

# Morse theory and Intersection Homology theory

by

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## Introduction

0.1 The intersection homology of a singular complex variety exhibits many of the same properties as the ordinary homology of a nonsingular variety. For example it satisfies Poincare duality, and the hard Lefschetz theorem, and conjecturally it has a pure Hodge decomposition. (See [CGM], introduction, for a discussion of this.) In this paper we add a further property to the list: a critical point of a Morse function has a Morse index.

The main results of classical Morse theory for ordinary homology and for a compact smooth variety  $M$  can be summarized as follows:

For an open, dense, set of functions  $f : M \rightarrow \mathbb{R}$  (called Morse functions), all values  $v \in \mathbb{R}$  have exactly one of the following properties (and only finitely many values have property 2 )

1) For small enough  $\epsilon$ ,  $M_{<v+\epsilon}$  is homeomorphic to  $M_{<v-\epsilon}$

2) There is an  $i$  (called the Morse index of the critical point  $p$  with critical value  $v$ ) such that for small enough  $\epsilon$ ,

$$H_k(M_{<v+\epsilon}, M_{<v-\epsilon}) = \begin{cases} 0 & \text{for } k \neq i \\ \mathbb{Z} & \text{for } k = i \end{cases}$$

Here  $M_{<c}$  denotes the inverse image by  $f$  of the open interval  $(-\infty, c)$ .

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The Morse theory we present here for intersection homology and for a compact Whitney stratified singular complex analytic variety  $X$  analytically embedded in a smooth variety  $M$ , implies the following:

For an open dense set of functions  $f : M \longrightarrow \mathbb{R}$  (called Morse functions in the sense of Lazzeri and Pignoni), all values  $v \in \mathbb{R}$  have exactly one of the following properties (and only finitely many values have property 2):

- 1) For small enough  $\varepsilon$ ,  $X_{<v+\varepsilon}$  is homeomorphic to  $X_{<v-\varepsilon}$  in a stratum perserving way.
- 2) There is an  $i$  (again called the Morse index of the critical point  $p$  with critical value  $v$ ) such that for small enough  $\varepsilon$ ,

$$\mathrm{IH}_k(X_{<v+\varepsilon}, X_{<v-\varepsilon}) = \begin{cases} 0 & \text{for } k \neq i \\ A_p & \text{for } k = i \end{cases}$$

Here  $A_p$  is called the Morse group of the critical point  $p$ ; it is not necessarily  $\mathbb{Z}$ . It depends only on the stratum of  $X$  containing  $p$ , not on  $f$ .

The existence of a Morse index is false for ordinary homology in the singular case:  $H_k(X_{<v+\varepsilon}, X_{<v-\varepsilon})$  may be nonzero for several different  $k$ . For an example, see section 4.5 example 3.

0.2 As in the case of a smooth complex analytic variety, convexity properties of  $f$  can be used to give a priori estimates for the Morse indices of its critical points. We will use this idea in this paper to obtain three results:

- 1) The Lefschetz hyperplane theorem holds for the intersection homology of a (singular) projective algebraic variety (section 5.4)
- 2) The intersection homology of a complex  $n$ -dimensional Stein space vanishes in dimensions  $>n$  (section 5.3)
- 3) The sheaf of intersection chains on a general fibre specializes (over a curve) to a perverse object [BBDG] on the special fibre (section 6.1).

Since our initial work on Morse theory, other methods have been used to obtain some of these results. Deligne observed that the sheaf theoretic method of Artin [A] could be used to prove (1). (See [GM2]). Malgrange [Mal], Kashiwara [K] and Bernstein [Ber] have used  $\mathcal{D}$  module techniques and Beilinson and Bernstein [BB] have used characteristic  $p$  techniques to prove (3). However the method of Morse theory has several advantages: it can be used to study the intersection homology with  $\mathbb{Z}$  coefficients (as well as  $\mathbb{Q}$  coefficients) and it applies to analytic (as well as algebraic) varieties.

0.3 we now give a more detailed statement of our main results. We assume that  $X$  is a purely  $n$  dimensional complex analytic variety, Whitney stratified with complex analytic strata, and embedded in a complex analytic manifold  $M$ .

Definition: A  $C^\infty$  function  $f : M \rightarrow \mathbb{R}$  is called a Morse function for  $X$  provided

- 1) For each stratum  $S$  of  $X$ , the function  $f|_S$  has only nondegenerate critical points.

(The critical points of  $f$  are the critical points of  $f|_S$  for some stratum  $S$ ; the critical values of  $f$  are the values of  $f$  at these points.)

- 2) At each critical point  $p \in X$ , the differential  $df(p)(\tau) \neq 0$  whenever  $\tau$  is a limit of tangent planes from some larger stratum  $T \neq S$
- 3) All critical values are distinct.

This definition is due to Lazzeri [L] and Pignoni [P].

If  $p$  is a critical point in the stratum  $S$ , then the Morse index  $i$  of  $f$  at  $p$  is  $c + \lambda$  where  $c$  is the complex co-dimension of  $S$  in  $X$  and  $\lambda$  is the classical Morse index of  $f|_S$ .

To determine the Morse group  $A_p$  we make the following construction. Choose a complex analytic manifold  $N$  meeting  $S$  transversely at  $p$  and a generic projection  $\pi : N \cap X \rightarrow \mathbb{C}$  sending  $p$  to  $0$ . For  $0 < \epsilon \ll \delta \ll 1$ , the complex link  $\mathcal{L}$  of  $S$  and its boundary  $\partial\mathcal{L}$  are

$$\mathcal{L} = \pi^{-1}(t) \cap B_\delta$$

$$\partial\mathcal{L} = \pi^{-1}(t) \cap \partial B_\delta$$

where  $0 < |t| < \epsilon$  and  $B_\delta$  is the intersection with  $X$  of a ball of radius  $\delta$  centered at  $p$  and  $\partial B_\delta$  is the intersection of  $X$  with the boundary of the ball. The Morse group  $A_p$  is the image of the variation map

$$(1-\mu) : \text{IH}_{c-1}(\mathcal{L}, \partial\mathcal{L}) \longrightarrow \text{IH}_{c-1}(\mathcal{L}) .$$

(The variation map associates to a chain  $Z$  in  $\mathcal{L}$  with boundary in  $\partial\mathcal{L}$  the chain  $Z - \mu Z$  where  $\mu$  is the monodromy transformation obtained by "carrying  $Z$  over a small loop around 0 in  $\mathbb{C}$ ". The chain  $Z - \mu Z$  is a cycle in  $\mathcal{L}$  since the monodromy is the identity near  $\partial\mathcal{L}$ . For a rigorous explanation of the variation map, see section 3.8).

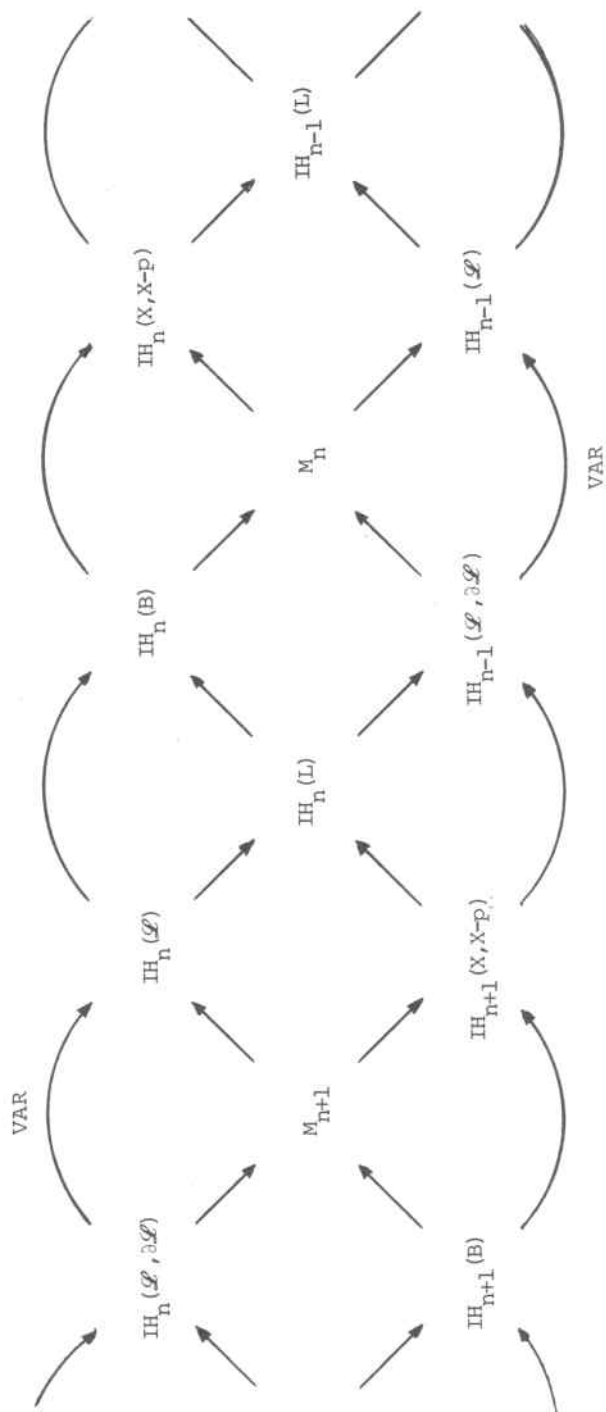
Theorem: Suppose  $f$  is a Morse function for  $X$  and  $f|X$  is proper.

- 1) If  $v$  is not a critical value, then for small enough  $\varepsilon$ ,  $X_{<v+\varepsilon}$  and  $X_{<v-\varepsilon}$  are homeomorphic by a stratum perserving map.
- 2) If  $v$  is the critical value for the critical point  $p$ , then for small enough  $\varepsilon$

$$\text{IH}_k(X_{<v+\varepsilon}, X_{<v-\varepsilon}) = \begin{cases} 0 & \text{if } k \neq i \\ A_p & \text{if } k = i \end{cases}$$

where  $i$  is the Morse index of  $f$  at  $p$  and  $A_p$  is the Morse group of the stratum through  $p$ .

0.4 Method of proof: In section §4.4 we will construct the following commutative diagram, where each sinusoidal curve is an exact sequence. (This is a local intersection homology version of the braid diagram in Clemens [C]). Here  $B$  is  $N \cap B_\delta$  and  $L$ , the link of  $S$ , is  $N \cap \partial B_\delta$ . The group  $\text{IH}_k(X_{<v+\varepsilon}, X_{<v-\varepsilon})$  is denoted  $M_k$ . The variation map is marked "VAR".



In the language of SGA IV,  $M_k$  is the group of " $R\phi$  vanishing cycles" and  $IH_k(\mathcal{L}, \partial\mathcal{L})$  is the group of " $R\psi$  vanishing cycles" for  $\pi$  at  $p$  (of the sheaf of intersection chains)

Remark: We may replace the symbol  $IH$  in this diagram with  $IH^{\bar{p}}$  for any perversity  $\bar{p}$ , or with homology or cohomology, and in each case the resulting diagram will commute and have exact sinusoidal curves.

The proof of the theorem is a simple diagram chase, once we know the following facts about intersection homology:

- a)  $IH_i(B) = 0$  for all  $i \geq n$ ;  $IH_i(B) = IH_i(L)$  for all  $i < n$ .
- b)  $IH_i(X, X-p) = 0$  for all  $i \leq n$ ;  $IH_i(X, X-p) \cong IH_{i-1}(L)$  for all  $i > n$
- c)  $IH_i(\mathcal{L}) = 0$  for all  $i > n - 1$
- d)  $IH_i(\mathcal{L}, \partial\mathcal{L}) = 0$  for all  $i < n - 1$

Facts (a) and (b) are restatements of the support conditions which characterize intersection homology (e.g. see [GM2]). Facts (c) and (d) are proven using Morse theory (and induction) for the function  $\pm g : \mathcal{L} \longrightarrow \mathbb{R}$  which is given by  $g(y) = \text{distance}(y, p)$

### §1. Intersection Homology Of Subanalytic Pseudomanifolds

The statements and proofs in this paper use relative intersection homology  $IH_k(X, A)$  where  $X$  is a stratified pseudomanifold and  $A$  is an open subset. The only properties of relative intersection homology that we will need are topological invariance (sec. 1.3), excision (sec. 1.5), the Kunneth theorem (sec. 1.6), and the local calculation and cocalculation (sec. 1.7).

