(v^{i-1})$ in a $2i$-dimensional space. These last numbers vanish since $v^i Sq^1 v^{i-1}=Sq^i(Sq^1 v^{i-1})=(Sq^1 v^{i-1})^2=0$ by argument (1). If $X$ is $4k+3$ dimensional then $v^{2k+1}Sq^1 v^{2k+1}=Sq^1 Sq^{2k} Sq^1 v^{2k+1}=v^1.Sq^{2k}Sq^1 v^{2k+1}=0$ since $X$ is orientable (cf. Corollary 10.2).

This leaves two interesting characteristic numbers: $v^k v^k=I\mathscr{X}(X)$ (mod 2) for $X^{2k}$ and $v^{2k}Sq^1 v^{2k}$ for $X^{4k+1}$. If $X$ is $4k$ dimensional, the Pontrjagin square gives rise to a further $\mathbb{Z}/(2)$ extension of the mod 2 Euler characteristic, as follows:

*Definition.* Let $H$ be a $\mathbb{Z}/(2)$ vectorspace with a nonsingular $\mathbb{Z}/(2)$-valued inner product $\langle\, ,\rangle$ and a function $\mathscr{P}$: $H\to\mathbb{Z}/(4)$ which satisfies

$$\mathscr{P}(x+y)=\mathscr{P}(x)+\mathscr{P}(y)+2\langle x, y\rangle$$

(which implies $\mathscr{P}(x)\equiv\langle x, x\rangle$ (mod 2)).

Define the *Pontrjagin invariant*

$$\pi(H)=\frac{1}{2\pi}\arg\sum_{x\in H}\exp\,(i\pi\mathscr{P}(x)/2)\in\mathbb{Z}/(4)$$

This sum is a fourth root of unity ([34], [32], [8]) and it is easy to see that $\dim(H)\equiv\pi(H)$ (mod 2). Therefore the cobordism invariant $\pi(X):=\pi(IH^{\bar{m}}_{2k}(X;\mathbb{Z}/(2)))$ is a $\mathbb{Z}/(2)$-extension of the invariant $I\mathscr{X}(X;\mathbb{Z}/(2))$. If $X$ is a manifold then $\pi(X)$ is the signature mod 4.

12.2. *Vanishing of the semicharacteristic*

There is some interesting algebra associated to the characteristic number $v^{2k}Sq^1 v^{2k}$; in particular (see [24])

PROPOSITION. *Suppose $X$ is a* $4k+1$ *dimensional orientable LSF space. Then the following four statements are equivalent:*

(1) $v^{2k}Sq^1(v^{2k})=0$

(2) *For all* $\xi\in IH^{\bar{m}}_{2k+1}(X)$, $Sq^{2k}Sq^1(\xi)=\xi.Sq^1(\xi)$

(3) *The rank of* $Sq^1$: $IH^{\bar{m}}_{2k+1}(X;\mathbb{Z}/(2))\to IH^{\bar{m}}_{2k}(X;\mathbb{Z}/(2))$ *is even.*

*Proof.* The same proof as in [24] works, provided we can establish the identity

$$\xi.Sq^1(\xi)=v^{2k}(X).Sq^1(\xi)$$

for all $\xi \in IH^{\bar{m}}_{2k+1}(X;\mathbb{Z}/(2))$. See [48] (or [6], [7]) for a proof in ordinary cohomology. We now give a direct geometric proof which works intersection homology. Let $x$ be an oriented P.L. chain which is a (mod 2) cycle such that $[x]=\xi$ and $[\partial x]=2.Sq^1(\xi)$, and let $y=\frac{1}{2}\partial x$. Choose a regular neighborhood $N$ of $\partial x$ and let $i$: $\partial N \to X$ denote the inclusion of the boundary of $N$ into $X$. We may assume that $\partial N$ is transverse to $x$. Thus $x \cap (X-N)$ is an orientable P.L. relative cycle in $IC^{\bar{m}}_{2k+1}(X-N, \partial N)$which represents a class $[x] \in IH^{\bar{m}}_{k+1}(X-N, \partial N; \mathbb{Z})$. Furthermore the class $[y] \in IH^{\bar{m}}_k(N;\mathbb{Z})$ pulls back to a class $[y'] \in IH^{\bar{m}}_k(\partial N; \mathbb{Z})$: just take $y'=r_*(y'')$ where $r$: $N-|y| \to \partial N$ is the retraction to the boundary, and $y''$ is any chain representative of $[y]$ which is dimensionally transverse [18] to $y$, i.e. such that $y \cap y'' = \phi$. If $\partial_*$: $IH^{\bar{m}}_{k+1}(X-N, \partial N; \mathbb{Z}) \to IH^{\bar{m}}_k(\partial N; \mathbb{Z})$ we have

$$\partial_*([x])=[x \cap \partial N]-2[y']$$

so

$$0=(\partial_*[x])^2=[x \cap \partial N]^2-4[x \cap \partial N].[y']+4[y']^2 \in \mathbb{Z}$$

However, $[x \cap \partial N]^2=0$ because a cycle representative $x'$ of $x$ may be chosen so that $x'$ is dimensionally transverse to $y$, and this implies $x' \cap N$ consists of finitely many transverse disks $D'_1 \cup D'_2 \cup \ldots \cup D'_r$ centered at nonsingular points $p'_1, p'_2, \ldots, p'_r$ of $y$. The boundaries $\partial D'_1 + \ldots + \partial D'_r$ represent the class $x \cap \partial N \in IH^{\bar{m}}_{2k}(\partial N; \mathbb{Z})$. A third cycle representative $x''$ of $x$ may be chosen so that the corresponding points of intersection $p''_1, \ldots, p''_s$ are disjoint from the points $p'_1, \ldots, p'_r$, and therefore the corresponding transverse disks $D''_i$ are disjoint from the disks $D'_j$. Thus

$$[x \cap \partial N]^2=[x' \cap \partial N].[x'' \cap \partial N]=[\phi]=0.$$

Finally, dividing by 4 and reducing mod 2 we have

$$Sq^{2k}Sq^1(\xi)=[y']^2=[x \cap \partial N].[y']=i^*(\xi).[y']=\xi.i_*(y')=\xi.Sq^1(\xi),$$

as desired.

12.3. *Vanishing of the Pontrjagin invariant.*

*Definition.* Let $H$ be a $\mathbb{Z}/(2)$-vectorspace with inner product $\langle , \rangle$ and a $\mathbb{Z}/(4)$-valued function $\mathscr{P}$ as in §11. We shall say that $H$ is hyperbolic if there exists a subspace $K \subset H$ such that $\mathscr{P}|K=0$ and $\dim(K)=\frac{1}{2}\dim(H)$.

PROPOSITION. *A $\mathbb{Z}/(2)$-vectorspace $H$ together with quadratic function $\mathscr{P}$ is hyperbolic if and only if its Pontrjagin invariant $\pi(H)$ vanishes. In fact, the $\mathbb{Z}/(4)$ invariant $\pi$ classifies triples $(H, \langle .,. \rangle, \mathscr{P})$ (which satisfy* 11.2(c) (i,ii)), *up to the orthogonal direct sum with hyperbolic spaces.*

*Proof.* The invariant $\pi$ is additive with respect to orthogonal direct sums. If $H$ is one dimensional with $x.x=1$ then $\mathscr{P}(x)=1$ or 3, so $\pi(H)=1$ or 3. Now suppose $\pi(H)=0$. We shall show that $H$ is hyperbolic. It is easy to see that $\dim(H)$ is congruent to $\pi(H)$ (mod 2). (see [33], [30], [8]), so $\dim(H)$ is even and there is a basis such that $H$ decomposes as an orthogonal sum of 2-dimensional subspaces with intersection matrix given by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (which we call type I) or $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ (which we call type II). Since $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ we can further assume that at most one factor of type II occurs. Since $\pi(H)$ is the sum of the Pontrjagin invariants of the factors, we consider each factor separately:

Let $A$ denote a factor of type I. For each $u \in A$, we have $\mathscr{P}(u)=0$ or 2. If $\mathscr{P}(u)=0$ for some nontrivial $u \in A$, then $A$ is hyperbolic. Furthermore, the orthogonal sum of two non-hyperbolic factors of type I is again hyperbolic, and the orthogonal sum of a nonhyperbolic type I factor with a nonhyperbolic type II factor is again hyperbolic. Thus we may assume there is at most a single nonhyperbolic factor, $B$, which we now examine. If $\mathscr{P}$ vanishes on some nonzero element $u \in B$, then $B$ is hyperbolic. Otherwise, it can be seen by direct computation that if $\mathscr{P}$ does not vanish on any (nonzero) element $u$ in this factor, then $\pi(B) \neq 0$.

## §13. COBORDISM OF LSF SPACES

13.1. *Statement of Results.*

THEOREM. *The cobordism groups of orientable* LSF *spaces are*

$$\Omega_i^{\mathrm{LSF}} = \begin{cases} \mathbb{Z}/(2) & \text{if } i=4k+1, \quad k \geq 1 \\ \mathbb{Z}/(4) & \text{if } i=4k, \quad k \geq 1 \\ 0 & \text{otherwise}, \quad i>0 \end{cases}$$

*where the nonzero invariants are given by* $v^{2k}Sq^1(v^{2k})$, *and the Pontrjagin invariant* $\pi(IH^{\bar m}_{2k}(X;\mathbb{Z}/2))$.

The proof appears in §15.