There are several constructions of intersection homology which may be adapted to the relative case. For example, using the piecewise linear construction of [GM0] and [GM1] one may define  $IH_k(X,A)$  to be the  $k^{\text{th}}$  homology group of the quotient chain complex  $IC_*(X)/IC_*(A)$ . Using the sheaf theoretic version of [GM2], one may define  $IH_k(X,A)$  to be  $H_c^{-k} Rj_* j^* \underline{IC}(X)$  where  $j$  is the inclusion of  $X - A$  in  $X$  and  $H_c$  denotes hypercohomology with compact supports. Readers familiar with one of the above constructions can verify the properties mentioned above and skip chapter 1.

We present here a construction of relative intersection homology which is defined when  $X$  is an open subset of a subanalytic pseudomanifold. We believe that there are several advantages to this approach. It gives a canonical group of chains (not requiring the choice of a PL structure), and it is more geometric than the sheaf theoretic approach.

### 1.1 Pseudomanifolds:

An oriented  $n$ -dimensional subanalytic pseudomanifold  $X$  is an open subset of a locally compact (real) subanalytic set, together with a Whitney subanalytic stratification  $X_0 \subset X_1 \subset \dots \subset X_{n-2} = \Sigma \subset X$ . Here,  $\dim_{\mathbb{R}} X_i = i$ ,  $\Sigma$  is the singular set, and each  $X_i - X_{i-1}$  is a union of finitely many  $i$ -dimensional analytic manifolds.  $X - \Sigma$  is oriented and dense in  $X$  and each  $x \in X_i - X_{i-1}$  has a neighborhood which is subanalytically isomorphic to  $E^i \times \text{cone}^\circ(L_x)$  by an isomorphism which takes strata to strata. Here,  $E^i$  is the open  $i$ -dimensional disc, and  $L_x$  is a subanalytic pseudomanifold of dimension  $n - i - 1$ , and  $\text{cone}^\circ$  denotes the open cone, i.e. the cone minus its base. We shall assume that  $X$  has only



strata of even codimension, i.e.,  $X_i = X_{i-1}$  if  $n-i$  is odd.

### 1.2 Chains

Suppose  $X$  is an oriented  $n$ -dimensional subanalytic pseudomanifold with strata of even codimension only. Let  $\underline{C}_i$  be the sheaf of  $i$ -dimensional subanalytic chains on  $X$  (see Hardt [H]) with integer coefficients. This sheaf assigns to any open subset  $U \subset X$  the group  $C_i^{BM}(U)$  of subanalytic chains  $\xi$  with closed support  $|\xi| \subset U$ . (These are chains with "infinite support", or "Borel-Moore" chains). The boundary operator gives  $\underline{C}_*$  the structure of a complex of (fine) sheaves. Define the subcomplex  $\underline{IC}_*$  by

$$IC_i^{BM}(U) = \{ \xi \in C_i^{BM}(U) \mid \dim|\xi| \cap X_m \leq i - (n-m)/2 - 1 \text{ and} \\ \dim|\partial\xi| \cap X_m \leq i - (n-m)/2 - 2 \text{ for all } m < n \}.$$

Definition:  $IH_i(X) = IH_i(X, \mathbb{Z})$  is the homology of the chain complex

$$\dots \longrightarrow IC_{i+1}(X) \longrightarrow IC_i(X) \longrightarrow IC_{i-1}(X) \longrightarrow \dots$$

where  $IC_i(X)$  denotes the subgroup of  $IC_i^{BM}(X)$  consisting of chains with compact support. Similarly,  $IH_i^{BM}(X)$  is the homology of the complex

$$\dots \longrightarrow IC_{i+1}^{BM}(X) \longrightarrow IC_i^{BM}(X) \longrightarrow IC_{i-1}^{BM}(X) \longrightarrow \dots$$

### 1.3 Relative Intersection Homology

If  $X$  is a (not necessarily compact) pseudomanifold, then any open subset  $U \subset X$  inherits the structure of a stratified pseudo-

manifold. Therefore  $IC_*(U)$  is defined as above and is a subcomplex of  $IC_*(X)$ . Define  $IC_*(X,U)$  to be the quotient complex  $IC_*(X)/IC_*(U)$ . This gives rise to the exact sequence

$$\dots \longrightarrow IH_i(U) \longrightarrow IH_i(X) \longrightarrow IH_i(X,U) \longrightarrow IH_{i-1}(U) \longrightarrow \dots$$

An inclusion of open subsets  $(X',U') \subset (X,U)$  induces a map between the corresponding long exact sequences. Similarly if  $V$  is open in  $U$ , and if  $U$  is open in  $X$ , then we have the long exact sequence

$$\dots \longrightarrow IH_i(U,V) \longrightarrow IH_i(X,V) \longrightarrow IH_i(X,U) \longrightarrow IH_{i-1}(U,V) \longrightarrow \dots$$

Theorem: ([GM1],[GM2])  $IH_i(X,U)$  is independent of the choice of stratification of  $X$  and is even a topological invariant of  $(X,U)$ , i.e. if  $f : X \longrightarrow X'$  is a homeomorphism of subanalytic pseudo-manifolds which takes the open subset  $U \subset X$  homeomorphically to  $U' \subset X'$  then  $f$  induces an isomorphism  $IH_i(X,U) \cong IH_i(X',U')$  for all  $i$ .

Proof: The proof follows from the identity

$$IH_i(X,U) = H_C^{-i}(Rj_{*\!}j^* \underline{IC})$$

(where  $j : X - U \subset X$  is the inclusion) and the topological invariance (up to quasi-isomorphism) of the complex of sheaves  $\underline{IC}$ . (see [GM2])

1.4 Remark on pseudomanifolds with boundary:

In this paper we will avoid taking the intersection homology of a pseudomanifold  $X$  with boundary  $\partial X$  (or with corners) by replacing it with the intersection homology of the interior  $X^\circ = X - \partial X$ . Similarly we will replace the relative intersection homology of  $(X, \partial X)$  with the intersection homology of the interior of  $X$  modulo a collared neighborhood of  $\partial X$ .

1.5 Excision Formula

If  $U$  is open in  $X$  and  $V$  is closed in  $U$ , then the inclusion  $(X-V, U-V) \longrightarrow (X, U)$  induces an excision isomorphism,

$$IH_i(X-V, U-V) \cong IH_i(X, U)$$

Proof: The proof is the same as for the excision formula in ordinary homology. The key ingredient is that each  $\underline{IC}_i$  is a fine sheaf.

1.6 Kunneth Formula

Let  $E^a$  denote the open disc  $\{y \in \mathbb{R}^a \mid \|y\| < 1\}$  and let  $\partial E^a \times E^1$  denote  $\{y \in \mathbb{R}^a \mid \frac{1}{2} < \|y\| < 1\}$  which is a collared neighborhood of the boundary of  $E^a$ . Suppose  $U$  is an open subset of a pseudomanifold  $X$ . Then

$$IH_i(X, U) \cong IH_i(X \times E^a, U \times E^a)$$

$$IH_i(X, U) \cong IH_{i+a}(X \times E^a, U \times E^a \cup X \times \partial E^a \times E^1)$$

1.7 The local calculation and cocalculation

Suppose  $x \in X$  is a point in some stratum  $S$  which has co-dimension  $2c$ . Let  $U$  be a neighborhood of  $x$  of the type considered in §1.1, ie  $U$  is homeomorphic to  $E^{n-2c} \times C^\circ(L_x)$  where  $C^\circ$  is the open cone and  $L_x$  is a pseudomanifold of dimension  $n - 2c - 1$ . Let  $\partial U \times E^1$  denote a collared neighborhood of the boundary of  $U$ , ie

$$\partial U \times E^1 = \partial E^{n-2c} \times E^1 \times C^\circ(L_x) \cup E^{n-2c} \times L_x \times E^1.$$

Then ([GM2]§2.2)

$$IH_i^{BM}(U) = IH_i(U, \partial U \times E^1) = \begin{cases} IH_{i-n+2c-1}(L_x) & \text{if } i \geq n - c + 1 \\ 0 & \text{if } i < n - c + 1 \end{cases}$$

$$IH_i(U) = \begin{cases} 0 & \text{if } i \geq c \\ IH_i(L_x) & \text{if } i < c \end{cases}$$

Proof: To calculate  $IH_i(U, \partial U \times E^1)$  it suffices to find  $IH_{i-n+2c}(C^\circ(L_x), L_x \times E^1)$  because of the Kunneth formula. If  $\xi$  is a cycle in  $IC_k(C^\circ(L_x), L_x \times E^1)$  and  $k \leq c$  then  $|\xi|$  does not intersect the cone point  $\{x\}$ . Thus "pushing along cone line" deforms this cycle into the neighborhood  $L_x \times E^1$  of the boundary. Thus  $IH_k(C^\circ(L_x), L_x \times E^1) = 0$ . However if  $k \geq c + 1$  then  $\xi$  is homologous to the cone over a cycle  $\xi'$  in  $L_x$  which can be obtained by intersecting  $\xi$  with  $L_x \times \{t\}$  for some  $t \in E^1$ .

Similarly  $IH_i(U) \cong IH_i(C^\circ(L_X))$ . Any compactly supported cycle  $\xi \in IC_i(C^\circ(L_X))$  is the boundary of  $\text{cone}(|\xi|)$  which is an allowable chain in  $IC_{i+1}(C^\circ(L_X))$  provided  $i \geq c$ . Otherwise  $\xi$  can be deformed into an  $i$ -dimensional cycle in  $L_X \times \{t\}$  for some  $t \in E^1$ .

### 1.8 Twisted Coefficients

Let  $X$  be a stratified subanalytic pseudomanifold with even codimension strata and with singularity set  $\Sigma$ . For any stratification of any  $i$ -dimensional chain  $\xi$  which satisfies the allowability conditions of §1.2, each  $i$ -dimensional stratum of  $\xi$  and each  $i - 1$  dimensional stratum of  $\xi$  will be contained in  $X - \Sigma$ . Thus we may speak of chains  $\xi$  with coefficients in  $T$ , whenever  $T$  is a local system of coefficients on  $X - \Sigma$ . (The fibre of  $T$  must be a regular Noetherian ring).

Definition: Let  $T$  be a local coefficient system in the sense of Steenrod [S] on  $X - \Sigma$ .  $\underline{IC}_i(T)$  is the sheaf on  $X$  whose sections over an open set  $U$  is  $IC_i^{BM}(U;T)$ , i.e. the group of  $i$ -dimensional subanalytic chains  $\xi$  with closed support in  $U$ , with coefficients in  $T$ , and which satisfy the allowability conditions of §1.2.  $IH_i(X;T)$  is the homology of the chain complex of sections with compact support of the sheaves  $\underline{IC}_*(T)$ .

Theorem:  $IH_i(X;T)$  exhibits the formal properties which were listed above: long exact sequence for a pair and triple, excision formula,

Kunneth formula, local calculation and co-calculation.

## §2. Real Morse Theory

### §2.1 Morse Functions on Subanalytic Sets

Suppose  $X$  is a subanalytic set. In order to obtain a class of "smooth" maps  $f : X \rightarrow \mathbb{R}$  and to obtain a suitable topology on this collection of maps, let us assume  $X$  is embedded in some smooth analytic manifold  $M$ . Fix a Whitney stratification of  $X$  by analytic submanifolds.

Definition: A function  $f : X \rightarrow \mathbb{R}$  is a Morse function provided

- a)  $f$  is the restriction of some smooth  $\tilde{f} : M \rightarrow \mathbb{R}$
- b) For each stratum  $A$  of  $X$ , the function  $f|_A$  has only non-degenerate critical points.
- c) All critical values are distinct.
- d) At each critical point  $x_0 \in A$ , the differential  $d\tilde{f}(x_0)(\tau) \neq 0$  whenever  $\tau$  is a limit of tangent planes from some larger stratum  $B > A$ .

Remarks: Every zero dimensional stratum is a critical point. The subset of smooth functions  $g : M \rightarrow \mathbb{R}$  which are Morse functions on  $X$  depends on the stratification of  $X$ .

This definition of Morse function was made by Lazzeri [L] and was amplified by Pignoni [P] who also proved:

Theorem: [P] If  $X$  is closed in  $M$  then the subset of smooth proper functions  $g : M \rightarrow \mathbb{R}$  whose restriction to  $X$  is Morse, forms an open dense subset of  $C^\infty_{\text{proper}}(M, \mathbb{R})$ . Each such restriction  $g|_X$  is topologically stable.

## 2.2 Morse Data

The main technical tool in this paper is the theorem in this section which was announced in [GM3]. The statements here are parallel to those in [GM3] except the spaces considered here are open subsets of pseudomanifolds (eg,  $X_{<a}$ ) instead of pseudomanifolds with boundaries (eg,  $X_{\leq a}$ ). Complete proofs of the theorem in this section will appear in [GM4].

Let  $X$  be a closed Whitney stratified subanalytic subset of some analytic manifold  $M$ . Suppose  $f : X \rightarrow \mathbb{R}$  is a proper Morse function which is the restriction of some smooth  $\tilde{f} : M \rightarrow \mathbb{R}$ . Fix a critical point  $x_0$  in some stratum  $A$  of  $X$ . Let  $\lambda$  be the Morse index of  $f|_A$  at  $x_0$ . Set  $v = f(x_0)$  and  $a = \dim(A)$ . Choose an analytic submanifold  $V \subset M$  which meets  $A$  transversally in the single point  $\{x_0\}$ .

Let  $B_\delta(x_0)$  be the closed ball of radius  $\delta$  with respect to some local coordinates, centered at  $x_0$  and let  $B_\delta^\circ(x_0)$  denote its interior. Choose  $\delta > 0$  to be so small that the following condition holds.

$\bar{N} = B_\delta(x_0) \cap V \cap X$  is compact,  $\delta B_\delta(x_0)$  meets each stratum of  $V \cap X$  transversely, and the same is true for all  $\delta' < \delta$ .

Shrink  $\delta$  if necessary so that the following condition also holds: there exists  $\epsilon > 0$  so that  $f$  is a Morse function on

$N \cap f^{-1}(v-2\varepsilon, v+2\varepsilon)$  with no critical points on any stratum of  $N$ , other than  $\{x_0\}$  (including strata of  $\delta B_\delta(x_0) \cap V \cap X$ ); and so that a similar statement is true for any  $\delta' < \delta$ .

Define  $X_{<a} = \{x \in X \mid f(x) < a\}$ . Let  $N = B_\delta^0(x_0) \cap V \cap X$  and let  $E^m$  denote the open  $m$ -disc with  $\partial E^m \times E^1$  denoting the complement of a smaller  $m$ -disc.

Theorem:  $X_{<v+\varepsilon}$  is homeomorphic to the space obtained from  $X_{<v-\varepsilon}$  by attaching

$$E^\lambda \times E^{a-\lambda} \times N_{<v+\varepsilon}$$

along

$$(\partial E^\lambda \times E^1 \times E^{a-\lambda} \times N_{<v+\varepsilon}) \cup (E^\lambda \times E^{a-\lambda} \times N_{<v-\varepsilon})$$

Furthermore, the homeomorphism types of the space  $N$ , and of the pair  $(N_{<v+\varepsilon}, N_{<v-\varepsilon})$  are independent of the choices of  $v$ ,  $\varepsilon$ , and  $\delta$ .

Corollary: Using the Kunneth formula and excision, we have

$$IH_i(X_{<v+\varepsilon}, X_{<v-\varepsilon}) = IH_{i-\lambda}(N_{<v+\varepsilon}, N_{<v-\varepsilon}) .$$

Remark: In the case that  $X$  is a complex analytic variety, the normal slice  $V$  can be chosen to be complex analytic also. Thus  $N$  is a complex analytic space with a subanalytic compactification  $\bar{N}$ . In the next chapter we will study the local geometric structure of this space  $N$ .



§3. Local Geometry of Complex Analytic Spaces

3.1. Introduction: In this section we study a particular neighborhood  $B$  and its boundary  $L$  of a zero dimensional stratum  $\{x_0\}$  in an analytic variety  $N$  by stratifying a generic projection  $p : N \rightarrow \mathbb{C}$  and intersecting  $B$  with the fibres of this projection. The generic fibre of  $p$  intersects the pair  $(B,L)$  in a pair  $(\mathcal{L}, \partial\mathcal{L})$  which we call the complex link (and its boundary) of the point  $x_0$ . There is a monodromy homeomorphism  $\mu : \mathcal{L} \rightarrow \mathcal{L}$  which is the identity on a collared neighborhood of  $\partial\mathcal{L}$  and from this information the pair  $(B,L)$  can be recovered up to homeomorphism as follows:  $\mathcal{L}$  is obtained from  $\mathcal{L} \times [0,1]$  by identifying  $(\ell,1)$  to  $(\mu(\ell),0)$  and then by attaching  $\partial\mathcal{L} \times D^2$  where  $D^2$  is a closed 2-disk along  $\partial\mathcal{L} \times S^1$ . Then  $B$  is homeomorphic to the cone over  $L$  with cone vertex at  $x_0$ .

If  $X$  is a complex analytic variety with a Whitney stratification and if  $x_0$  is a point in some stratum  $S$  then this discussion will apply (for example, in section §4.4) to a normal slice  $N$  through  $X$  which is transversal to  $S$  at  $x_0$ .

3.2 Generic Projections

Let  $N$  be a Whitney stratified (not necessarily closed) complex analytic variety in some  $\mathbb{C}^m$ . Suppose  $\{x_0\}$  is a zero-dimensional stratum of  $N$ . Let  $p : N \rightarrow \mathbb{C}$  be the restriction of a linear projection such that  $p(x_0) = 0$  and  $\operatorname{Re}(p)$  is a Morse function near  $x_0$ . This means that  $dp(x_0)(\tau) \neq 0$  whenever  $\tau$  is a limit of tangent planes from a larger stratum.

The set of such projections is open and dense in the space of all linear maps  $\mathbb{C}^m \rightarrow \mathbb{C}$ . (See Pignoni [P]).

Notation: Let  $B_r(x_0)$ ,  $B_r^O(x_0)$ ,  $\partial B_r(x_0)$  denote respectively the closed ball, open ball, and boundary of the ball of radius  $r$  centered at  $x_0$ . Let  $D_r$ ,  $D_r^O$ ,  $\partial D_r$  denote respectively the closed disc, open disc, and boundary of the disc of radius  $r$  centered at  $0 \in \mathbb{C}$ .

### 3.3 Choice of $\epsilon$ and $\delta$

Definition: The real number  $\delta > 0$  is small if  $B_\delta(x_0) \cap N$  is compact,  $\partial B_\delta(x_0)$  is transverse to each stratum of  $N$ , and so that the same is true for all  $\delta'$  with  $0 < \delta' < \delta$ .

If  $\delta$  is small, the real number  $\epsilon > 0$  is subordinate to  $\delta$  (written  $\epsilon \ll \delta$ ) if  $\text{Re}(p)$  has no critical points (other than  $x_0$ ) on any stratum of  $N \cap B_\delta(x_0) \cap p^{-1}(D_{2\epsilon}^O)$  (including the strata in  $N \cap \partial B_\delta(x_0) \cap p^{-1}(D_{2\epsilon}^O)$ ). This is equivalent to the statement that for any  $\xi \in D_{2\epsilon}^O$ ,  $p^{-1}(\xi)$  is transverse to each stratum of  $N \cap B_\delta(x_0)$  with the single exception that  $p^{-1}(0)$  fails to be transverse to the stratum  $\{x_0\}$ .

The number  $\delta > 0$  is very small (written  $\delta \ll 1$ ) if it is small, and for any  $\delta' < \delta$  there exists an  $\epsilon > 0$  which is subordinate to  $\delta'$ . (Such a  $\delta$  exists because  $\text{Re}(p)$  is a Morse function).

Choose  $\delta > 0$  very small and choose  $\epsilon > 0$  subordinate to  $\delta$ . Let  $D = D_\epsilon$  and fix  $\xi \in D - \{0\}$ .

### 3.4 Definition of the Complex Link and related spaces

1. The complex link, its interior and boundary:

$$\mathcal{L} = p^{-1}(\xi) \cap B_\delta(x_0) \qquad \mathcal{L}^0 = p^{-1}(\xi) \cap B_\delta^0(x_0)$$

$$\partial\mathcal{L} = p^{-1}(\xi) \cap \partial B_\delta(x_0)$$

2. The particular neighborhood of  $x_0$  and its interior:

$$B = p^{-1}(D) \cap B_\delta(x_0) \qquad B^0 = p^{-1}(D^0) \cap B_\delta^0(x_0)$$

3. The "horizontal" and "vertical" parts of the real link, and their interiors:

$$L_h = p^{-1}(D) \cap \partial B_\delta(x_0) \qquad L_h^0 = p^{-1}(D^0) \cap \partial B_\delta^0(x_0)$$

$$L_v = p^{-1}(\partial D) \cap B_\delta(x_0) \qquad L_v^0 = p^{-1}(\partial D) \cap B_\delta^0(x_0)$$

4. The boundaries of the horizontal and vertical parts of the real link:

$$\partial L_v = \partial L_h = p^{-1}(\partial D) \cap \partial B_\delta(x_0)$$

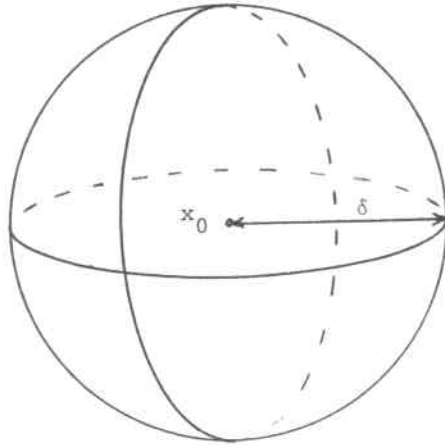
5. The real link:

$$L = \partial B = L_v \cup L_h$$

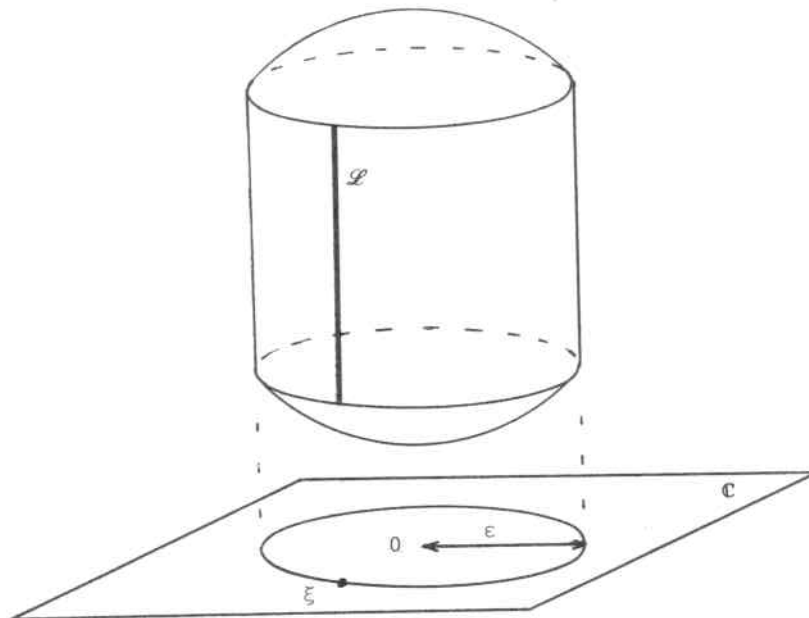
6. The cut off spaces:

$$B_{<0} = p^{-1}(\{z \mid \operatorname{Re} z < 0\}) \cap B \qquad B_{<0}^0 = B_{<0} \cap B_\delta^0(x_0)$$

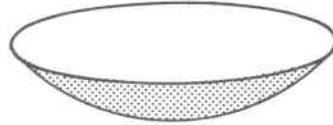
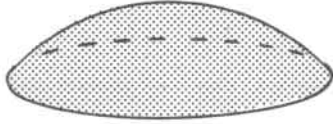
$$L_{<0} = B_{<0} \cap L$$



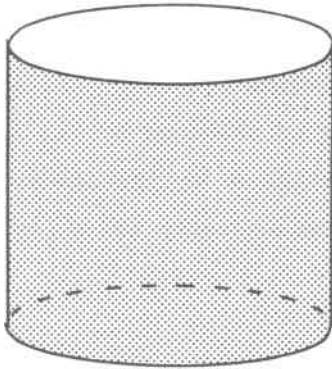
The ball  $B_\delta(x_0)$  in the normal slice



The particular neighborhood B



The horizontal part  $L_h$  of the real link



The vertical part  $L_v$  of the real link

3.5 Topological properties of the neighborhood

The results of this section were announced in [GM3]. The proofs involve delicate stratification theoretic arguments using controlled vector fields, and will be published in [GM5]. Previous results in this direction are due to Milnor [Mi2] and Lê [Lê].