13.2. *Preparing a cycle for surgery.*

LEMMA A. *Suppose $X$ is a $2k$-dimensional LSF space. Let $\xi \in IH^{\bar m}_k(X;\mathbb{Z}/(2))$ be a class such that $Sq^1(\xi)=0$ and $\xi.\xi=0$. Then $\xi$ has a representative P.L. cycle $|\xi| \subset X$ with a regular neighborhood $N(\xi)$ such that $IH^{\bar m}_j(N;\mathbb{Z}/(2))=0$ for $j \geqslant k+1$, $IH^{\bar m}_k(N;\mathbb{Z}/(2))=\mathbb{Z}/(2)$ and such that the operation*

$$Sq^1\colon IH^{\bar m}_k(\partial N;\mathbb{Z}/(2)) \to IH^{\bar m}_{k-1}(\partial N;\mathbb{Z}/(2))$$

*vanishes.*

*Proof.* See [37] or 17.3 for the computation of the intersection homology of $N$. From the Bockstein sequence (§6) we see that the mod 2 class $\xi$ admits a P.L. cycle representative $|\xi|$ with no boundary (mod 4), since $Sq^1(\xi)=0$. Consequently the operation $Sq^1$: $IH^{\bar m}_k(N;\mathbb{Z}/(2)) \to IH^{\bar m}_{k-1}(N;\mathbb{Z}/(2))$ vanishes. Since $\xi.\xi=0$, the long exact sequence for the pair $IH^{\bar m}_*(N,\partial N;\mathbb{Z}/(2))$ breaks into dually paired short exact sequences,

$$\begin{array}{ccccccccc} 0 & \to & IH^{\bar m}_{k+1}(N,\partial) & \xrightarrow{\partial_*} & IH^{\bar m}_k(\partial) & \xrightarrow{i_*} & IH^{\bar m}_k(N) & \longrightarrow & 0 \\ & & \downarrow Sq^1 & & \downarrow Sq^1 & & \downarrow Sq^1 & & \\ 0 & \longrightarrow & IH^{\bar m}_k(N,\partial) & \xrightarrow{\partial_*} & IH^{\bar m}_{k-1}(\partial) & \xrightarrow{i_*} & IH^{\bar m}_{k-1}(N) & \longrightarrow & 0 \end{array}$$

For any $a \in IH^{\bar m}_{k+1}(N,\partial;\mathbb{Z}/(2))$ we have $Sq^1\partial_*(a)=0$ because, for any $b \in IH^{\bar m}_k(\partial N)$, $(Sq^1\partial_*(a)).b=(\partial_*Sq^1(a)).b=a.Sq^1i_*(b)+Sq^1(a.i_*b)=a.0+v^1(X).a.i_*(b)=0$ since $X$ is orientable. Therefore the rank of $Sq^1\colon IH^{\bar m}_k(\partial N) \to IH^{\bar m}_{k-1}(\partial N)$ is at most one. If $k$ is odd then dim $(\partial N)=4l+1$ and since $\partial N$ is the boundary of an LSF space $N$ we have $v^k(\partial N)\,Sq^1v^k(\partial N)=0$, so by 12.2, the rank of $Sq^1$ is even. This implies $Sq^1$ vanishes. If $k$ is even then for any class

$\xi' \in IH_k^{\bar{m}}(\partial N)$ such that $i_*(\xi') = \xi$, $Sq^1(\xi').\partial_*(c) = 0$ whenever $c \in IH_{k+1}^{\bar{m}}(N, \partial N)$. So it suffices to show that $Sq^1(\xi').\xi' = 0$. However by 11.2(c) (iii) we have $0 = \mathscr{P}_2(\xi) = \mathscr{P}_2(i_*(\xi')) = 2i_*(\xi'.Sq^1(\xi')) \in \mathbb{Z}/(4)$, which implies $\xi'.Sq^1(\xi') = 0$ as claimed.

LEMMA B. *Suppose $X$ is a $2k+1$ dimensional LSF space and $\xi \in IH_k^{\bar{m}}(X; \mathbb{Z}/(2))$ is in the kernel of $Sq^1$. Then $\xi$ has a P.L. cycle representative $|\xi|$ with a regular neighborhood $N(\xi)$ such that $IH_k^{\bar{m}}(N; \mathbb{Z}/(2)) = \mathbb{Z}/(2)$, $IH_k^{\bar{m}}(\partial N; \mathbb{Z}/(2)) = \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$, and such that the operation*

$$Sq^1 : IH_k^{\bar{m}}(\partial N; \mathbb{Z}/(2)) \to IH_{k-1}^{\bar{m}}(\partial N; \mathbb{Z}/(2))$$

*vanishes.*

*Proof.* As in Lemma A, we may choose $|\xi|$ so that $Sq^1$ vanishes on $IH_k^{\bar{m}}(N(\xi)) \cong \mathbb{Z}/(2)$, and as usual $IH_{k+1}^{\bar{m}}(N) = 0$. By Poincaré duality, the long exact sequence for the pair $IH_*^{\bar{m}}(N, \partial N)$ breaks into short exact sequences. For any $a \in IH_k^{\bar{m}}(\partial N)$ and for any $b \in IH_{k+1}^{\bar{m}}(\partial N)$ we have $Sq^1(a).b = Sq^1(a).\partial_*(b')$ for some $b' \in IH_{k+2}^{\bar{m}}(N, \partial N)$ (because $\partial_*$ is surjective). Thus, $Sq^1(a).b = (i_* Sq^1(a)).b' = (Sq^1(i_* a)).b' = 0$ (by 4.2). By Poincaré duality, this implies $Sq^1(a) = 0$.

## §14. SURGERY IN THREE SPECIAL CASES

Throughout this chapter, $\mathbb{Z}/(2)$ coefficients will be assumed.

14.1. PROPOSITION. *Suppose $X$ is a $2k$-dimensional orientable LSF space, $\xi \in IH_k^{\bar{m}}(X; \mathbb{Z}/(2))$, and $Sq^1(\xi) = 0$. If $k$ is odd, assume that $\xi \cdot \xi = Sq^k(\xi) = 0$; if $k$ is even then assume that $\mathscr{P}_2(\xi) = 0$ (which implies that $\xi \cdot \xi = 0$). Then $X$ is LSF-cobordant to an orientable LSF space $X'$ such that*

$$IH_j^{\bar{m}}(X'; \mathbb{Z}/(2)) = \begin{cases} IH_j^{\bar{m}}(X) & \text{for } j \neq k \\ \xi^{\perp}/\langle \xi \rangle & \text{for } j = k \end{cases}$$

*and in particular, $\dim IH_k^{\bar{m}}(X') = \dim IH_k^{\bar{m}}(X) - 2$. Furthermore, if $IH_k^{\bar{m}}(X) = \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ then the cobordism $Z$ between $X$ and $X'$ may be chosen so that the space*

$$Y = Z \cup_{X'} \text{cone } (X')$$

*is an LSF-space which satisfies $X = \partial Y$ and*

$$IH_j^{\bar{m}}(Y; \mathbb{Z}/(2)) = \begin{cases} 0 & (j \geq k+1) \\ IH_k^{\bar{m}}(X)/\langle \xi \rangle & (j = k) \\ IH_j^{\bar{m}}(X) & (j \leq k-1) \end{cases}$$

*Proof.* We will perform surgery on the class $\xi$. By Lemma A (13.2) there is a cycle representative $|\xi|$ of $\xi$ with a regular neighborhood $N$ such that $Sq^1$ vanishes on $IH_k^{\bar{m}}(\partial N)$. Thus the space $X' = (X - N) \cup_{\partial N} \text{cone}(\partial N)$ is an $LSF$-space. We claim that the cobordism

$$Z = X \times [0, 1] \cup_{N \times \{1\}} \text{cone}(N \cup_{\partial} c(\partial))$$

is an allowable $LSF$-cobordism between $X$ and $X'$ which satisfies the conditions above (and where we abbreviate $\partial N$ by $\partial$ and cone$(\partial N)$ by $c(\partial)$).

*Z is allowable*: It suffices to check that the new singularity $c(N \cup_{\partial} c(\partial))$ is an $LSF$-allowable singularity, i.e. that $IH_k^{\bar{m}}(N \cup_{\partial} c(\partial); \mathbb{Z}/(2)) = 0$. In fact, using the long exact sequence for the pair $IH_*^{\bar{m}}(N, \partial)$, the fact that $\xi \cdot \xi = 0$, and the Mayer Vietoris sequence for the union $N \cup c(\partial)$, it is easy to see that

$$IH_j^{\bar{m}}(N \cup_\partial c(\partial); \mathbb{Z}/(2)) = \begin{cases} IH_j^{\bar{m}}(N, \partial) & \text{for } j > k \\ 0 & \text{for } j = k \\ IH_j^{\bar{m}}(N) & \text{for } j < k \end{cases}$$

*Computation of* $IH_k^{\bar{m}}(X')$: (This is the usual surgery calculation as in [37], [29].) Recall [18], [37] that $IH_k^{\bar{m}}(X') \cong \text{Image } (IH_k^{\bar{m}}(X-N) \to IH_k^{\bar{m}}(X-N, \partial N))$, and that this map factors

$$\begin{array}{ccccc} & & IH_k^{\bar{m}}(N) & & \\ & & \downarrow & & \\ IH_k^{\bar{m}}(X-N) & \to & IH_k^{\bar{m}}(X) & \xrightarrow{f} & IH_k^{\bar{m}}(X, X-N) \cong \mathbb{Z}/(2) \\ & \searrow^{I} & \downarrow^{g} & & \\ & & IH_k^{\bar{m}}(X, N) \cong IH_k^{\bar{m}}(X-N, \partial N) & & \end{array}$$

where $f(y) = \xi \cdot y$ so $\ker(f) = \xi^\perp$. Therefore $\text{Image}(I) = g(\ker(f)) = \xi^\perp/\langle \xi \rangle$.

*Computation of* $IH_*^{\bar{m}}(Z)$ *and* $IH_j^{\bar{m}}(X')$ $(j \neq k)$. By comparing the Mayer Vietoris sequences for the unions $X = (X-N) \cup_\partial N$, and $X' = (X-N) \cup_\partial c(\partial)$, (noting that $f$ is surjective), and by comparing the Mayer Vietoris sequences for the union

$$Z = (X \times I) \cup_{N \times \{1\}} c(N \cup_\partial c(\partial)) = (X' \times I) \cup_{c(\partial) \times \{0\}} c(N \cup_\partial c(\partial))$$

it is easy to see that, for $j \neq k$ we have

$$IH_j^{\bar{m}}(X) \cong IH_j^{\bar{m}}(X') \cong IH_j^{\bar{m}}(Z) \cong \begin{cases} IH_j^{\bar{m}}(X, N) & \text{for } j \geq k+1 \\ IH_j^{\bar{m}}(X-N) & \text{for } j \leq k-1 \end{cases}$$

and

$$IH_k^{\bar{m}}(Z) \cong IH_k^{\bar{m}}(X)/\langle \xi \rangle$$

*Computation of* $IH_*^{\bar{m}}(Y)$: Use the Mayer Vietoris sequence for the union

$$Y = Z \cup_{X'} c(X')$$

and the computations above, to verify that

$$IH_j^{\bar{m}}(Y) = \begin{cases} 0 & \text{if } j \geq k+1 \\ IH_j^{\bar{m}}(Z) & \text{if } j \leq k \end{cases}$$

14.2. PROPOSITION. *Suppose $X$ is a $k+1$ dimensional orientable LSF-space, $\xi \in IH_k^{\bar{m}}(X; \mathbb{Z}/(2))$, $Sq^k(\xi) = 0$ and $Sq^1(\xi) = 0$. Then $X$ is LSF-cobordant to an orientable LSF space $X'$ such that $IH_{k+1}^{\bar{m}}(X') = \xi^\perp$ and $IH_k^{\bar{m}}(X') = IH_k^{\bar{m}}(X)/\langle \xi \rangle$. If $IH_k^{\bar{m}}(X)$ was one dimensional, then $IH_k^{\bar{m}}(X') = 0$ and the cobordism $Z$ between $X$ and $X'$ may be chosen so that the space*

$$W = Z \cup_{X'} c(X')$$

*is an orientable LSF-space with $\partial W = X$ and*

$$IH_j^{\bar{m}}(W) = \begin{cases} 0 & \textit{for } j \geq k+2 \\ \mathbb{Z}/(2) & \textit{for } j = k+1 \\ 0 & \textit{for } j = k \\ IH_j^{\bar{m}}(X) & \textit{for } j \leq k-1 \end{cases}$$

*Proof.* We will perform surgery on $\xi$. By Lemma B (13.2) there is a cycle representative $|\xi|$ of $\xi$ with a regular neighborhood $N$ such that $IH_k^{\bar{m}}(N; \mathbb{Z}/(2)) \cong \mathbb{Z}/(2))$ and such that $Sq^1$ vanishes on $IH_k^{\bar{m}}(\partial N) \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$. Furthermore (see 4.3) the class $\xi$ pulls back to a class $\xi' \in IH_k^{\bar{m}}(\partial N)$ such that $\xi' \cdot \xi' = 0$. If $k$ is even, the class $\xi'$ may be chosen so that $\mathscr{P}(\xi') = 0 \in \mathbb{Z}/(4)$: since $\xi' \cdot \xi' = 0$ either $\mathscr{P}(\xi') = 0$ or 2. In the latter case we can replace $\xi'$ with the class $\xi' + \xi^*$ where $\xi^* \in IH_i^{\bar{m}}(\partial N)$ is the class represented by the boundary of a disk which is perpendicular to a smooth point of $|\xi|$. Then $\xi' \cdot \xi^* = 1$ and $\mathscr{P}(\xi^*) = 0$, so $\mathscr{P}(\xi' + \xi^*) = \mathscr{P}(\xi') + \mathscr{P}(\xi^*) + 2\xi' \cdot \xi^* = 4 = 0 \pmod 4$.

Applying the special case 14.1 to $\partial N$ we can find a particular null cobordism $Y$ of $\partial N$. Thus the space $X' = (X - N) \cup_{\partial N} Y$ is an LSF-space. We claim that $IH_k^{\bar{m}}(X') = IH_k^{\bar{m}}(X)/\langle \xi \rangle$ and that the space

$$Z = (X \times I) \cup_{N \times \{1\}} \operatorname{cone}(N \cup_\partial Y)$$

is an LSF-allowable cobordism between $X$ and $X'$ which satisfies the conditions in the statement of proposition 14.2.

*Z is allowable*: It suffices to check that cone$(N \cup_\partial Y)$ is an LSF-allowable singularity, i.e. that $Sq^1$ vanishes on $IH_{k+1}^{\bar{m}}(N \cup_\partial Y)$. We will show that $IH_{k+1}^{\bar{m}}(N \cup_\partial Y)$ is actually 0. An examination of the long exact sequences for the pairs $IH_*^{\bar{m}}(X, N)$ and $IH_*^{\bar{m}}(X, X - N)$ reveals

$$IH_j^{\bar{m}}(X) = \begin{cases} IH_j^{\bar{m}}(X, N) & \text{for } j \geq k+1 \\ IH_j^{\bar{m}}(X - N) & \text{for } j \leq k \end{cases}$$

This fact, together with the computation of $IH_*^{\bar{m}}(Y)$ (14.1) fit into the long exact sequence for the pair $IH_*^{\bar{m}}(N, \partial N)$ and the Mayer Vietoris sequence for the union $IH_k^{\bar{m}}(N \cup_\partial Y)$ giving

$$IH_j^{\bar{m}}(N \cup_\partial Y) = \begin{cases} IH_{j-1}^{\bar{m}}(\partial N) & j \geq k+2 \\ 0 & j = k,\ k+1 \\ IH_j^{\bar{m}}(N) & j \leq k-1 \end{cases}$$

*Calculation of $IH_*^{\bar{m}}(X')$*: Consider the long exact sequence for the pair

$$\begin{array}{ccccccc} IH_{k+2}^{\bar{m}}(X, X-N) & \to & IH_{k+1}^{\bar{m}}(X-N) & \to & IH_{k+1}^{\bar{m}}(X) & \xrightarrow{\alpha} & IH_{k+1}^{\bar{m}}(X, X-N) \\ \| & & \uparrow \beta & & & & \| \\ IH_{k+2}^{\bar{m}}(N, \partial N) & \xrightarrow{\cong} & IH_{k+1}^{\bar{m}}(\partial N) & & & & IH_{k+1}^{\bar{m}}(N, \partial N) \end{array}$$

The map $\alpha$ is given by multiplication with $\xi$. Therefore $\xi^\perp = \operatorname{coker}(\beta)$. Use this fact in the Mayer Vietoris sequences for the unions $X = (X - N) \cup_\partial N$ and $X' = (X - N) \cup_\partial Y$ to compute

$$IH_j^{\bar{m}}(X') = \begin{cases} IH_j^{\bar{m}}(X) & \text{for } j \geq k+2 \\ \xi^\perp & \text{for } j = k+1 \\ IH_j^{\bar{m}}(X)/\langle \xi \rangle & \text{for } j = k \\ IH_k^{\bar{m}}(X) & \text{for } j \leq k-1 \end{cases}$$

*Computation of $IH_*^{\bar{m}}(Z)$*: Using the Mayer Vietoris sequence for the union

$$Z = (X \times I) \cup_{N \times \{1\}} c(N \cup_\partial Y) = (X' \times I) \cup_{Y \times \{0\}} c(N \cup_\partial Y)$$

it is easy to compute

$$IH_j^{\bar{m}}(Z) = \begin{cases} IH_j^{\bar{m}}(X) & \text{for } j \geq k+2 \\ IH_k^{\bar{m}}(Y) = IH_k^{\bar{m}}(\partial N)/\langle \xi \rangle & \text{for } j = k+1 \\ 0 & \text{for } j = k \\ IH_j^{\bar{m}}(X) & \text{for } j \leqq k-1 \end{cases}$$

*Computation of* $IH_*^{\bar{m}}(W)$: Use the Mayer Vietoris sequence for the union

$$W = Z \cup_{X'} \text{cone}(X')$$

to find

$$IH_j^{\bar{m}}(W) = \begin{cases} 0 & \text{for } j \geq k+2 \\ IH_j^{\bar{m}}(Z) & \text{for } j \leq k+1 \end{cases}$$

14.3. Proposition. *Suppose $X$ is an orientable $2k+2$ dimensional LSF-space and $\xi \in IH_k^{\bar{m}}$ $(X;\ \mathbb{Z}/(2))$. Suppose $Sq^k(\xi) = 0$ and $Sq^1(\xi) = 0$. Then $X$ is LSF-cobordant to an orientable LSF-space $X'$ such that $IH_k^{\bar{m}}(X') = IH_k^{\bar{m}}(X)/\langle \xi \rangle$, together with the induced operations of $Sq^k$ and $Sq^1$.*

*Proof.* As in Lemma A (13.2), $\xi$ may be represented by a cycle $|\xi|$ with a regular neighborhood $N$ in $X$, such that $IH_k^{\bar{m}}(N;\ \mathbb{Z}/(2)) = \mathbb{Z}/(2)$ and

$$IH_j^{\bar{m}}(\partial N) = \begin{cases} IH_{j+1}^{\bar{m}}(N, \partial N) & \text{for } j \geq k+1 \\ IH_j^{\bar{m}}(N) & \text{for } j \leq k \end{cases}$$

and so that both $Sq^1$ and $Sq^k$ vanish on $IH_k^{\bar{m}}(\partial N;\ \mathbb{Z}/(2)) = \langle \xi \rangle$.

*Remark.* The argument in Lemma A (13.2) applies in this case to show that $Sq^1$ vanishes on $IH_{k+1}^{\bar{m}}(\partial N)$ so we could try to perform surgery on $\xi$ by replacing $N$ with cone($\partial N$). Although this operation is LSF-allowable, it will not kill the class $\xi$.

Define $X' = (X - N) \cup_{\partial N} W$, where $W$ is the null cobordism of $\partial W$ from 14.2. We will show that $X$ and $X'$ are cobordant, and that $X'$ has the required properties. The cobordism between them is

$$Z = (X \times I) \cup_{N \times \{1\}} \text{cone}(N \cup_{\partial N} W)$$

and to show it is $LSF$-allowable we must show that $IH_{k+1}^{\bar{m}}(N \cup W) = 0$. In fact, a simple calculation with the Mayer Vietoris sequence, together with the homology of $W$ as computed 14.2 gives

$$IH_j^{\bar{m}}(N \cup_\partial W) = \begin{cases} IH_{j-1}^{\bar{m}}(\partial N) & \text{for } j \geq k+3 \\ 0 & \text{for } j = k,\ k+1,\ k+2 \\ IH_j^{\bar{m}}(N) & \text{for } j \leq k-1 \end{cases}$$

Comparison of the Mayer Vietoris sequences for $X = (X - N) \cup_{\partial N} N$ and for $X' = (X - N) \cup_{\partial N} W$ gives

$$IH_j^{\bar{m}}(X') = \begin{cases} IH_j^{\bar{m}}(X) & \text{for } j \neq k \\ IH_j^{\bar{m}}(X)/\langle \xi \rangle & \text{for } j = k \end{cases}$$

Furthermore the operations $Sq^1$ and $Sq^k$ are left unchanged on any class $\eta \in IH_k^{\bar{m}}(X)$ which is different from $\xi$ because such classes admit cycle representatives $|\eta|$ such that $|\eta| \cap |\xi| = \phi$

by transversality. Therefore the geometric homology operations as applied to $|\eta|$ are unaffected by the surgery.

## §15. PROOF OF THEOREM 13.1

Each of the invariants $\pi$ and $v^kSq^1v^k$ has a nonzero value on some smooth orientable manifold $M$ (in fact, $\pi(M)$ is the signature modulo 4). Therefore we need only show that if $X$ is an orientable LSF-pseudomanifold and if the relevant invariant of $X$ vanishes, then $X$ is a boundary.

15.1. Proposition. *Suppose $X$ is an orientable $2k+1$ dimensional LSF-space and suppose $v^kSq^1v^k(X) = 0$ (which always happens if $k$ is odd, by* 12.2). *Then $X$ is a boundary.*

*Proof.* The proof proceeds in two steps.

*Step 1.* We will show that $X$ is cobordant to an orientable $LSF$-space $X'$ such that

$$(\operatorname{Ker} Sq^k)\cap(\operatorname{Im} Sq^1) = 0 \text{ in } IH_k^{\bar{m}}(X'; \mathbb{Z}/(2))$$

For if $\xi = Sq^1(\eta)\in IH_k^{\bar{m}}(X)$ and if $Sq^k(\xi) = 0$ then $Sq^1(\xi) = 0$ so we can apply proposition 14.2 to kill $\xi$ by surgery. This gives rise to a cobordant space $X'$ such that $IH_k^{\bar{m}}(X') \cong IH_k^{\bar{m}}(X)/\langle\xi\rangle$. This procedure may be repeated until $Sq^k$ is injective on the image of $Sq^1$.