Theorem: (a) The projection

$$p : B - p^{-1}(0) \longrightarrow D_\epsilon - \{0\}$$

is a topological fiber bundle.

(b) There is a homeomorphism

$$q : \text{Cone}(L) \longrightarrow B$$

of the cone on  $L$  to  $B$  which

takes the vertex of the cone to  $x_0$ , takes the base of the cone to  $L$  by the identity and takes  $\text{Cone}(L_{<0})$  homeomorphically to  $B_{<0} \cup \{x_0\}$

(c) There is an embedding

$$r : \partial\mathcal{L} \times [0,1) \times D \longrightarrow B$$

which takes  $\partial\mathcal{L} \times \{0\} \times D$  homeomorphically to  $L_h$  and such that  $p \circ r : \partial\mathcal{L} \times [0,1) \times D \longrightarrow D$  is the projection on the third factor.

We denote the part of the collared neighborhood of  $\partial\mathcal{L}$  that lies in  $L^0$  by  $\partial\mathcal{L} \times E^1 \subset \mathcal{L}^0$

(d) The pair  $(B_{<\frac{\varepsilon}{2}}^O, B_{<-\frac{\varepsilon}{2}}^O)$  is homeomorphic to the pair  $(B_\delta^O(x_0)_{<\eta}, B_\delta^O(x_0)_{<-\eta})$  (for some  $\eta$  sufficiently small), where  $B_\delta^O(x_0)_{<a} = \{x \in B_\delta^O(x_0) \mid \operatorname{Re}(p(x)) < a\}$  and similarly for  $B_{<a}^O$ .

Remark: Part (c) of this theorem says that the fibration of part (a) is trivial near  $L_h$ , and that the trivialization can be chosen to extend over the whole disk  $D$ . It also says that  $L_h$  has a collared neighborhood in  $B$ ,  $\partial\mathcal{L}$  has a collared neighborhood in  $\mathcal{L}$ , and these collared neighborhoods are compatible.

Theorem: The topological type of each of the spaces and maps defined in the preceding sections is independent of the choices of  $p$ ,  $\varepsilon$ , or  $\delta$ . If  $N$  was obtained as the normal slice at  $x_0$  to a connected stratum  $S$  in some Whitney stratified complex analytic variety  $X$ , then the topological type of these spaces and maps is also independent of the choice of  $N$  or  $x_0$ .

### 3.6 Intersection homology of the neighborhood

The following are corollaries of the first theorem of the last section (3.5).

Corollary 1: There is an isomorphism

$$\operatorname{IH}_i(B^O - \{x_0\}, B_{<0}) = \operatorname{IH}_{i-1}(\mathcal{L}^O, \partial\mathcal{L} \times E^1)$$

for all  $i$

Proof: We prove this corollary in four steps.

Step 1:

$$IH_i(B^O - \{x_0\}, B_{<0}^O) = IH_i(L, L_{<0})$$

In fact there is a homeomorphism of pairs of topological spaces

$$(B^O - \{x_0\}, B_{<0}^O) = (L, L_{<0}) \times E^1$$

as may be seen from Theorem 3.5, part 2. The isomorphism above follows then from the Kunneth theorem.

Step 2: Let

$$\begin{aligned} L_c &= L_{<0} \cup L_h \cup r(\partial\mathcal{L} \times [0, \frac{1}{2}] \times \partial D) \\ &= L_{<0} \cup r(\partial\mathcal{L} \times \{0\} \times D \cup \partial\mathcal{L} \times [0, \frac{1}{2}] \times \partial D) \end{aligned}$$

We claim  $(L, L_{<0})$  is homeomorphic to  $(L, L_c)$  therefore

$$IH_i(L, L_{<0}) \cong IH_i(L, L_c)$$

In order to see this, consider the subspace of  $L$ ,

$$\tilde{L}_h = r(\partial\mathcal{L} \times \{0\} \times D \cup \partial\mathcal{L} \times [0, 1] \times D)$$

This is an open neighborhood of  $L_h$  in  $L$ . The homeomorphism  $L_h \cong \partial\mathcal{L} \times D$  extends to a homeomorphism  $\tilde{L}_h \cong \partial\mathcal{L} \times D_{3\varepsilon}^O$  where

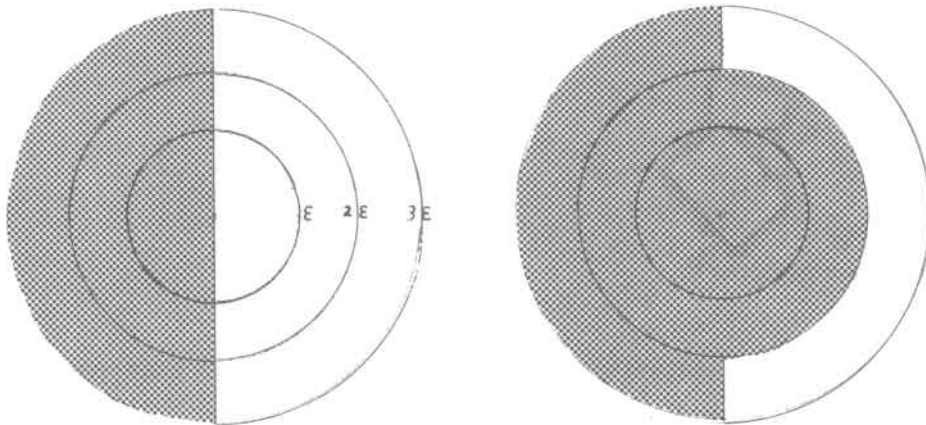


$D_{3\varepsilon}^0 = \{z \in \mathbb{C} \mid |z| < 3\varepsilon\}$ , in such a way that the collaring lines project to radial lines in  $\mathbb{C}$ . Algebraically,

$$\{r(\partial\mathcal{L} \times [0,1] \times \partial D) \longrightarrow \partial\mathcal{L} \times \mathbb{C} \quad (*)$$

$$r(\ell, t, \xi) \longrightarrow (\ell, (1+2t)\xi)$$

There is a homeomorphism of the following pairs of open subsets of  $\mathbb{C}$  (where the larger space is the  $3\varepsilon$  disc and the subspace is shaded)



which is the identity in a neighborhood of the  $3\varepsilon$  circle. Applying this homeomorphism with (\*) above establishes the claim.

Step 3. Consider the pair of subspaces of  $\mathbb{C}$ ,

$$I = \{z \in \partial D \mid -\frac{1}{2} \varepsilon < \operatorname{Re}(z)\}$$

$$J = \{z \in \partial D \mid -\frac{1}{2} \varepsilon < \operatorname{Re}(z) < 0\}$$

The restriction of the fibration  $p : L_V^{\circ} \longrightarrow \partial D$  to the open interval  $I$  is trivial. By theorem 3.5(c) the trivialization  $T : \mathcal{L}^{\circ} \times I \longrightarrow p^{-1}(I) \cap L_V^{\circ}$  can be chosen so as to take  $(\partial \mathcal{L} \times E^1) \times I$  homeomorphically to  $r(\partial \mathcal{L} \times (0, \frac{1}{2}) \times I)$ . The pair  $(L^{\circ}, \partial \mathcal{L} \times E^1) \times (I, J)$  is then taken by  $T$  to the pair  $(L, L_c)$ . By excision,

$$\operatorname{IH}_i(L, L_c) = \operatorname{IH}_i((\mathcal{L}^{\circ}, \partial \mathcal{L} \times E^1) \times (I, J))$$

Step 4.

$$\operatorname{IH}_i((\mathcal{L}^{\circ}, \partial \mathcal{L} \times E^1) \times (I, J)) = \operatorname{IH}_{i-1}(\mathcal{L}^{\circ}, \partial \mathcal{L} \times E^1)$$

by the Kunneth formula.

Remark: A careful analysis of the above argument shows that the resulting isomorphism of

$$\operatorname{IH}_i(B^{\circ} - \{x_0\}, B_{<0}^{\circ}) \text{ with } \operatorname{IH}_{i-1}(L^{\circ}, \partial \mathcal{L}^{\circ} \times E^1)$$

is canonical.

Corollary 2. There is an isomorphism

$$\mathrm{IH}_i(B_\delta^0(x_0)_{<\eta}, B_\delta^0(x_0)_{<-\eta}) \cong \mathrm{IH}_i(B^0, B_{<0}^0)$$

Remark: By the results of §2.2, the left hand side of this equation is the Morse group.

Proof of corollary 2. From theorem 3.5(d) we have,

$$\mathrm{IH}_i(B_\delta^0(x_0)_{<\eta}, B_\delta^0(x_0)_{<-\eta}) \cong \mathrm{IH}_i(B_{<\frac{\varepsilon}{2}}^0, B_{<-\frac{\varepsilon}{2}}^0)$$

The inclusion of pairs  $(B_{<\frac{\varepsilon}{2}}^0, B_{<-\frac{\varepsilon}{2}}^0) \longrightarrow (B^0, B_{<0}^0)$  induces an isomorphism

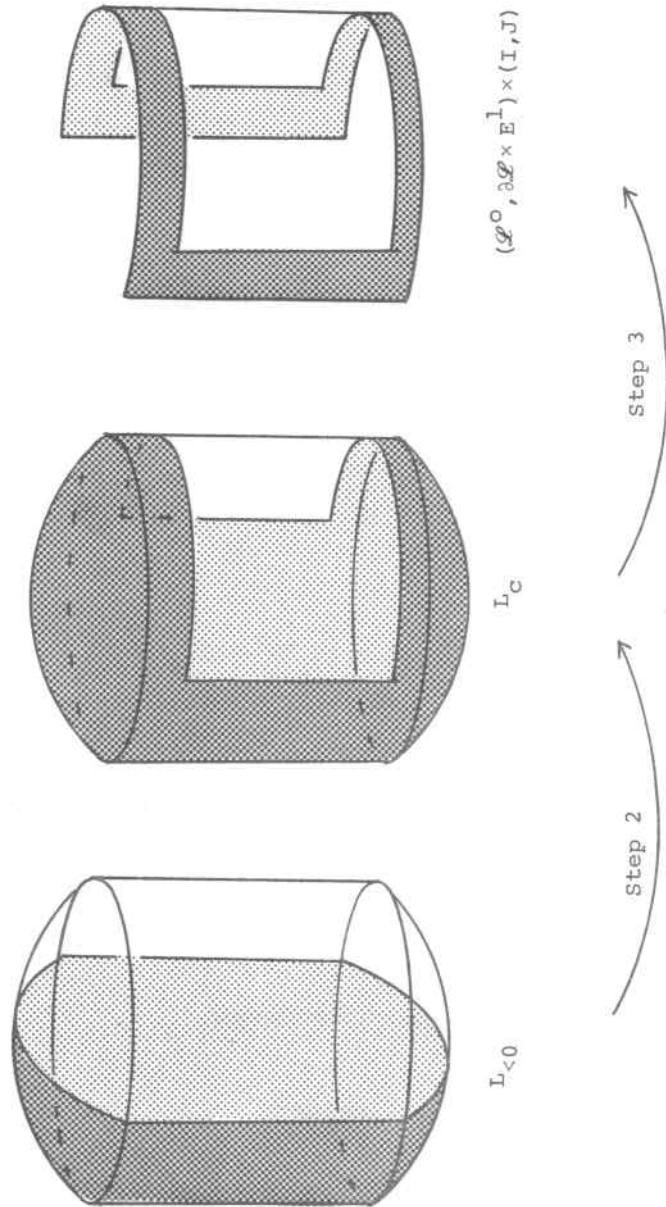
$$\mathrm{IH}_i(B_{<\frac{\varepsilon}{2}}^0, B_{<-\frac{\varepsilon}{2}}^0) \cong \mathrm{IH}_i(B^0, B_{<0}^0)$$

for the following reason: from theorem 3.5(a) the inclusions  $B_{<\varepsilon}^0 \longrightarrow B^0$  and  $B_{<-\varepsilon}^0 \longrightarrow B_{<0}^0$  are isotopic through inclusions to homeomorphisms. Therefore these inclusions induce isomorphisms on intersection homology. The corollary now follows from the five lemma.

Corollary 3. There is a canonical isomorphism

$$\mathrm{IH}_i(B_{<0}^0) \cong \mathrm{IH}_i(\mathcal{L}^0)$$

Proof: By theorem 3.6 part 1  $B_{<0}^0 \approx \mathcal{L}^0 \times E^2$ .



Proof of Corollary 1

3.7 Variation Map

Definition: The variation map is the composition

$$\begin{aligned}
 & \text{IH}_i(\mathcal{L}, \partial\mathcal{L}) \\
 & \cong \text{IH}_i(\mathcal{L}^0, \partial\mathcal{L} \times E^1) && \text{(see §1.4)} \\
 & \cong \text{IH}_{i+1}(B_{<0}^0 - \{x_0\}, B_{<0}^0) && (\text{§3.6 corollary 1}) \\
 & \quad \downarrow \partial_* \\
 & \cong \text{IH}_i(B_{<0}^0) \\
 & \cong \text{IH}_i(\mathcal{L}^0) && (\text{§3.6 corollary 3})
 \end{aligned}$$

Intuitive Description: This homomorphism may be defined rigorously in the subanalytic context as follows: by Hardt [H] the monodromy  $\mu : \mathcal{L} \rightarrow \mathcal{L}$  may be taken to be a subanalytic homeomorphism which fixes a collared neighborhood  $U$  of  $\partial\mathcal{L}$  in  $\mathcal{L}$ . The subcomplex  $\tilde{\text{IC}}_*(\mathcal{L}^0, U)$  of  $\text{IC}_*(\mathcal{L}^0, U)$  consisting of chains which respect the product structure of  $U$  has the same homology as  $\text{IC}_*(\mathcal{L}^0, U)$ . However, for any  $\xi \in \tilde{\text{IC}}_i(\mathcal{L}^0, U)$ ,  $\xi - \mu(\xi)$  is an absolute chain in  $\text{IC}_i(\mathcal{L}^0)$ . Thus  $(I - \mu)$  induces a homomorphism  $\text{IH}_i(\mathcal{L}^0, U) \rightarrow \text{IH}_i(\mathcal{L}^0)$ . This homomorphism agrees with the composition defined above.

§4. The Main Theorem

§4.1 Proposition A<sub>n</sub>

Let  $X$  be a complex purely  $n$  dimensional analytic Whitney stratified space and let  $f : X \rightarrow \mathbb{R}$  be a proper Morse function with a nondegenerate critical point  $x_0$ . Let  $S$  be the stratum which contains  $x_0$ , and let  $c \geq 0$  be the complex codimension of  $S$  in  $X$ . Define  $m$ , the Morse index of  $f$  at  $x_0$ , to be  $\lambda + c$  where  $\lambda$  is the Morse index of  $f|_S$  at  $x_0$ . Suppose the region  $[v-\epsilon, v+\epsilon)$  contains no critical values other than  $v = f(x_0)$ . Then

$$IH_i(X_{<v+\epsilon}, X_{<v-\epsilon}) = 0 \text{ unless } i = m$$

Furthermore, if  $c > 0$  then  $IH_m(X_{<v+\epsilon}, X_{<v-\epsilon})$  is isomorphic to the image of the variation map

$$(I-\mu) : IH_{c-1}(\mathcal{L}^0, \partial\mathcal{L} \times E^1) \rightarrow IH_{c-1}(\mathcal{L}^0)$$

where  $(\mathcal{L}, \partial\mathcal{L})$  is the complex link and its boundary which is associated to the connected component of  $S$  which contains  $x_0$ .

Remark 1: This isomorphism can be made canonical if

- a) the transversal slice  $N$  is chosen through  $x_0$
- b) the projection  $N \rightarrow \mathbb{C}$  is taken to be the complexification of the differential  $df(x_0)$ , where  $\tilde{f}$  is a smooth extension of  $f$  to some  $\mathbb{C}^m$  which contains  $X$

c) the complex link, monodromy, and variation map are constructed from this particular projection.

Remark 2: The Morse index  $m = \lambda + c$  can be viewed more symmetrically as follows: relabel  $IH_{\star}(X)$  according to the Beilinson - Bernstein-Deligne-Gabber scheme,  $IH_i^{\star}(X) \equiv IH_{i-n}(X)$ . Then

$$IH_i^{\star}(X_{<v+\epsilon}, X_{<v-\epsilon}) = 0 \text{ unless } i = I/2$$

where  $I$  is the index of the Hessian of  $f|_S$  at  $x_0$ , i.e.,  $I$  is the number of positive eigenvalues of the Hessian minus the number of negative eigenvalues.

#### 4.2 Proposition $B_n$

Let  $Y$  be a complex analytic Whitney stratified space of any pure dimension and let  $S$  be a stratum of  $Y$  whose complex codimension is  $c = n + 1 > 0$ . Choose a point  $y_0 \in S$  and let  $\mathcal{L}$  be the complex link of  $S$  at the point  $y_0$  (which will have complex dimension  $n$ ). Then

$$IH_i(\mathcal{L}^{\circ}) = 0 \text{ for all } i \geq n + 1$$

$$IH_i(\mathcal{L}^{\circ}, \partial\mathcal{L} \times E^1) = 0 \text{ for all } i \leq n - 1 .$$

#### 4.3 Proof that $A_n$ implies $B_n$

As in [GM3] we consider the proper function  $f : \mathcal{L}^{\circ} \longrightarrow (0, \delta)$  given by

$$f(y) = \text{distance between } y_0 \text{ and } y .$$

By a slight ( $C^\infty$  close) modification of  $f$  we may assume it is a Morse function and has the property that for any stratum  $A$  of  $\mathcal{L}$ ,  $\text{index}(f|_A) \leq \dim_{\mathbb{C}} A$  at any critical point in  $A$ . Proposition  $A_n$  applied to this  $f$  implies that for each critical value  $v$ ,

$$\text{IH}_i(\mathcal{L}^O_{<v+\epsilon}, \mathcal{L}^O_{<v-\epsilon}) = 0 \text{ for all } i > \dim_{\mathbb{C}} A + \text{cod}_{\mathbb{C}} A = n$$

where  $A$  is the stratum which contains the critical point.

To prove the second statement, consider the function  $f : \mathcal{L}^O \rightarrow (-\delta, 0)$  given by

$$f(y) = - \text{distance between } y_0 \text{ and } y .$$

This can be approximated by a Morse function whose restriction to each stratum  $A$  has critical points of index  $\geq \dim_{\mathbb{C}} A$  and for which  $\mathcal{L}^O_{<-\delta} = \partial \mathcal{L} \times E^1$  and  $\mathcal{L}^O_{<0} = \mathcal{L}^O$ .

Repeated application of proposition  $A_n$  (at each critical point) gives

$$\text{IH}_i(\mathcal{L}^O, \partial \mathcal{L} \times E^1) = 0 \text{ for all } i < n .$$

#### 4.4 Proof that $B_k$ for all $k < n$ applies $A_n$ .

If the critical point  $x_0$  lies in the nonsingular part of  $X$  then the conclusion of  $A_n$  is clear since  $X_{<v+\epsilon}$  is obtained from

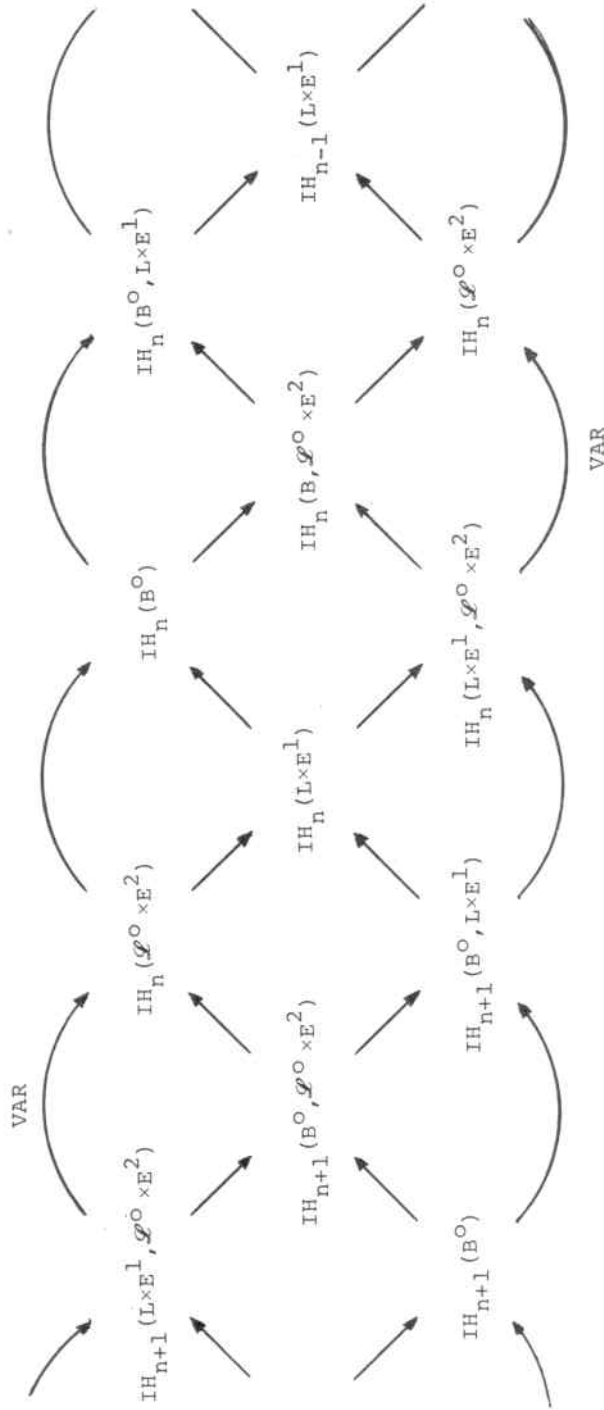


$X_{<v-\epsilon}$  by attaching  $E^\lambda \times E^{n-\lambda}$  along  $\partial E^\lambda \times E^1 \times E^{n-\lambda}$ .

Thus we may assume  $x_0$  lies in the singularity set of  $X$ . By restricting  $f$  to a suitably chosen normal slice through  $x_0$ , applying theorem 2.2 and the Kunneth formula, we may reduce to the case that  $\{x_0\}$  is a zero dimensional stratum of  $X$ ,  $c = n = \dim_{\mathbb{C}}(X)$  and  $\lambda = 0$ .

The main idea of the proof is to analyze the triple of spaces  $(B^0, B^0 - \{x_0\}, B_{<0}^0)$  where  $B^0$  is the particular neighborhood of  $x_0$  (§3.4) and  $B_{<0}^0$  is the associated cut off space. By theorem 3.5(b),  $B^0 - \{x_0\}$  is homeomorphic to  $L \times E^1$ , and by 3.5(a),  $B_{<0}^0$  is homeomorphic to  $\mathcal{L}^0 \times E^2$ . So we denote this triple by  $(B^0, L \times E^1, \mathcal{L}^0 \times E^2)$ .