*Step 2.* If $Sq^k$: $IH_k^{\bar{m}}(X)\to\mathbb{Z}/(2)$ is injective on the image of $Sq^1$, then the rank of $Sq^1$ is either 0 or 1. We will now show that $Sq^1$ vanishes, and hence $Y=\text{cone}(X)$ is an $LSF$-allowable null cobordism. If $k$ is odd then (10.2) $Sq^k$ vanishes, so $\operatorname{Im}(Sq^1)\cong(\operatorname{Ker} Sq^k)\cap(\operatorname{Im} Sq^1)=0$. If $k$ is even then (12.2) the vanishing of the characteristic number $v^kSq^1(v^k)$ implies that the rank of $Sq^1$ is even, so it cannot be 1.

15.2. Proposition. *Suppose $X$ is a $2k$-dimensional orientable $LSF$-space and*

(a) *if $k$ is even then $\pi(X)=0\in\mathbb{Z}/(4)$*
(b) *if $k$ is odd then $I\mathcal{X}(X;\mathbb{Z}/(2))=0\in\mathbb{Z}/(2)$.*

*Then $X$ is a boundary.*

*Proof.* The proof proceeds in 3 steps:

*Step 1.* $X$ is cobordant to an $LSF$-space $X'$ such that $(\operatorname{Ker} Sq^{k-1})\cap(\operatorname{Im} Sq^1)=0$ in $IH_{k-1}^{\bar{m}}(X')$. This is proven by repeated application of Proposition 14.3 to classes $\xi\in(\operatorname{Ker} Sq^{k-1})\cap(\operatorname{Im} Sq^1)$.

*Step 2.* The space $X'$ is LSF-cobordant to an $LSF$-space $X''$ such that

$$(\operatorname{Ker} Sq^{k-1})\cap(\operatorname{Im} Sq^1) = 0 \text{ in } IH_{k-1}^{\bar{m}}(X'')$$

and

$k$ even $\Rightarrow$ $\quad(\operatorname{Ker}\mathcal{P})\cap(\operatorname{Ker} Sq^1) = 0$ in $IH_k^{\bar{m}}(X'')$

$k$ odd $\Rightarrow$ $\quad(\operatorname{Ker} Sq^k)\cap(\operatorname{Ker} Sq^1) = 0$ in $IH_k^{\bar{m}}(X'')$

(where $\operatorname{Ker}\mathcal{P} = \{\xi\,|\,\mathcal{P}(\xi) = 0\}$).

*Proof.* Apply Proposition 14.1 to kill classes $\xi \in (\operatorname{Ker}\mathscr{P}) \cap (\operatorname{Ker} Sq^1)$ (or $\operatorname{Ker} Sq^k \cap \operatorname{Ker} Sq^1$) without changing $IH^{\bar{m}}_{k-1}(X')$.

*Step 3.* We will show that $IH^{\bar{m}}_k(X''; \mathbb{Z}/(2)) = 0$, so $Y = \operatorname{cone}(X'')$ is an $LSF$-allowable null cobordism. Since $\dim IH^{\bar{m}}_k(X'') = I\mathscr{X}(X) = 0 \pmod 2$, the dimension of this group is even. Consider first the case that k is odd. Then (10.2) $Sq^k = 0$ so ker $Sq^1 = 0$. Thus $Sq^1$ is injective, so the composition

$$Sq^{k-1} Sq^1 : IH^{\bar{m}}_k(X'') \to \mathbb{Z}/(2)$$

is injective. Therefore $\dim IH^{\bar{m}}_k(X'') \le 1$, so it must be 0.

Now consider the case that $k$ is even. Then $k-1$ is odd so $Sq^{k-1}$: $IH^{\bar{m}}_{k-1}(X'') \to \mathbb{Z}(2)$ vanishes (10.2) so the conclusion of Step 1 is that $Sq^1$: $IH^{\bar{m}}_k(X'') \to IH^{\bar{m}}_{k-1}(X'')$ also vanishes. But $\pi(X'') = 0$ so $IH^{\bar{m}}_k(X'')$ is $\mathscr{P}$-hyperbolic (12.3), i.e.

$$\tfrac{1}{2}\dim IH^{\bar{m}}_k(X'') = \dim \ker(\mathscr{P}) = \dim(\operatorname{Ker}\mathscr{P} \cap \operatorname{Ker} Sq^1) = 0$$

so $IH^{\bar{m}}_k(X'') = 0$. In case $k$ is odd, $I\mathscr{X}(X; \mathbb{Z}/2)$ is always zero by 10.7.

## §16. $\bar{s}$-DUALITY SPACES

16.1. *Definitions.* Consider the special perversity

$$\bar{s} = (0, 0, 0, 1, 1, 2, 2, 3, 3, \ldots)$$

and its dual, the cospecial preversity

$$\bar{r} = (0, 1, 2, 2, 3, 3, 4, 4, \ldots)$$

A pseudomanifold $X$ is an *$\bar{s}$-duality space* if, for each stratum of odd codimension $c = 2k + 1$ we have

$$IH^k_{\bar{m}}(L; \mathbb{Z}/(2)) = 0$$

and for each stratum of even codimension $c = 2k$, we have

$$IH^{k-1}_{\bar{m}}(L; \mathbb{Z}/(2)) = IH^k_{\bar{m}}(L; \mathbb{Z}/(2)) = 0$$

where $L$ is the link of the stratum. We shall make the further technical assumption that $X$ admits a stratification with no strata of codimension 1, 2, 3, or 4.

*Remark.* The pseudomanifold assumption implies that there are no strata of codimension 1. The normality assumption implies that there are no strata of codimension 2. The $\bar{s}$-duality space assumption implies that the link of a codimension 3 stratum is a smooth homology 2-sphere, so it is a smooth sphere, and so there are no strata of codimension 3. The $\bar{s}$-duality space assumption on the link of a codimension 4 stratum imply that the link is a smooth $\mathbb{Z}/(2)$-homology three-sphere, of which there are many. So our technical assumption consists of a single condition on strata of codimension 4. It must be made so that the (infinitely generated) cobordism groups of homology three-spheres do not appear in our cobordism theory.

16.2. Proposition. *Let X be an $\bar{s}$-duality space. Then*

(a) *The natural homomorphisms*

$$IH^i_{\bar{s}}(X; \mathbb{Z}/(2)) \to IH^i_{\bar{m}}(X; \mathbb{Z}/(2)) \to IH^i_{\bar{n}}(X; \mathbb{Z}/(2)) \to IH^i_{\bar{r}}(X; \mathbb{Z}/(2))$$

*are isomorphisms (and consequently the perversity $\bar m$ involved in the definition of an $\bar s$ perversity space could be replaced by $\bar s$, $\bar n$, or $\bar r$).*

(b) *The Wu classes $v^i(X)$ lift canonically to $IH^i_{\bar s}(X; \mathbb{Z}/(2))$*

(c) *X admits an $\bar m$-Bockstein homomorphism*

$$\beta\colon IH^i_{\bar m}(X; \mathbb{Z}/(2)) \to IH^i_{\bar m}(X; \mathbb{Z})$$

*whose mod 2 reduction is the Steenrod operation*

$$Sq^1\colon IH^i_{\bar m}(X; \mathbb{Z}/(2)) \to IH^{i+1}_{\bar m}(X; \mathbb{Z}/(2))$$

*and there is a Bockstein exact sequence,*

$$\ldots \to IH^i_{\bar m}(X; \mathbb{Z}) \to IH^i_{\bar m}(X; \mathbb{Z}) \to IH^i_{\bar m}(X; \mathbb{Z}/(2)) \xrightarrow{\beta} IH^{i+1}_{\bar m}(X; \mathbb{Z}) \to \ldots$$

*which gives rise to the short exact universal coefficient sequence*

$$0 \to IH^i_{\bar m}(X; \mathbb{Z}) \otimes \mathbb{Z}/(2) \to IH^i_{\bar m}(X; \mathbb{Z}/(2)) \to \operatorname{Tor}(IH^{i+1}_{\bar m}(X; \mathbb{Z}), \mathbb{Z}/(2)) \to 0$$

*(and $\mathbb{Z}$ may be replaced by the localization $\mathbb{Z}_{(2)}$ in these sequences).*

Proof. Part (a) follows from [19] 5.5. Parts (b) and (c) are direct consequences of §5 and §6 of this paper. The functor $\otimes\, \mathbb{Z}_{(2)}$ is flat so $\mathbb{Z}$ may be replaced by $\mathbb{Z}_{(2)}$.

16.3. *Linking Pairing.* In the following section we will use the notation $T^{\bar m}_k$ to denote the torsion subgroup of $IH^{\bar m}_k(X; \mathbb{Z}_{(2)})$. In this paragraph we will assume that $X$ is an oriented $\bar s$-duality space of dimension $n$.

Lemma. *The middle intersection homology of X satisfies Poincaré duality over $\mathbb{Z}_{(2)}$, i.e. there is a commutative diagram with exact columns and isomorphisms along the rows,*

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
T^{\bar m}_i & \xrightarrow{\cong} & \operatorname{Ext}(IH^{\bar m}_{n-i-1}(X;\mathbb{Z}_{(2)}), \mathbb{Z}_{(2)}) \cong \operatorname{Hom}(T^{\bar m}_{n-i-1}, \mathbb{Q}/\mathbb{Z}_{(2)}) \\
\downarrow & & \downarrow \\
IH^{\bar m}_i(X; \mathbb{Z}_{(2)}) & \xrightarrow{\cong} & IH^{\bar m}_i(X; \mathbb{Z}_{(2)}) \\
\downarrow & & \downarrow \\
IH^{\bar m}_i(X; \mathbb{Z}_{(2)})/T^{\bar m}_i & \xrightarrow{\cong} & \operatorname{Hom}(IH^{\bar m}_{n-i}(X; \mathbb{Z}_{(2)}), \mathbb{Z}_{(2)}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

*which gives rise to a nondegenerate linking pairing*

$$T^{\bar m}_k(X) \times T^{\bar m}_{n-k-1}(X) \to \mathbb{Q}/\mathbb{Z}_{(2)}$$

*where $\mathbb{Z}_{(2)}$ denotes the localization of $\mathbb{Z}$ at 2.*

*Proof of Lemma.* The $\bar s$-duality space assumptions imply that for every stratum of odd codimension $c = 2l + 1$ we have $IH^{\bar m}_l(L; \mathbb{Z}_{(2)}) = 0$ and for every stratum of even codimension $c = 2l$ we have $IH_l^{\bar{m}}(L; \mathbb{Z}_{(2)}) = 0$. Thus the assumptions of [20] 4.4 are satisfied (but with $\mathbb{Z}$ replaced by $\mathbb{Z}_{(2)}$) and we conclude that the linking pairing

$$T_k^{\bar{m}}(X) \otimes T_{n-k-1}^{\bar{m}}(X) \to \mathbb{Q}/\mathbb{Z}_{(2)}$$

is nonsingular, where $T_k^{\bar{m}}(X)$ denotes the torsion subgroup of $IH_k^{\bar{m}}(X; \mathbb{Z}_{(2)}) \cong IH_k^{\bar{m}}(X; \mathbb{Z})_{(2)}$.