The Morse group  $IH_1(B_\delta^0(x_0)_{<\eta}, B_\delta^0(x_0)_{<-\eta})$  is isomorphic to the intersection homology of the pair  $(B^0, \mathcal{L}^0 \times E^2)$  by corollary (2) of §3.6. Consider the long exact sequence for the triple  $(\mathcal{L}^0 \times E^2 \subset L \times E^1 \subset B^0)$  (§1.3) and the long exact sequences for the pairs  $(L \times E^1, \mathcal{L}^0 \times E^2)$ ,  $(B^0, \mathcal{L}^0 \times E^2)$ , and  $(B^0, L \times E^1)$ . These fit together in the following commutative braid diagram which has exact sinusoidal curves.



The argument now proceeds by a diagram chase plus these facts:

$$\left. \begin{array}{l} \text{IH}_i(B^\circ, L \times E^1) = 0 \quad \text{for } i \leq n \\ \text{IH}_i(B^\circ) = 0 \quad \text{for } i \geq n \end{array} \right\} \text{from §1.7 and §3.5(a)}$$

$$\text{IH}_i(\mathcal{L}^\circ) = 0 \quad \text{for } i \geq n \quad \text{from proposition } B_k \text{ for } k < n$$

$$\begin{aligned} \text{IH}_i(L, \mathcal{L}^\circ \times E^1) = \text{IH}_{i-1}(\mathcal{L}^\circ, \partial \mathcal{L}^\circ \times E^1) = 0 \quad \text{for } i \leq n - 1 \\ \text{from §3.6 corollary and proposition } B_k \text{ for } k < n \end{aligned}$$

Remark: The braid diagram of this proof is naturally isomorphic to the braid diagram in the introduction to this paper because:

- a) The identification  $\text{IH}_{i+1}(L, \mathcal{L}^\circ \times E^1) = \text{IH}_i(\mathcal{L}^\circ, \partial \mathcal{L}^\circ \times E^1)$  is made in §3.6. This identification can be chosen compatibly with the variation map defined in §3.8.
- b) The "Morse group"  $\text{IH}_i(X_{<\epsilon}, X_{<-\epsilon})$  is identified with  $\text{IH}_i(B^\circ, \mathcal{L}^\circ \times E^2)$  in corollary 2 of §3.6 as remarked above.
- c) The local intersection homology  $\text{IH}_i(X, X-x_0)$  is identified with  $\text{IH}_i(B^\circ, L \times E^1)$  by excision and theorem 3.5 part (b).
- d) The remaining identifications follow from the Kunneth formula of section §1.6.

#### §4.5 Examples:

1. Suppose  $C$  is a complex algebraic curve with a singular point  $x_0$ . The complex link of  $C$  at  $x_0$  consists of  $m$  points,

where  $m$  is the multiplicity of  $C$  at  $x_0$ . The kernel of the variation map has dimension  $b$ , where  $b$  is the number of analytic branches of  $C$  at  $x_0$ . Therefore the Morse group at  $x_0$  has rank  $m - b$ .

2. Consider  $\mathbb{C}^2 = \{(z_1, z_2, 0, 0) \in \mathbb{C}^4 \mid z_1, z_2 \in \mathbb{C}\}$ . Let  $X$  be the singular Schubert variety

$$X = \{P \in G_2(\mathbb{C}^4) \mid \dim_{\mathbb{C}}(P \cap \mathbb{C}^2) \geq 1\}$$

with isolated singular point  $\{x_0\} =$  the plane  $\mathbb{C}^2$ . The Morse group at  $x_0$  is 0 for the following reason:

$\tilde{X} = \{(L, P) \in G_1(\mathbb{C}^2) \times G_2(\mathbb{C}^4) \mid P \cap \mathbb{C}^2 = L\}$  is a small resolution ([GM2]) of  $X$ , with projection  $\pi : \tilde{X} \rightarrow X$  given by  $\pi(L, P) = P$ .

Therefore  $\pi_* \underline{C}_* \cong \underline{IC}_*$ , i.e. the complex of sheaves of intersection chains on  $X$  is just the pushforward of the complex of sheaves of ordinary chains on  $\tilde{X}$ . Let  $f : X \rightarrow \mathbb{R}$  be a Morse function. Then (by a simple calculation)  $f \circ \pi : \tilde{X} \rightarrow \mathbb{R}$  is a Morse function with no critical points on  $\pi^{-1}\{x_0\}$ . Consequently

$$IH_i(X_{<v+\epsilon}, X_{<v-\epsilon}) = H_i(\tilde{X}_{<v+\epsilon}, \tilde{X}_{<v-\epsilon}) = 0$$

where  $v = f(x_0)$  is the critical value.

3. Let  $V$  be a complex algebraic variety embedded in  $\mathbb{C}P^n$ . Let  $X$  be the cone over  $V$  in  $\mathbb{C}P^{n+1}$ . Then the complex link  $\mathcal{L}$  of the cone point  $p$  in  $X$  is homeomorphic to the complement of a generic hyperplane section of  $V$ .

By the results of [GM3] , for an open dense set of functions  $f : \mathbb{C}P^{n+1} \longrightarrow \mathbb{R}$  , for  $v = f(p)$  and for small enough  $\epsilon$  , we have

$$H_k(X_{<v+\epsilon} , X_{<v-\epsilon}) \cong \tilde{H}_{k-1}(\mathcal{L})$$

where  $\tilde{H}$  denotes reduced homology. Choosing  $V$  with enough homology, we get a counterexample to the existence of a Morse index in ordinary homology. For example if  $V$  is the disjoint union of two non-singular curves at least one of which is not a  $\mathbb{C}P^1$  linearly embedded in  $\mathbb{C}P^n$  ,  $H_k(X_{<v+\epsilon} , X_{<v-\epsilon})$  is nonzero for  $k = 1$  and  $2$  .

## §5. Consequences and Extensions of the Main Theorem

### §5.1 Twisted Coefficients

Suppose  $X$  is a Whitney stratified complex analytic variety and  $T$  is a local system of coefficients ( $[S]$ ) on the nonsingular part  $X - \Sigma$  , whose fibre is a regular Noetherian ring. Then  $T$  restricts to a local system of coefficients (also denoted  $T$ ) on the nonsingular part of each of the spaces considered in §4 (eg  $B^0, \mathcal{L}^0, N^0$ , etc.). Propositions  $A_n$  and  $B_n$  remain true if  $IH_*$  is now interpreted as the intersection homology with coefficients in  $T$  (§1.8) because only the formal properties from §1 of intersection homology were used in the proofs of  $A_n$  and  $B_n$  .

### 5.2 Intersection Homology of the link

Proposition: Suppose  $X$  is a stratified complex analytic variety and

$x_0$  is a point in a stratum  $S$  of complex codimension  $c > 0$ . Then

$$IH_i(L) = \begin{cases} IH_{i-1}(\mathcal{L}^0, \partial\mathcal{L} \times E^1) & \text{for } i > c \\ \ker(I-\mu) & \text{for } i = c \\ \text{coker}(I-\mu) & \text{for } i = c - 1 \\ IH_i(\mathcal{L}^0) & \text{for } i < c - 1 \end{cases}$$

where  $L$  is the link of  $S$  at  $x_0$ ,  $\mathcal{L}$  is the complex link of  $S$  at  $x_0$ , and  $(I-\mu) : IH_{c-1}(\mathcal{L}, \partial\mathcal{L}) \longrightarrow IH_{c-1}(\mathcal{L})$  is the variation map (§3.7).

The proof is immediate from propositions  $A_n, B_n$  of §4 and the braid diagram.

### 5.3 Intersection Homology of Stein spaces

Theorem: Let  $X$  be an  $n$ -dimensional Stein complex analytic variety. Then  $IH_i(X) = 0$  for all  $i > n$

Proof: Embed  $X$  as a closed analytic subspace in  $\mathbb{C}^N$  for some  $N$ . Choose a point  $p \in X$  such that

$$f(x) = \text{distance}(p, X)$$

is a Morse function. Then for any stratum  $A$  of  $X$  and for any critical point  $x \in A$ , we have

$$\text{index}(f|_A, x) \leq \dim_{\mathbb{C}} A$$

so  $IH_i(X_{<v+\epsilon}, X_{<v-\epsilon}) = 0$  for all  $i > n$ , by theorem  $A_n$  (where

$v = f(x)$  .

#### 5.4 Lefschetz Hyperplane Theorem

Let  $X$  be a complex  $n$  dimensional algebraic variety embedded in complex projective space  $\mathbb{C}P^N$  and let  $Y \subset X$  be a generic hyperplane section of  $X$ , i.e.  $Y = X \cap H$  where  $H$  is a hyperplane which is transverse to each stratum of a Whitney stratification of  $X$ .

Theorem: The inclusion  $Y \longrightarrow X$  induces isomorphisms  $IH_i(Y) \cong IH_i(X)$  for all  $i < n - 1$  and a surjection  $IH_{n-1}(Y) \longrightarrow IH_{n-1}(X)$  .

Proof: The transversality assumption implies that  $Y$  has a product neighborhood (which we can denote  $Y \times E^2$ ) in  $X$  . By the Kunneth formula and the long exact sequence for the pair  $(X, Y \times E^2)$  it suffices to show that  $IH_i(X, Y \times E^2) = 0$  for all  $i < n$  .

Stratify  $X$  . Let  $f : X \longrightarrow \mathbb{R}$  be a Morse perturbation of the function  $x \longrightarrow \text{distance}(x, H)$  where distance is taken with respect to the usual (Fubini-Study) metric on  $\mathbb{C}P^N$  . Then  $X_{<\delta} \cong Y \times E^2$  for some  $\delta > 0$  and  $X_{<m} = X$  for some  $m > 0$  . Furthermore, for any stratum  $A$  of  $X$  and for any critical point  $x_0 \in A$  we have

$$\text{index}(f|_A, x_0) \geq \dim_{\mathbb{C}} A .$$

If  $v = f(x_0)$  then for some  $\epsilon > 0$  we have

$$IH_i(X_{<v+\epsilon}, X_{<v-\epsilon}) = 0 \text{ for all } i < n$$

Apply this formula to each critical point in  $X - Y \times E^2$  to obtain the result.

Remark: The Lefschetz hyperplane theorem was discovered independently by P. Deligne who used sheaf theory and the method of Artin ([A]) in his proof. (See [GM2] §7).

### 5.5 Morse Inequalities

An argument identical to the standard one (eg Milnor [Mi]) can be used to derive Morse inequalities for the intersection homology groups. However each critical point must be counted with a multiplicity which is the rank of the Morse group at that point:

Suppose  $X$  is a complex analytic variety with a Whitney stratification. For each  $x \in X$  define the rank of the variation at  $x$ ,

$$\ell(x) = \text{rank} \left( (I-\mu) : \text{IH}_*(\mathcal{L}^0, \partial\mathcal{L} \times E^1) \longrightarrow \text{IH}_*(\mathcal{L}^0) \right)$$

where  $\mathcal{L}$  is the complex link associated to the stratum which contains  $x$  (define  $\ell(x) = 1$  if  $x$  is a nonsingular point of  $X$ ).

Theorem: Suppose  $X$  is compact and  $f : X \longrightarrow \mathbb{R}$  is a Morse function. Define  $b_j = \text{rank}(\text{IH}_j(X))$  and for each nonnegative integer  $m$  define

$$R_m = \sum \{ \ell(x) \mid x \text{ is a critical point of } f \text{ and } \text{codim}(S) + \text{index}(f|_S, x) = m, \text{ where } S \text{ is the stratum containing } x \}$$



Then the following Morse inequalities hold:

$$\begin{aligned} R_0 &\geq I b_0 \\ R_1 - R_0 &\geq I b_1 - I b_0 \\ R_2 - R_1 + R_0 &\geq I b_2 - I b_1 + I b_0 \end{aligned}$$

etc.

and 
$$\sum_{i=0}^{2n} (-1)^i R_i = \sum_{i=0}^{2n} (-1)^i I b_i$$

where  $n$  is the complex dimension of  $X$ .

### 5.6 Other Perversities

Propositions  $A_n$  and  $B_n$  of section 4 must be modified if we wish to consider any perversity  $\bar{p}$  other than the "middle" perversity. However the proof of the modified propositions proceeds exactly as before but with changes in the numbering. We give the results here: the intersection homology Morse group is 0 except in a range of dimensions which depends on the perversity.

Definition: Let  $p = (p(2), p(3), \dots)$  be a perversity ([GM1]) i.e.  $p(2) = 0$  and  $p(j) \leq p(j+1) \leq p(j) + 1$ .

Define  $d(0) = d(1) = 0$  and for  $k \geq 2$ ,

$$d(k) = \sup\{j - 1 - p(j) \mid 2 \leq j \leq k\}$$

but set  $d(k) = 0$  if all these numbers are negative.

Define  $h(0) = h(1) = 0$  and for  $k \geq 2$ ,

$$h(k) = \sup\{p(j) - j \mid 2 \leq j \leq k\}$$

but set  $h(k) = 0$  if all these numbers are negative.

Proposition  $A'_n$ : Suppose  $X$  is a complex  $n$  dimensional Whitney stratified analytic variety and  $f : X \rightarrow \mathbb{R}$  is a proper Morse function. Let  $x_0$  be an isolated critical point with critical value  $v$ . Say  $S$  is the stratum which contains  $x_0$ ,  $\lambda$  is the Morse index of  $f|_S$  at  $x_0$  and  $c \geq 0$  is the complex codimension of  $S$  in  $X$ . Suppose  $[v-\epsilon, v+\epsilon)$  contains no critical values except  $v$ .

Then

$$\overline{IH}_i^p(X_{<v+\epsilon}, X_{<v-\epsilon}) = 0$$

unless  $c - h(2c) + \lambda \leq i \leq c + d(2c) + \lambda$ .

Proposition  $B'_n$ : Let  $Y$  be a complex analytic stratified space and let  $S$  be a stratum of  $Y$  whose codimension is  $c = n + 1 > 0$ . Fix  $y \in S$  and let  $\mathcal{L}$  be the complex link of  $S$  at the point  $y$ . Then

$$IH_i^p(\mathcal{L}^0) = 0 \text{ for all } i \geq n + 1 + d(2n)$$

$$IH_i^p(\mathcal{L}^0, \partial\mathcal{L} \times E^1) = 0 \text{ for all } i \leq n - 1 - h(2n).$$

Corollary: Let  $X$  be an  $n$ -dimensional Stein complex analytic variety.

Then  $\overline{IH}_1^{\overline{D}}(X) = 0$  for all  $i \geq n + 1 + d(2n)$

Corollary: Let  $Y$  be a generic hyperplane section of a complex  $n$ -dimensional projective algebraic variety  $X$ . Then the map ([GM2] §5.4)

$$\overline{IH}_1^{\overline{D}}(Y) \longrightarrow \overline{IH}_1^{\overline{D}}(X)$$

is an isomorphism for all  $i < n - 1 - h(2n)$  and is a surjection for  $i = n - 1 - h(2n)$ .

§6. Specialization over a Curve

6.1 Introduction. Throughout this section  $X$  will denote an irreducible complex analytic variety and  $f : X \longrightarrow D^0$  will be a proper analytic map to the (open) unit disc in the complex plane. We will assume that  $f$  is the restriction of a smooth proper nonsingular analytic map  $\tilde{f} : M \longrightarrow D^0$  where  $M$  is a smooth variety which contains  $X$ . We will furthermore assume that  $X$  and  $D^0$  have been analytically Whitney stratified so that  $f$  takes each stratum of  $X$  submersively to a stratum of  $D^0$ , and so that the origin  $0 \in D^0$  is the only zero dimensional stratum in the target. (It follows that  $f : X \longrightarrow D^0$  satisfies condition  $A_f$  of Thom ([T],[Ma])).

For any  $t \in D^0$ , let  $X_t = f^{-1}(t)$ .

We will construct a continuous map

$$\psi_t : X_t \longrightarrow X_0$$

for  $|t|$  sufficiently small, which we call the specialization map.

Our main theorem is that the "complex of vanishing intersection homology cycles"  $R\psi_{t*}(\underline{IC}^*)$  is a perverse sheaf, i.e. an object in the abelian category described in [BBDG].

The specialization map  $\psi$  depends on the choice of stratification of  $f$ . It is a fact (which we do not prove here) that the quasi-isomorphism class of  $R\psi_{t*}(\underline{IC}^*)$  does not depend on the choice of stratification of  $X$ , and it agrees with the  $R\psi(\underline{IC}^*)$  from SGA 4.

Method of proof:

The map  $\psi_t : X_t \longrightarrow X_0$  will be the restriction to  $X_t$  of a "canonical retraction"  $\psi : U \longrightarrow X_0$  of a neighborhood  $U$  of  $X_0$  to  $X_0$ . This retraction corresponds to the intuitive notion of collapsing a tubular neighborhood of each stratum of  $X_0$  down to that stratum; however special techniques from stratification theory are needed to make such a retraction continuous. This retraction is locally trivial over each stratum of  $X_0$ , so  $R\psi_{t*}(\underline{IC}')$  is constructible. We must show that the stalk cohomology of this complex vanishes in a certain range of dimensions. However the stalk cohomology of this complex of sheaves at a point  $x$  in a stratum  $A$  of  $X_0$  is just  $IH_*^{BM}(\pi_A^{-1}(x) \cap \text{Ball} \cap X_t)$  where  $\pi_A : T_A \longrightarrow A$  is the projection of a tubular neighborhood of  $A$  to  $A$ .

We wish to apply Morse theory to calculate this intersection homology group. Unfortunately  $\pi_A^{-1}(x)$  is not necessarily a complex analytic space. Therefore we find a homeomorphism

$$\pi_A^{-1}(x) \cap \text{Ball} \cap X_t = T \cap \text{Ball}' \cap X_t'$$

where  $T$  is a complex analytic normal slice through  $A$  at  $x$ . We apply Morse theory to this second space, concluding that its intersection homology vanishes in the desired range of dimensions, and the same holds for the first space since  $IH_*$  is a homeomorphism invariant.

The second axiom of perverse sheaves amounts to showing that the intersection homology with compact support also vanishes in a

certain range of dimensions. The same Morse-theoretic methods apply in this case.

6.2. Definition of  $\psi_t : X_t \longrightarrow X_0$  .

Choose a system of control data  $\{T_A, \pi_A, \rho_A\}$  on  $X$  . (See [T],[Ma]). This consists of tubular neighborhoods  $T_A = T_A(2\varepsilon)$  in  $M$  of each stratum  $A$  , together with projections  $\pi_A : T_A \longrightarrow A$  and "tubular functions"  $\rho_A : T_A \longrightarrow [0, 2\varepsilon]$  such that whenever  $A \subset \bar{B}$  the following commutation relations

$$\pi_A \pi_B = \pi_B \pi_A , \quad \rho_A \pi_B = \rho_A$$

hold on  $T_A \cap T_B$  . Recall that  $\rho_A(x) = \langle \phi^{-1}(x), \phi^{-1}(x) \rangle$  where  $\phi : E_A \longrightarrow M$  is a smooth embedding of the normal bundle of  $A$  into  $M$  , and  $\langle , \rangle$  is a smoothly varying inner product on the fibres of  $E_A$  .

Control data satisfies the following transversality property: for any pair of strata  $A, B$  in  $X$  , where  $A \subset \bar{B}$  , and for any  $\varepsilon' \leq 2\varepsilon$  and for any  $x \in A$  ,

$$\pi_A^{-1}(x) \cap \rho_A^{-1}(\varepsilon') \text{ is transverse to } B \text{ in } M . \quad (*)$$

For any stratum  $A$  in  $X_0$  , define the  $\varepsilon$ -interior of  $A$  to be the compact subset

$$A^\circ = A - \bigcup_B \rho_B^{-1} [0, \varepsilon)$$

where the union is taken over all strata  $B \subset \bar{A}$ . We claim there exists  $\eta > 0$  such that for any  $x \in A^\circ$  and for any  $t \in \mathbb{C} - \{0\}$  with  $|t| \leq \varepsilon$ ,  $\tilde{f}^{-1}(t)$  is transverse to  $\pi_A^{-1}(x) \cap \rho_A^{-1}(\varepsilon)$ . (Such an  $\eta$  will be called "sufficiently small").

Proof of claim: Suppose there is no such  $\eta$ . Then there is a sequence  $t_i \in \mathbb{C} - \{0\}$  which converges to 0, a sequence  $x_i \in A^\circ$  which converges to some  $x_0 \in A^\circ$ , and a sequence  $y_i \in \pi_A^{-1}(x_i) \cap \rho_A^{-1}(\varepsilon) \cap X_{t_i}$  where transversality fails (since  $A^\circ$  is compact). We may assume the  $y_i$  all lie in the same stratum  $D$  of  $X$  and converge to some  $y_0$  in a stratum  $C$  of  $X_0$ . We may also assume the tangent planes  $T_{y_i}(X_{t_i} \cap D)$  converge to some plane  $\tau$ . However by condition  $A_f$  of Thom,

$$T_{y_0} C \subset \tau \text{ which is not transverse to } T_{y_0}(\pi_A^{-1}(x_0) \cap \rho_A^{-1}(\varepsilon))$$

This contradicts (\*).

Define  $\psi : U(\varepsilon) \rightarrow X_0$  to be the continuous "retraction" defined in [Gor] §7.1, where

$$U(\varepsilon) = \bigcup_A \{y \in T_A \mid \rho_A(y) \leq \varepsilon\}$$

this union being taken over all strata  $A$  of  $X_0$ . ( $\psi$  is not actually a retraction in the strict sense of the word because its restriction to  $X$  is not the identity, but is only homotopic to the identity).

Definition:  $\psi_t : X_t \longrightarrow X_0$  is the restriction of  $\psi$  to the fibre  $X_t$  where  $|t| \leq \eta$  and  $\eta$  is sufficiently small.

It is a fact which we do not prove (or use) here that the topological type of the map  $\psi_t$  is independent of the choice of control data, subordinate family of lines, or parameter values  $t$ .

Remark: For any stratum  $A$  of  $X_0$  and for any  $x \in A$  there is a unique  $x' \in A^0$  such that

$$\psi^{-1}(x) = \pi_A^{-1}(x') \cap U(\varepsilon)$$

$$\psi_t^{-1}(x) = \pi_A^{-1}(x') \cap U(\varepsilon) \cap X_t$$

and this  $x'$  depends continuously on  $x$ . This can be seen from the construction of  $\psi$  in [Gor].

### 6.3 Specialization Data

Fix a connected component  $A$  of a stratum of  $X_0$ . Specialization data for  $A$  is a collection

$$\{p, \langle \rangle, m, \delta, \eta, t\}$$

with the following properties: (1)  $p \in A$ . (2)  $\langle \rangle$  is a positive definite inner product on the vectorspace  $V = T_p M / T_p A$ . (3)  $m : V \longrightarrow M$  is a smooth map such that  $m(0) = p$ , and  $m$  is transverse to  $A$  at  $0$ , i.e.  $dm(0)(V) + T_p(A) = T_p(M)$ . (4)  $\delta > 0$  is a real number such that  $m$  is an embedding on



$\{v \in V \mid \langle v, v \rangle \leq \delta\}$  and for all  $\delta' \leq \delta$  the set

$$\partial B_{\delta'} = \{m(v) \mid \langle v, v \rangle = \delta'\}$$

is transverse to every stratum of  $X$ . (5)  $\eta > 0$  is a real number such that for all  $s \in \mathbb{C} - \{0\}$  with  $|s| \leq \eta$ ,  $\tilde{f}^{-1}(s)$  is transverse to  $\partial B_{\delta} \cap X$ . (6)  $t \in \mathbb{C} - \{0\}$  and  $|t| < \eta$ .

The trace of the specialization data  $\{p, \langle \rangle, m, \delta, \eta, t\}$  is the (stratified) topological space  $\tilde{f}^{-1}(t) \cap X \cap B_{\delta}(p)$  where  $B_{\delta}(p) = \{m(v) \mid \langle v, v \rangle \leq \delta\}$ .