16.4. *Orientable $\bar{s}$-duality spaces.*

*Definition.* A pseudomanifold $X$ is an orientable $\bar{s}$-duality space if it is both orientable and also an $\bar{s}$-duality space.

*Remarks.* For such spaces the first Wu class $v^1$ is defined but is always 0. The following characteristic numbers may also be formed: $v^i v^{n-i}$, $v^{n-i-1} Sq^1(v^i)$, $Sq^1(v^i) Sq^1(v^{n-i-2})$. This is because each $v^i$ lifts to the middle intersection homology, and the operation $Sq^1$ preserves the middle intersection homology. If $X$ is a $4k$ dimensional orientable $\bar{s}$-duality space then the nondegenerate intersection pairing on the middle interesection homology.

$$IH_{2k}^{\bar{m}}(X; \mathbb{Z}_{(2)}) \times IH_{2k}^{\bar{m}}(X; \mathbb{Z}_{(2)}) \to \mathbb{Z}_{(2)}$$

is a symmetric bilinear pairing and so it determines a class $\sigma(X) \in W(\mathbb{Z}_{(2)})$ in the Witt group of symmetric bilinear form modules over $\mathbb{Z}_{(2)}$ (see [33], [49]). Each of the above numbers is a cobordism invariant for cobordisms between $\bar{s}$-duality spaces. We remark that the *signature* $\operatorname{sig}(X) \in \mathbb{Z}$ is the image of the invariant $\sigma(X)$ under the natural homomorphism

$$W(\mathbb{Z}_{(2)}) \to W(\mathbb{R}) \cong \mathbb{Z}$$

If $k = 1$, then the pairing above is nonsingular over $\mathbb{Z}$ (see (16.1)) and so determines a class $\operatorname{sig}(X) \in W(\mathbb{Z}) \cong \mathbb{Z}$.

Proposition. *If $X^n$ is an orientable $\bar{s}$-duality space, then the following relations hold between the above characteristic numbers:*

(1) $$Sq^1(v^i) Sq^1(v^{n-i-2}) = 0$$

(2) $$v^i v^{n-i} = 0 \text{ unless } n = 2i \text{ is even}$$

(3) $$v^i Sq^1(v^{n-i-1}) = 0 \text{ unless } n = 4k+1 \text{ and } i = n-i-1 = 2k$$

(4) $$\operatorname{sig}(X) (\operatorname{mod} 2) = v^{2k} . v^{2k} \text{ if } n = 4k$$

*Proof.* See 12.1. Part (4) is the standard observation that the signature and Euler characteristic are congruent mod 2.

16.5. *Cobordism of locally orientable $\bar{s}$-duality spaces*

Theorem. *An orientable $\bar{s}$-duality space $X$ is null-cobordant (by an orientable $\bar{s}$-duality cobordism) iff all its characteristic numbers (from 16.4) are zero. In fact, the cobordism groups $\Omega_i^{\bar{s}}$ of orientable i-dimensional $\bar{s}$-duality spaces are precisely the (higher) Mischenko–Ranicki–Witt symmetric L-groups* ([31], [35], [34]), if $i \neq 4$,

$$\Omega_i^{\bar{s}} = L^i(\mathbb{Z}_{(2)}) = \begin{cases} W(\mathbb{Z}_{(2)}) & \text{if } i = 4k \\ \mathbb{Z}/(2) & \text{if } i = 4k+1,\ k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

*with the invariants given by $\sigma(X)$ and $v^{2k} Sq^1 v^{2k}$, respectively. If $i = 4$, $\Omega_i^{\bar{s}} \cong \mathbb{Z}$ given by* $\operatorname{sig}(X)$.

*Proof.* We will consider 4 cases:

*Case 0*: dim $(X) \leq 4$. This case will be left to the reader.

*Case 1a*: $X$ is orientable, $\dim(X) = 2k - 1 \geq 5$ is odd, $IH_k(X; \mathbb{Z}/(2)) \cong \mathbb{Z}/(2)$, and the Bockstein $\beta$ vanishes on $IH_k(X; \mathbb{Z}/(2))$. This case will be treated in §18.

*Case 1b*: $X$ is orientable and $\dim(X) = 2k \geq 6$ is even. This case will be treated in §19 (and will use the result from Case 1a).

*Case 1c*: $X$ is orientable and $\dim(X) = 2k + 1 \geq 5$ is odd. This case will be treated in §20 (and will use the result from Cases 1a and 1b).

## §17. PREPARING A CYCLE FOR SURGERY

17.1. In order to perform surgery above the middle dimension on a cycle $\xi$, it is necessary to choose the cycle very carefully in order to control the intersection homology of a regular neighborhood $N$ of $\xi$. Throughout this chapter we will assume the pseudomanifold $X$ is compact and orientable, so we may use homology notation $IH_i^{\bar{p}}(X)$ instead of cohomology notation $IH_{\bar{p}}^{n-i}(X)$.

*Definition.* A cycle representative $|\xi|$ of a homology class $\xi \in IH_i^{\bar{m}}(X; R)$ will be said to be *R-prepared* if

$$IH_j^{\bar{m}}(N; R) = \begin{cases} 0 & \text{for } j > i \\ R & \text{for } j = i \\ 0 & \text{for } j = i - 1 \end{cases}$$

where $N$ is a regular neighborhood of $|\xi|$ and where $R$ denotes any of the coefficient rings $\mathbb{Z}$, $\mathbb{Z}/(2)$, or $\mathbb{Z}_{(2)}$.

17.2. PROPOSITION. *Suppose $X$ is an n-dimensional orientable pseudomanifold, $\xi \in IH_i^{\bar{p}}(X; \mathbb{Z}/(2))$, $\beta(\xi) = 0$, and $n \geqq 5$, $i \geqq 3$ and $n - i \geq 2$.*

(a) *If $H_1(X - \Sigma; \mathbb{Z}_{(2)}) = 0$ then $\xi$ may be represented by a prepared cycle $|\xi|$*

(b) *If $H_1(X - \Sigma; \mathbb{Z}_{(2)}) \neq 0$ then $X$ is cobordant (by a cobordism which is trivial in a neighborhood of the singular set $\Sigma$) to an orientable pseudomanifold $X'$ such that $H_1(X' - \Sigma(X'); \mathbb{Z}_{(2)}) = 0$ such that the inclusions induce isomorphisms*

$$IH_j^{\bar{m}}(X; \mathbb{Z}/(2)) \cong IH_j^{\bar{m}}(W; \mathbb{Z}/(2)) \text{ for all } j \geq 2$$

$$IH_j^{\bar{n}}(X'; \mathbb{Z}/(2)) \cong IH_j^{\bar{m}}(W; \mathbb{Z}/(2)) \text{ for all } j \leq n - 2$$

*In particular,* $IH_j^{\bar{m}}(X; \mathbb{Z}/(2)) \cong IH_j^{\bar{m}}(X'; \mathbb{Z}/(2))$ for $2 \leq j \leq n - 2$.

The proofs will take the rest of this section.

17.3. LEMMA. *Suppose $X$ is an n-dimensional* P.L. *pseudomanifold, $|\xi|$ is any $(\bar{p}, i)$-allowable* [18] *piecewise linear subset of $X$ and $N$ is a regular neighborhood of $|\xi|$, with boundary $\partial N$. Let $r\colon N \to |\xi|$ denote a* P.L. *retraction. Then for any coefficient ring R we have*

(1) *r induces an isomorphism* ([37])

$$r_*\colon IH_i^{\bar{p}}(N; R) \cong H_i(|\xi|; R)$$

(2) *and an injection* ([33])

$$r_*: IH^{\bar{p}}_{i-1}(N; R) \to H_{i-1}(|\xi|; R)$$

(3) *and*

$$IH^{\bar{m}}_k(N; R) = 0 \text{ for } k > i.$$

*Proof.* We may take $r$ to be a simplicial collapse, so it fits into a deformation retraction

$$D: N \times [0, 1] \to N$$

such that $D|N \times (0, 1]$ is stratum preserving (and even simplex preserving), $D|N \times \{0\} = r$, and $D|N \times \{1\} = \text{id}$. For any $i-1$ dimensional P.L. cycle $\eta \in IC^{\bar{p}}_{i-1}(N; \mathbb{Z})$, the cochain $D(|\eta| \times I)$ is $(\bar{p}, i)$-allowable in $N$ because, for any stratum $S$, we have

$$\dim(D(|\eta| \times I) \cap S) \leq \max(1 + \dim(|\eta| \cap S), \dim(|\xi| \cap S)) \leq i - \text{cod}(S) + p(\text{cod}(S)).$$

Therefore, $D(|\eta| \times I)$ is a homology between $\eta$ and the $i-1$ cycle $r(\eta)$ in $|\xi|$. Now suppose that $r_*(\eta) = 0$, i.e. there is a chain $\omega \in C_i(|\xi|; \mathbb{Z})$ such that $\partial\omega = r(\eta)$. Then $\omega$ is also $(\bar{p}, i)$ allowable in $N$ (since it is contained in $|\xi|$) and

$$\partial(D(\eta) + \omega) = \eta$$

which shows that $\eta$ is homologous to zero. This proves part (2) of the Proposition.

The same argument gives parts (1) and (3) of the Proposition, but for part (1), the inclusion of cycles induces a natural map

$$H_i(|\xi|; R) \to IH^{\bar{p}}_i(N; R)$$

which is the inverse of

$$r_*: IH^{\bar{p}}_i(N; R) \to H_i(|\xi|; R)$$

17.4. *Remark.* The same proof gives the following more general statement: Let $\xi \in IC^{\bar{p}}_i(X; \mathbb{Z})$ be a cycle with support $|\xi|$ and regular neighborhood $N$. Fix $j \leq i$, and choose a perversity $\bar{q}$ such that $q(c) \geq p(c) + i - j$ for all $c$. Then the retraction $r: N \to |\xi|$ induces an isomorphism

$$r_*: IH^{\bar{q}}_j(N; R) \to H_j(|\xi|; R)$$

and an injection

$$r_*: IH^{\bar{q}}_{j-1}(N; R) \to H_{j-1}(|\xi|; R).$$

17.5. Lemma ([33]). *Suppose $X$ is a normal, orientable $n$ dimensional pseudomanifold, $H_1(X - \Sigma; \mathbb{Z}_{(2)}) = 0$ and that $\xi \in IC^{\bar{p}}_i(X; \mathbb{Z}_{(2)})$ is an orientable cycle. If $n \geq 5$, $i \geq 3$ and $n - i \geq 2$ then $\xi$ is homologous to a cycle $\xi'$ which is R-prepared, i.e. such that*

$$H_i(|\xi'|; R) \cong R$$
$$H_{i-1}(|\xi'|; R) = 0$$

*where $R = \mathbb{Z}_{(2)}$ or $\mathbb{Z}/(2)$.*

*Proof.* (See [33] for details.) Make $|\xi|$ into an irreducible normal pseudomanifold (by piping, as in [37]) so that $\xi - \Sigma(|\xi|) \subset X - \Sigma$. Any loop in $\xi - \Sigma(|\xi|)$ bounds an embedded 2-manifold $M$ in $X - \Sigma$ such that $M \cap |\xi| = \partial M$ and it is possible to perform ambient surgery on $M$ since it may be chosen with trivial normal bundle in $X - \Sigma$. This kills $H_1(|\xi|)$ and by duality it also kills $H_{i-1}(|\xi|)$. (If $n = 5$, then $M$ may intersect $|\xi| - \Sigma(|\xi|)$ in isolated points. For each such point, it is possible to drag $M$ across $|\xi| - \Sigma(|\xi|)$ to create a new intersection point of opposite sign. The Whitney trick then makes $M$ and $|\xi| - \Sigma(|\xi|)$ disjoint.)