Proposition: For any choice of  $p, \langle \rangle, m$  satisfying (1), (2), (3) above, there exist  $\delta, \eta$  and  $t$  which satisfy (4), (5) and (6). If  $\{p_0, \langle \rangle_0, m_0, \delta_0, \eta_0, t_0\}$  and  $\{p_1, \langle \rangle_1, m_1, \delta_1, \eta_1, t_1\}$  are two choices of specialization data, then there is a smooth one-parameter family of specialization data connecting them.

Corollary 1: The homeomorphism type of the trace of specialization data is independent of the choice of  $p, \langle \rangle, m, \delta, \eta$ , or  $t$ .

Proof of Corollary 1: given two choices of specialization data, find a one-parameter family  $\{p_s, \langle \rangle_s, m_s, \delta_s, \eta_s, t_s\}$  between them. The set  $Y \subset M \times [0,1]$  which is given by  $Y_s = \tilde{f}^{-1}(t_s) \cap X \cap B_{\delta_s}(p_s)$  projects by a stratified submersion to the unit interval  $[0,1]$ . The first isotopy lemma of Thom gives a homeomorphism from  $Y_0$  to  $Y_1$ .

Corollary 2: The fibre at any point  $p \in X_0$  of the specialization map  $\psi_t : X_t \longrightarrow X_0$  is homeomorphic to the closure (in  $X_t$ ) of a

complex analytic subspace. Fibres over nearby points can be given compatible analytic structures.

Proof of Corollary 2: The fibre at  $p$  of  $\psi_t$  is (by definition) the trace of the specialization data  $\{p', \langle \rangle, \phi, 2\epsilon, \eta, t\}$  where  $|t| \leq \eta$ ,  $\eta$  is sufficiently small, and  $\langle \rangle, \phi, \epsilon$  are determined by the system of control data on  $X$ , and  $p'$  lies in the same stratum as  $p$ .

Let  $A$  denote the stratum of  $X_0$  which contains  $p$ . Choose  $\delta' > 0$  and a map  $m : T_p, M/T_p, A \rightarrow M$  which satisfies property (3) of §6.3 and which is a complex analytic embedding on the set  $\{v \in T_p, M/T_p, A \mid \langle v, v \rangle < 2\delta'\}$ . According to proposition 6.3 there exists  $\delta \leq \delta'$ ,  $w > 0$  and  $s \in \mathbb{C} - \{0\}$  such that  $\{p', \langle \rangle, m, \delta, w, s\}$  satisfies properties 1 through 6 of §6.3. The trace of this specialization data,  $\tilde{f}^{-1}(s) \cap X \cap B_\delta(p')$  is therefore homeomorphic to  $\psi_t^{-1}(p)$  (by corollary 1) and is also the closure in  $X_t$  of the analytic subspace  $\tilde{f}^{-1}(s) \cap X \cap T$  where

$$T = m(\{v \in T_p, M/T_p, A \mid \langle v, v \rangle < \delta\}) .$$

#### 6.4 Proof of Proposition 6.3.

(a) Existence of  $\delta, \eta$  and  $t$ : given  $p, \langle \rangle$  and  $m$ , there exists  $\delta > 0$  satisfying (4) because  $dm(0)$  has maximal rank, and the stratification of  $X$  satisfies Whitney's condition B. The proof that  $\eta$  and  $t$  exist satisfying (5) and (6) is exactly the proof of the claim in §6.2 (i.e. if no such  $\eta$  exists then condition  $A_f$  of Thom is violated).

Remarks: The set of  $(\delta, \eta, t)$  which satisfy (4), (5) and (6) is open in  $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{C} - \{0\}$ . In fact, given the first  $k$  elements of a collection of specialization data, satisfying properties (1)...(k), it is possible to find the remaining  $6-k$  elements satisfying properties (k-1)...(6), and the set of such choices is open in the space of all possible choices of those  $6-k$  elements.

(b) existence of one parameter families of specialization data:

Given  $\{p_0, \langle \rangle_0, m_0, \delta_0, \eta_0, t_0\}$  and  $\{p_1, \langle \rangle_1, m_1, \eta_1, t_1\}$  satisfying properties (1) through (6) of §6.3, choose a smooth path  $p(s)$  from  $p_0$  to  $p_1$ . Choose a connexion on the normal bundle of the stratum  $A$  in  $M$  so we can identify the normal spaces  $V_s = T_{p(s)}M/T_{p(s)}A$  with a fixed vectorspace  $V$ . It is easy to find a one parameter family of inner products  $\langle \rangle(s)$  on  $V$  which connect  $\langle \rangle_0$  to  $\langle \rangle_1$ . Choose a one-parameter family of maps  $m(s) : V \rightarrow M$  such that  $m(s)(0) = p(s)$  and  $dm(s)(0)$  has maximal rank, and  $m(0) = m_0, m(1) = m_1$ .

Uniform choice of  $\delta$ : We claim there exists  $\bar{\delta} > 0$  such that for any  $s \in [0,1]$  the collection

$$\{p(s), \langle \rangle(s), m(s), \bar{\delta}\}$$

satisfies property (4) of §6.3. For, suppose not. Then there is a sequence  $(s_i, \delta_i) \rightarrow (s_0, 0)$  such that (4) fails for each  $i$ . This means either (a)  $dm(s_i)(v_i) = 0$  for some  $v_i \in V, \langle v_i, v_i \rangle < \delta_i$ , or else (b)  $\partial B_{\delta_i}(p(s_i))$  fails to be transverse to some stratum of  $X$ . If (a) occurs for infinitely many values of  $i$  then this contradicts

the fact that  $dm(s_0)(0)$  has maximal rank. If (b) occurs for infinitely many values of  $i$  then this contradicts Whitney's condition B for the stratification of  $X$  (at the point  $p(s_0)$ ).

Uniform choice of  $\eta$ : We claim there exists  $\bar{\eta} > 0$  such that for every  $s \in [0,1]$  the collection

$$\{p(s), \langle \rangle(s), m(s), \bar{\delta}, \bar{\eta}\}$$

satisfies condition (5) of §6.3. For, suppose not. Then there is a sequence  $\eta_i \rightarrow 0$  which fail to satisfy condition (5), i.e. there is a sequence of points  $t_i \in \mathbb{C} - \{0\}$  with  $|t_i| < \eta_i$ , and points  $y_i \in \tilde{f}^{-1}(t_i) \cap \partial B_{\delta}(p(s_i)) \cap X$  where transversality fails. By taking subsequences if necessary, we may assume the  $y_i$  all lie in the same stratum  $D$  of  $X$ , they converge to some  $y_0$  in a stratum  $C$  of  $X_0$ , and the planes  $T_{y_i}(\tilde{f}^{-1}(t_i)) \cap T_{y_i}D$  converge to some plane  $\tau$ .

By Thom's condition  $A_f$ ,

$T_{y_0}C \subset \tau$  which fails to be transverse to  $T_{y_0}(\partial B_{\bar{\delta}}(p(s_0)))$ . This contradicts property (4) which was already verified.

It is now clear how to construct a one parameter family of specialization data: shrink  $t$  from  $t_0$  to some  $\bar{t}$  where  $|\bar{t}| < \bar{\eta}$ . Then shrink  $\eta$  from  $\eta_0$  to  $\bar{\eta}$ . Then shrink  $\delta$  from  $\delta_0$  to  $\bar{\delta}$ . Then move  $p, \langle \rangle, m$  along the path  $p(s), \langle \rangle(s), m(s)$ . Then increase  $\delta$  from  $\bar{\delta}$  to  $\bar{\delta}_1$ , increase  $\eta$  from  $\bar{\eta}$  to  $\eta_1$  and finally move  $t$  from  $\bar{t}$  to  $t_1$ .

6.5. Specialization of Perverse Objects [BBDG]

Suppose  $Y$  is a complex  $n$ -dimensional Whitney stratified analytic variety. Let  $R$  be a regular Noetherian ring and let  $\underline{S}^*$  be a bounded complex of sheaves of  $R$ -modules on  $Y$  whose cohomology sheaves  $\underline{H}^i(\underline{S}^*)$  are locally constant on each stratum of  $Y$ .  $\underline{S}^*$  is an object in the abelian category of perverse sheaves if, for any stratum  $A$  of  $Y$ , any point  $y \in A$ , and any sufficiently small conical neighborhood  $U$  of  $y$  (of the type considered in §1.1), the following support and cosupport conditions hold:

$$H^i(j_*\underline{S}^*) = H^i(U; \underline{S}^*) = 0 \quad \text{for all } i > c$$

$$H^i(j^!\underline{S}^*) = H_c^i(U; \underline{S}^*) = 0 \quad \text{for all } i < 2n - c$$

where  $c$  is the complex codimension of  $A$  in  $Y$ , and  $j : \{y\} \rightarrow Y$  is the inclusion of the point.

In order to make the complex of sheaves  $\underline{IC}_*$  into a perverse object we must use this shift of indices :

$$\underline{IC}^{2n-i} \cong \underline{IC}_i$$

Theorem: Suppose  $\pi : X \rightarrow D^0$  is a proper analytic map to the unit disc  $D \subset \mathbb{C}$  (as in §6.1) and  $\psi_t : X_t \rightarrow X_0$  is the specialization map of §6.2. Then  $R\psi_{t*}(\underline{IC}^*)$  is a perverse object.

Proof: Since  $\underline{IC}^*$  is a complex of fine sheaves we may write  $\psi_{t*}(\underline{IC}^*)$  instead of  $R\psi_{t*}(\underline{IC}^*)$ .

The argument in §6.5 shows that  $\psi_{t*}(\underline{IC}^*)$  is constructible, i.e. that its cohomology sheaves are locally trivial on each stratum of  $X_0$ . The whole map  $\psi_t$  is locally trivial over each stratum of  $X_0$ .

We must verify the support and cosupport conditions. Suppose  $x \in X_0$  is a point in a stratum  $A$  of complex codimension  $c$ . By taking an analytic normal slice through  $A$  at  $x$  and applying the Kunnetth formula, we may assume  $A = \{x_0\}$  is a zero dimensional stratum (so  $c = n$ ).

The stalk cohomology is:

$$H^i(j^*\psi_{t*}\underline{IC}^*) = IH_{2n-i}^{BM}(\psi_t^{-1}(x)) = IH_{2n-i}(X_t, X_t - \psi_t^{-1}(x))$$

(by Godement [God] 4.17.1 or Bredon [B] theorem IV 6.1 p. 141 and theorem II 12.1 p. 59).

According to corollary 2 of §6.3,  $\psi_t^{-1}(x)$  is homeomorphic to some analytic space  $T \cap B_\delta(x') \cap X_t$ . By excision  $IH_{2n-i}(X_t, X_t - \psi_t^{-1}(x)) = IH_{2n-i}(T \cap X_t \cap B_\delta^O, T \cap X_t \cap (\partial B_\delta \times E^1))$ . Use the Morse theory for a Morse perturbation of the function

$$f(y) = - \text{distance}(y, x)$$

as in §4.3 or §5.4, to see that this group vanishes whenever  $2n - i < n$ .

Similarly the costalk cohomology is

$$H^i(j^!\psi_{t*}\underline{IC}^*) = IH_{2n-i}^{BM}(X_t, X_t - \psi_t^{-1}(x)) = IH_{2n-i}(\psi_t^{-1}(x))$$

by Bredon [B] prop. V 7.2 p. 206 and theorem II 12.1 p. 59. Replace  $\psi_t^{-1}(x)$  with  $T \cap B_\delta(x') \cap X_t$  as above and use a Morse function close to

$$f(y) = \text{distance } (y, x)$$

as in §4.3 or §5.3 to see that this group vanishes whenever  $2n - i > n$ .

Corollary:  $R\psi_{t*}$  takes perverse objects to perverse objects.

Proof: The simple perverse objects on  $X_t$  are the intersection chains  $\underline{IC}^*$  (with twisted coefficients) of subvarieties of  $X_t$ . The specialization of any such complex of sheaves is a perverse object on  $X_0$  according to theorem 6.6. It follows that the same is true for extensions of such sheaves.

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