17.6. *Proof of Proposition* 17.2. Part (a) follows from 17.3 and 17.5. Since $X$ is orientable, we can kill $H_1(X - \Sigma; \mathbb{Z}_{(2)})$ by doing surgery on embedded circles. For $2 \le i \le n-2$ this will not change $IH_i^{\bar{p}}(X; \mathbb{Z}/(2))$ (although it may change $IH_2^{\bar{m}}(X; \mathbb{Z}_{(2)})$).

## §18. CASE 1a: SURGERY ABOVE THE MIDDLE DIMENSION

18.1. In this section we prove the following special case of Theorem 16.5:

PROPOSITION. *Suppose $X$ is a compact orientable locally orientable $\bar{s}$-duality space of dimension* $\dim(X) = 2k - 1 \ge 5$. *Suppose that $\xi \in IH_k^{\bar{m}}(X; \mathbb{Z}/(2))$ and that the Bockstein $\beta$ vanishes on $\xi$. Then there is an orientable locally orientable $\bar{s}$-duality cobordism between X and a space $X'$ such that $IH_k^{\bar{m}}(X'; \mathbb{Z}/(2)) \cong IH_k^{\bar{m}}(X; \mathbb{Z}/(2))/\langle \xi \rangle$.*

*Remarks.* Since $X$ is orientable and compact, we will use intersection homology notation $IH_i^{\bar{m}}(X)$ instead of intersection cohomology notation $IH_{\bar{p}}^{n-i}(X)$. The above assumptions imply that all the characteristic numbers (from 16.4) vanish.

18.2. *Proof.* We will perform surgery (above the middle dimension) on the class $\xi \in IH_k^{\bar{m}}(X; \mathbb{Z}/(2))$. By 17.2 we may assume (by replacing $X$ with a cobordant space, if necessary, changing at most $IH_1^{\bar{m}}(X; \mathbb{Z}/(2))$ and $IH_{n-1}^{\bar{m}}(X; \mathbb{Z}/(2)))$ that $\xi$ has a cycle representative with support $|\xi|$ which has a regular neighborhood $N \subset X$ such that $IH_k^{\bar{m}}(N; \mathbb{Z}/(2)) = \mathbb{Z}/(2)$ and $IH_{k-1}^{\bar{m}}(N; \mathbb{Z}/(2)) = 0$. From the long exact sequence for the pair $IH_k^{\bar{m}}(N, \partial N; \mathbb{Z}/(2))$, we see that $\xi$ comes from a class $\xi' \in IH_k^{\bar{m}}(\partial N; \mathbb{Z}/(2))$ and also that $IH_{k-1}^{\bar{m}}(\partial N; \mathbb{Z}/(2)) = 0$. Consider the pseudomanifold

$$X' = (X - N) \cup_{\partial N} \operatorname{cone}(\partial N)$$

We will show

(i) $X'$ is orientable, locally orientable, and is an $\bar{s}$-duality space
(ii) $X'$ is cobordant to $X$ (by an orientable locally orientable $\bar{s}$-duality cobordism)

$$W = (X \times I) \cup_{N \times \{0\}} \operatorname{cone}(N \cup_{\partial N} \operatorname{cone}(\partial N))$$

(iii) $IH_k^{\bar{m}}(X'; \mathbb{Z}/(2)) \cong IH_k^{\bar{m}}(X; \mathbb{Z}/(2))/\langle \xi \rangle$ and $IH_{k-1}^{\bar{m}}(X'; \mathbb{Z}/(2)) = \xi^{\perp}$

In particular, if dim $(IH_k^{\bar{m}}(X; \mathbb{Z}/(2)) = 1$ then $V = W \cup_{X'} \operatorname{cone}(X')$ is an orientable locally orientable $\bar{s}$-duality space such that $X = \partial V$.

18.3. *Proof of (i).* The new singularity is the cone point in $\operatorname{cone}(\partial N)$, but

$$IH_{k-1}^{\bar{m}}(\partial N; \mathbb{Z}/(2)) = 0$$

so the cone on $\partial N$ is an $\bar{s}$-duality space.

18.4. *Proof of* (ii). We have replaced the neighborhood $N$ by $\operatorname{cone}(\partial N)$, so the cobordism $W$ between $X$ and $X'$ is a thickening of $X \cup_N \operatorname{cone}(N \cup_{\partial N} \operatorname{cone}(\partial N))$. Then $W$ is an orientable locally orientable pseudomanifold whose boundary is $X \cup X'$. To see that $W$ is an $\bar{s}$-duality space we must show that

$$IH_k^{\bar{m}}(N \cup_{\partial N} \operatorname{cone}(\partial N); \mathbb{Z}/(2)) = 0$$

But this follows from the Mayer Vietoris sequence for the union. In fact, we compute that

$$IH_k^{\bar{m}}(N \cup \mathrm{cone}(\partial N)) = \begin{cases} IH_j^{\bar{m}}(N, \partial N) & j \geq k+1 \\ 0 & j = k,\ k-1 \\ IH_j^{\bar{m}}(N) & j \leq k-2 \end{cases}$$

18.5. *Proof of* (iii). Compare the Mayer Vietoris sequence for the union

$$X = (X - N) \cup_{\partial N} N \quad \text{and} \quad X' = (X - N) \cup_{\partial N} \mathrm{cone}(\partial N)$$

to see that

$$IH_j^{\bar{m}}(X'; \mathbb{Z}/(2)) = \begin{cases} \mathrm{coker}(IH_k^{\bar{m}}(\partial N) \to IH_k^{\bar{m}}(X - N) & j = k \\ IH_{k-1}^{\bar{m}}(X - N) & j = k-1 \\ IH_j^{\bar{m}}(X) & \text{otherwise} \end{cases}$$

Then use the surgery diagram (14.1) to evaluate the special cases $j = k$, $k-1$.

18.6. *Further properties of the cobordism.* If $IH_k^{\bar{m}}(X; \mathbb{Z}/(2)) \cong \mathbb{Z}/(2)$ then

$$V = W \cup_{X'} \mathrm{cone}(X')$$

is a null cobordism of $X$. Using Mayer Vietoris we compute

$$IH_j^{\bar{m}}(W; \mathbb{Z}/(2)) \cong \begin{cases} IH_j^{\bar{m}}(X) & j \geq k+1 \\ 0 & j = k \\ IH_j^{\bar{m}}(X) & j \leq k-1 \end{cases}$$

and use Mayer Vietoris again to find

$$IH_j^{\bar{m}}(V; \mathbb{Z}/(2)) \cong \begin{cases} 0 & j \geq k \\ IH_j^{\bar{m}}(W) & j \leq k-1 \end{cases}$$

## §19. PROOF OF CASE 1b

19.1. In this section we will prove the following result:

PROPOSITION. *Suppose $X$ is a compact orientable $\bar{s}$-duality space with $n = \dim(X) = 2k \geq 6$ and suppose the Witt class $\sigma(X) \in W(\mathbb{Z}_{(2)})$ is 0 (which always happens if $k$ is odd). Then there is an orientable $\bar{s}$-duality space $W$ with boundary,*

$$X = \partial W.$$

19.2. *Remarks.* Since $X$ is compact and orientable, we will use intersection homology notation $IH_i^{\bar{m}}(X)$ instead of intersection cohomology $IH_{\bar{m}}^{n-i}(X)$ notation. If $IH_k^{\bar{m}}(X; \mathbb{Z}/(2)) = 0$ then we are done since $W = \mathrm{cone}(X)$ is an orientable $\bar{s}$-duality space. So we will kill $IH_k^{\bar{m}}(X; \mathbb{Z}/(2))$ by surgery.

19.3. LEMMA. *Suppose $X$ is an orientable $\bar{s}$-duality space of dimension $2k \geq 6$, and that $\xi \in IH_k^{\bar{m}}(X; \mathbb{Z}/(2))$ satisfies $\beta(\xi) = 0$. Suppose that some lift $\tilde{\xi} \in IH_k^{\bar{m}}(X; \mathbb{Z}_{(2)})$ of $\xi$ satisfies $\tilde{\xi} \cdot \tilde{\xi} = 0 \in \mathbb{Z}_{(2)}$. Then $X$ is $\bar{s}$-cobordant to an $\bar{s}$-duality space $X'$ such that*

$$IH_k^{\bar{m}}(X'; \mathbb{Z}/(2)) \cong \xi^{\perp}/\langle \xi \rangle$$

*Proof.* Let $i:\partial N \to N$ be the inclusion of the boundary of a regular neighborhood of $|\tilde{\xi}|$ in $X$. Then there is a class $\tilde{\xi}' \in IH_k^{\bar{m}}(\partial N; \mathbb{Z}_{(2)})$ such that $i_*(\tilde{\xi}') = \tilde{\xi}$ because the homomorphism

$$IH_k^{\bar{m}}(N; \mathbb{Z}_{(2)}) \to IH_k^{\bar{m}}(N, \partial N; \mathbb{Z}_{(2)})$$

is given by multiplication with $\tilde{\xi}$. Let $\xi' = \tilde{\xi}' \ (\mathrm{mod}\ 2) \in IH_k(\partial N; \mathbb{Z}/(2))$. Since $\beta(\xi') = 0$, and $\dim(\partial N) \geq 5$, we can apply case (1a) to $\partial N$ and obtain a particular null cobordism $V$ of $\partial N$. We claim that

(i) $X' = (X - U) \cup_{\partial N} V$ is cobordant to $X$ by an orientable locally orientable $\bar{s}$-duality space,

$$Z = (X \times I) \cup_{N \times \{0\}} \mathrm{cone}(N \cup V)$$

(ii) $\dim IH_k^{\bar{m}}(X'; \mathbb{Z}/(2)) = \dim IH_k^{\bar{m}}(X; \mathbb{Z}/(2)) - 2$.

*Proof of* (i). We must show that the cone on $N \cup V$ is allowable, i.e. that $IH_k^{\bar{m}}(N \cup V) = 0$. However, this follows immediately from the Mayer Vietoris sequence for the union $N \cup V$, together with the calculation of 18.6. In fact, we calculate

$$IH_j^{\bar{m}}(N \cup_{\partial} V; \mathbb{Z}/(2)) \cong \begin{cases} IH_{j-1}^{\bar{m}}(\partial N) \cong IH_j^{\bar{m}}(N, \partial) & j \geq k+2 \\ 0 & j = k+1,\ k,\ k-1 \\ IH_j^{\bar{m}}(N) & j \leq k-2 \end{cases}$$

*Proof of* (ii). From the Mayer Vietoris sequence for the union $X' = (X - N) \cup_{\partial N} V$ we see that $IH_k^{\bar{m}}(X'; \mathbb{Z}/(2)) \cong \mathrm{coker}(IH_k^{\bar{m}}(\partial N) \to IH_k^{\bar{m}}(X - N))$. Following the computation of 14.1 we have

$$IH_j^{\bar{m}}(X'; \mathbb{Z}/(2)) = \begin{cases} IH_j^{\bar{m}}(X) & \text{if } j \neq k \\ \xi^{\perp}/\langle \xi \rangle & \text{if } j = k \end{cases}$$

*Proof of Proposition* 19.1. Since $\sigma(X) = 0 \in W(\mathbb{Z}_{(2)})$ there is a self-annihilating subspace $S \subset IH_k(X; \mathbb{Z}_{(2)})$ of half the dimension. Thus, any basis element $\tilde{\xi} \in S$ satisfies $\tilde{\xi} \cdot \tilde{\xi} = 0$, and $\xi = \tilde{\xi}(\mathrm{mod}\ 2) \in IH_k(X; \mathbb{Z}/(2))$ is nonzero, but $\beta(\xi) = 0$. Proposition 19.3 may be repeatedly applied to kill such classes $\xi$. Similarly if $\tilde{\xi} \in T_k^{\bar{m}}(X; \mathbb{Z}_{(2)})$ is a torsion class whose mod 2 reduction $\xi = \tilde{\xi}(\mathrm{mod}\ 2)$ is nonzero, then $\tilde{\xi} \cdot \tilde{\xi} = 0$, $\beta(\tilde{\xi}) = 0$, and so Prop. 19.3 may be applied again to kill $\xi$. Eventually we obtain a cobordism $Z$ between $X$ and an $\bar{s}$-duality space $X'$ such that $IH_k^{\bar{m}}(X; \mathbb{Z}_{(2)}) = 0$. By the universal coefficient theorem (16.2) this implies that $\beta$: $IH_k^{\bar{m}}(X'; \mathbb{Z}/(2)) \to \mathrm{Tor}(IH_{k-1}^{\bar{m}}(X'; \mathbb{Z}_{(2)}), \mathbb{Z}/(2))$ is an isomorphism. It follows that $IH_k^{\bar{m}}(X'; \mathbb{Z}/(2)) = 0$ because $\mathrm{Tor}(IH_{k-1}^{\bar{m}}(X'; \mathbb{Z}_{(2)}), \mathbb{Z}/(2)) \subset T_{k-1}^{\bar{m}}(X')$ which is dually paired under the linking pairing (16.3) with $T_k(X'; \mathbb{Z}_{(2)}) \subset IH_k^{\bar{m}}(X'; \mathbb{Z}_{(2)}) = 0$. Thus $W = Z \cup_{X'} \mathrm{cone}(X')$ is an $\bar{s}$-allowable null cobordism of $X$.

19.4. *Further properties of the cobordism.* Suppose $X$ and $\xi$ satisfy the hypotheses of 19.3 and that $\dim (IH_k(X; \mathbb{Z}/(2)) = 2$. Then $W = Z \cup_{X'} \mathrm{cone}(X')$ satisfies

$$IH_k^{\bar{m}}(W; \mathbb{Z}/(2)) \cong IH_k^{\bar{m}}(X; \mathbb{Z}/(2))/\langle \xi \rangle.$$

*Proof.* Use the Mayer Vietoris sequence for the union

$$Z = (X \times I) \cup_N \mathrm{cone}(N \cup V)$$

to find
$$IH_j^{\bar{m}}(Z;\mathbb{Z}/(2)) \cong \begin{cases} IH_j^{\bar{m}}(X) & j \geq k+1 \\ IH_k(X)/\langle \xi \rangle & j = k \\ IH_j(X) & j \leq k-1 \end{cases}$$

then use Mayer Vietoris again to find

$$IH_j^{\bar{m}}(W;\mathbb{Z}/(2)) \cong \begin{cases} 0 & j = k+1 \\ IH_j^{\bar{m}}(Z) & j \leq k \end{cases}$$

## §20. PROOF OF CASE 1c

20.1. In this chapter we will prove the following result:

PROPOSITION. *Suppose $X$ is a compact orientable $\bar{s}$-duality space of odd dimension, $dim(X) = 2k+1 \geq 5$, and suppose the characteristic number $v^k Sq^1(v^k)$ vanishes. Then there is an orientable $\bar{s}$-duality space $W$ with boundary,*

$$X = \partial W.$$

20.2. *Remark.* Since $X$ is orientable we may continue to use homology notation instead of cohomology notation.

20.3. LEMMA. *Suppose $X$ is a compact orientable locally orientable $\bar{s}$-duality space and $dim(X) = 2k+1 \geq 5$. Then $X$ is cobordant to a pseudomanifold $X'$ such that the Bockstein $\beta$: $IH_k^{\bar{m}}(X';\mathbb{Z}/(2)) \to IH_{k-1}^{\bar{m}}(X';\mathbb{Z}_{(2)})$ vanishes.*

*Proof.* By the long exact Bockstein sequence (6.3) the target group of $\beta$ is

$$\mathrm{Tor}_{\mathbb{Z}}(IH_{k-1}^{\bar{m}}(X;\mathbb{Z}_{(2)}), \mathbb{Z}/(2)) \subset IH_{k-1}^{\bar{m}}(X;\mathbb{Z}_{(2)})$$

which is dually paired (under the linking pairing of 16.3) with

$$T_{k+1}^{\bar{m}} \otimes \mathbb{Z}/(2)$$

where $T_{k+1}^{\bar{m}}$ denotes the torsion subgroup of $IH_{k+1}^{\bar{m}}(X;\mathbb{Z}_{(2)})$. In fact, we can kill the group

$$IH_{k+1}^{\bar{m}}(X;\mathbb{Z}_{(2)}) \otimes \mathbb{Z}/(2) = \ker(\beta) \subset IH_{k+1}^{\bar{m}}(X;\mathbb{Z}/(2))$$

by surgery (see case 1(a), §18), and this implies that $IH_{k+1}^{\bar{m}}(X;\mathbb{Z}_{(2)}) = 0$.

20.4. LEMMA. *Suppose $X$ is an orientable $\bar{s}$-duality space of dimension $2k+1 \geqslant 5$, and suppose that the Bockstein $\beta$ vanishes on $IH_k^{\bar{m}}(X;\mathbb{Z}/(2))$. Then $X$ is cobordant to a space $X'$ such that*

$$Sq^k: IH_k^{\bar{m}}(X';\mathbb{Z}/(2)) \to \mathbb{Z}/(2)$$

*is an injection.*

*Proof.* First we consider the case $k \geqslant 3$. Let $\xi \in IH_k^{\bar{m}}(X;\mathbb{Z}/(2))$ and suppose that $\beta(\xi) = 0$ and that $Sq^k(\xi) = 0$. We will kill the class $\xi$ by surgery. It follows (from the long exact sequence for the homology of a regular neighborhood $N$ of $|\xi|$ and its boundary) that $\xi$ comes from a class $\xi' \in IH_k^{\bar{m}}(\partial N;\mathbb{Z}/(2))$, and that

$$IH_k^{\bar{m}}(\partial N;\mathbb{Z}/(2)) \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$$

We wish to apply 19.3 to $(\partial N, \xi')$ and so we verify the hypotheses:

(a) $\beta(\xi')) = 0$ because $|\xi'|$ may be chosen to be an orientable pseudomanifold—in fact we may take $\xi' = r_*(A)$ where $r: N - |\xi| \to \partial N$ is the retraction to the boundary of the regular neighborhood, and where $A$ is an orientable cycle representative of $\xi$ which is dimensionally transverse [18] to $|\xi|$, i.e. $A \cap |\xi| = \phi$.

(b) a lift $\tilde{\xi}' \in IH_k^{\bar{m}}(\partial N; \mathbb{Z}_{(2)})$ may be found such that $\tilde{\xi}' \cdot \tilde{\xi}' = 0 \in \mathbb{Z}_{(2)}$ by using the same trick as in 14.2 (i.e. replace $\tilde{\xi}'$ by $\tilde{\xi}' + m\xi^*$ where $\xi^*$ denotes the boundary of a disk which is transverse to a nonsingular point of $|\xi|$).

Applying 19.3 to $\partial N$ (which has dimension $\geq 6$) we obtain a null cobordism $W$, such that $\partial W = \partial N$ and (from 19.4), $IH_k^{\bar{m}}(W; \mathbb{Z}(2)) \cong IH_k^{\bar{m}}(\partial N; \mathbb{Z}/(2))/\langle \xi \rangle$. This implies that the space

$$X' = (X - N) \cup_{\partial N} W$$

is an $\bar{s}$-duality space which is cobordant to $X$ by the (orientable) cobordism

$$\text{cone}(N \cup_{\partial N} W) \cup_{N \times \{1\}} (X \times I)$$

because

$$IH_k^{\bar{m}}(N \cup_{\partial N} W; \mathbb{Z}/(2)) = IH_{k+1}^{\bar{m}}(N \cup_{\partial N} W; \mathbb{Z}/(2)) = 0$$

as may be computed from the Mayer Vietoris sequence. We claim that $IH_k^{\bar{m}}(X'; \mathbb{Z}(2) \cong \xi^{\perp}/\langle \xi \rangle$, together with the induced operations of $Sq^k$ and $\beta$.

*Proof of claim.* By comparing the Mayer Vietoris sequence for the union $X = (X - N) \cup_\partial N$ and $X' = (X - N) \cup_\partial W$ we find

$$IH_j^{\bar{m}}(X'; \mathbb{Z}/(2)) \cong \begin{cases} IH_j^{\bar{m}}(X) & \text{for } j \geq k+2 \\ \text{coker } (IH_j^{\bar{m}}(\partial N) \to IH_j^{\bar{m}}(X - N) & \text{for } j = k+1, k \\ IH_j^{\bar{m}}(X) & \text{for } j \leq k-1 \end{cases}$$

and using the long exact sequences for $IH_*^{\bar{m}}(X, X - N)$ and for $IH_*^{\bar{m}}(N, \partial N)$ we find $IH_{k+1}^{\bar{m}}(X'; \mathbb{Z}/(2)) = \xi^{\perp}$ and $IH_k^{\bar{m}}(X'; \mathbb{Z}/(2)) = IH_k^{\bar{m}}(X)/\langle \xi \rangle$. It remains to show that $\beta$ vanishes on $IH_k^{\bar{m}}(X - N)$ and on $IH_k^{\bar{m}}(W)$, by Mayer Vietoris. But the single $k$-dimensional class $\xi^*$ in $W$ is represented by a $k$-sphere in $\partial N$ which is the boundary of a transverse disk, i.e. a disk which is transverse to a smooth point of $|\xi|$. Thus $\xi^*$ is orientable so $\beta(\xi^*) = 0$. Since $\beta$ vanishes on $IH_k^{\bar{m}}(X)$ and $\beta$ is natural, it remains to show that the homomorphism

$$IH_{k-1}^{\bar{m}}(X - N; \mathbb{Z}_{(2)}) \to IH_{k-1}^{\bar{m}}(X; \mathbb{Z}_{(2)})$$

is injective. However the kernel of this map is the image of $IH_k^{\bar{m}}(X, X - N)$ which can be calculated as follows (see 16.3):

$$\begin{aligned} IH_k^{\bar{m}}(X, X - N; \mathbb{Z}_{(2)}) &\cong IH_k^{\bar{m}}(N, \partial N; \mathbb{Z}_{(2)}) \\ &\cong \text{Hom}(IH_{k+1}^{\bar{m}}(N; \mathbb{Z}_{(2)}), \mathbb{Z}_{(2)}) \oplus \text{Ext}^1(IH_k^{\bar{m}}(N; \mathbb{Z}_{(2)}), \mathbb{Z}_{(2)}) \end{aligned}$$

But the first group vanishes since $N$ is a neighborhood of a $k$-cycle, and the second group vanishes since $IH_k^{\bar{m}}(N; \mathbb{Z}_{(2)}) \cong \mathbb{Z}_{(2)}$ is torsion-free.

Now we consider separately the case $k = 2$. The singularity set $\Sigma$ has codimension $> 4$ (see 16.1). Since $X$ is orientable, we can assume $\pi_1(X - \Sigma) = 0$ by doing surgery on embedded circles. This gives rise to the second surjection in the following diagram:

$$H_2(X - \Sigma) \twoheadrightarrow IH_2^{\bar{m}}(X; \mathbb{Z}) \twoheadrightarrow IH_2^{\bar{m}}(X; \mathbb{Z}/(2)) \xrightarrow{\beta} IH_1^{\bar{m}}(X; \mathbb{Z}) = H_1(X - \Sigma; \mathbb{Z}) = 0$$

By the Hurewicz theorem, $\pi_2(X-\Sigma) \cong H_2(X-\Sigma)$ so the cycle $\xi$ can be represented by a sphere which is embedded in $X-\Sigma$. Since $\beta(\xi)=0$ and $Sq^2(\xi)=0$ it follows that the normal bundle $\nu$ of this sphere (which is classified by $w^2(\nu)=Sq^2(\xi)=0$) is trivial. Therefore we can perform usual surgery on this embedded sphere, and it is straightforward to check that this kills the class $\xi$.

20.5. LEMMA. *Suppose $X$ is an orientable $\bar{s}$-duality space of dimension $2k+1 \geq 5$ and suppose that*

$$Sq^k: IH_k^{\bar{m}}(X; \mathbb{Z}/(2)) \to \mathbb{Z}/(2)$$

*is injective and that $v^k Sq^1(v^k)=0$. Then $X$ bounds.*

*Proof.* If $IH_k^{\bar{m}}(X; \mathbb{Z}/(2))=0$ then $X=\partial(\text{cone}(X))$ and we are done. This happens if $k$ is odd, by Corollary 10.2. Otherwise, $k$ is even and $IH_{k+1}^{\bar{m}}(X; \mathbb{Z}/(2))=\mathbb{Z}/(2)$ (by duality) and is generated by the Wu class $v^k$. Furthermore, duality plus the equation $v^k Sq^1(v^k)=0$ implies that $Sq^1(v^k)=0$. Now there are three cases:

(1) $IH_k^{\bar{m}}(X; \mathbb{Z}_{(2)})=0$

(2) $IH_k^{\bar{m}}(X; \mathbb{Z}_{(2)})=\mathbb{Z}_{(2)}$

(3) $IH_k(X; \mathbb{Z}_{(2)})=\mathbb{Z}/(2^r)$ for some $r \geq 1$.

We will show that in each of these cases, $\beta$: $IH_{k+1}(X; \mathbb{Z}/(2)) \to IH_k(X; \mathbb{Z}_{(2)})$ vanishes, so by 18.1 we will be finished. In cases (1) and (2) we have $\beta=0$ (since $\beta$ takes values in the two-torsion). In case (3), the linking pairing of 16.3 is a nondegenerate skew symmetric bilinear form on $\mathbb{Z}/(2^r)\times\mathbb{Z}/(2^r)\to\mathbb{Q}/\mathbb{Z}_{(2)}$ which implies $r=1$. In other words, the reduction mod 2,

$$IH_k(X; \mathbb{Z}_{(2)}) \to IH_k(X; \mathbb{Z}/(2))$$

is an isomorphism. Therefore $\beta(v^k)=\beta(v^k)(\text{mod } 2)=Sq^1(v^k)=0$, so $\beta$ vanishes on $IH_{k+1}(X; \mathbb{Z}/(2))$.

20.6. *Proof of Theorem* 16.5. We will review the four cases:

(1) $\Omega_{4k}^{\bar{s}} \cong W(\mathbb{Z}_{(2)})$: By Proposition 19.1, if $\sigma(X)=0 \in W(\mathbb{Z}_{(2)})$ then $X$ is a boundary. The invariants can all be realized by plumbing, as in [37].

(2) $\Omega_{4k+1}^{\bar{s}} \cong \mathbb{Z}/(2)$: If $\dim(X)=4k+1$ and $k\geq 1$ and $v^{2k}Sq^1 v^{2k}(X)=0$ then by propositions 20.3, 20.4, and 20.5, $X$ is a boundary. (For $k=0$ this group is 0). For $k \geq 1$ there are manifolds which realize this characteristic number ([32] p. 467.)

(3) $\Omega_{4k+2}^{\bar{s}}=0$: This is covered by Proposition 19.1.

(4) $\Omega_{4k+3}^{\bar{s}}=0$: If $\dim(X)=4k+3$ then (as in 16.4) the characteristic number $v^{2k}Sq^1 v^{2k}$ vanishes, so by 20.3, 20.4, and 20.5, $X$ is a boundary.

## §21. OPEN PROBLEMS

21.1. *Remove the orientability assumption.* We do not know the cobordism groups of *LSF*-spaces or of $\bar{s}$-duality spaces, but the problem is interesting. The cobordism groups of locally orientable *LSF* spaces (resp. $\bar{s}$-duality spaces) may be computed by the same Stong sequences (as in §10) in terms of the cobordism groups of orientable *LSF* (resp. $\bar{s}$-duality) spaces. However we do not know how to compute the cobordism groups for the unoriented theories. In the $\bar{s}$-duality space case, the same characteristic numbers can be defined, but only by a very strange procedure: The Wu class $v^1$ always lifts to the perversity $\bar{1}$ (7.4) and the natural homomorphism

$$IH_*^{\bar{s}}(X; \mathbb{Z}/(2)) \to IH_*^{\bar{s}+\bar{1}}(X; \mathbb{Z}/(2))$$

is an isomorphism for $\bar{s}$-duality spaces (16.2). Thus, for example, the number

$$v^i v^{n-i-2}(v^1)^2$$

can be constructed by first multiplying $v^i \cdot v^1 \in IH_{n-i-1}^{\bar{s}+\bar{1}}(X)$, then lifting this class back to $IH_{n-i-1}^{\bar{s}}(X)$, then multiplying again by $v^1$, lifting back to $IH_{n-i-2}^{\bar{s}}(X)$, then multiplying by $v^{n-i-2}$. We do not know whether this procedure (of multiplying, then reducing perversity, then multiplying again) is an associative product. If it is nonassociative, then there may be new characteristic numbers for $\bar{s}$-duality spaces which do not occur for manifolds.

21.2. *$\bar{p}$-Duality Spaces.* Let $\bar{p} \leq \bar{m}$ be a perversity. A pseudomanifold $X$ is a $\bar{p}$-duality space if, for each stratum of codimension $c$, we have

$$IH_{p(c)+1}^{\bar{m}}(L; \mathbb{Z}/(2)) = IH_{p(c)+2}^{\bar{m}}(L; \mathbb{Z}/(2)) = \ldots = IH_{c-2-p(c)}^{\bar{m}}(L; \mathbb{Z}/(2)) = 0$$

where $L$ is the link of the stratum. For $\bar{p}$-duality spaces the natural homomorphism

$$IH_*^{\bar{p}}(X; \mathbb{Z}/(2)) \to IH_*^{\bar{c}-2-\bar{p}}(X; \mathbb{Z}/(2))$$

is an isomorphism. For $\bar{p} = \bar{m}$, these are the (mod 2) Witt spaces of [37]; for $\bar{p} = \bar{0}$ these are (mod 2) homology manifolds.

*Problem.* Find the cobordism groups of $\bar{p}$-duality spaces.

*Remarks.* As $\bar{p}$ increases, the class of $\bar{p}$-duality spaces approaches the class of mod 2 homology manifolds, and we conjecture that the cobordism groups only get larger, i.e. if $\bar{p} \leq \bar{q}$ then $\Omega^{\bar{p}} \to \Omega^{\bar{q}}$ is a surjection. The computation of the cobordism groups of (mod 2) homology manifolds does not seem to have been made, although [4], [21], and [22] contain many nontrivial invariants which do not appear for manifolds. This already happens for $\bar{s}$-duality spaces: the Hasse invariants which appear in $W(\mathbb{Z}_{(2)})$ (see 16) do not appear in cobordism of manifolds. It would be interesting to see if geometric surgery methods like the ones developed here could be used to make the computation of the cobordism groups of homology manifolds.

*Conjecture.* Suppose there exists $c_0 \geq 4$ such that $\bar{p}(c) = \bar{m}(c)$ for all $c \leq c_0$. Then the cobordism class of a $\bar{p}$-duality space $X$ is determined by the single characteristic number $I\mathcal{X}(X; \mathbb{Z}/(2))$, the intersection homology Euler characteristic of $X$.

21.3. *Rational and integral characteristic numbers.* It would be interesting to find examples of spaces for which some Chern or Pontrjagin numbers could be formed. Our techniques do not give a solid approach to this question because we use cohomology operations (in intersection homology) to construct the Whitney classes in intersection homology, and we do not know of a similar method for constructing Pontrjagin classes. However the computation of the $\mathbb{Q}$- and $\mathbb{Z}$-Witt space cobordism groups ([37], [33]) represent first steps in this direction.

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