LOOP PRODUCTS AND CLOSED GEODESICS

MARK GORESKY and NANCY HINGSTON

Abstract

The critical points of the length function on the free loop space $\Lambda(M)$ *of a compact Riemannian manifold* M *are the closed geodesics on* M*. The length function gives a filtration of the homology of* $\Lambda(M)$*, and we show that the Chas-Sullivan product*

$$H_i(\Lambda) \times H_j(\Lambda) \xrightarrow{*} H_{i+j-n}(\Lambda)$$

is compatible with this filtration. We obtain a very simple expression for the associated graded homology ring $\mathrm{Gr}H_*(\Lambda(M))$ *when all geodesics are closed, or when all geodesics are nondegenerate. We also interpret Sullivan's coproduct* $\vee$ *(see [Su1], [Su2]) on* $C_*(\Lambda)$ *as a product in cohomology*

$$H^i(\Lambda, \Lambda_0) \times H^j(\Lambda, \Lambda_0) \xrightarrow{\circledast} H^{i+j+n-1}(\Lambda, \Lambda_0)$$

(where $\Lambda_0 = M$ *is the constant loop). We show that* $\circledast$ *is also compatible with the length filtration, and we obtain a similar expression for the ring* $\mathrm{Gr}H^*(\Lambda, \Lambda_0)$*. The nonvanishing of products* σ^{*n} *and* $\tau^{\circledast n}$ *is shown to be determined by the rate at which the Morse index grows when a geodesic is iterated. We determine the full ring structure* $(H^*(\Lambda, \Lambda_0), \circledast)$ *for spheres* $M = S^n$*,* $n \geq 3$*.*

Contents

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1. Introduction

1.1

Let M be a smooth compact manifold without boundary. In [CS], M. Chas and D. Sullivan constructed a new product structure

$$H_i(\Lambda) \times H_j(\Lambda) \xrightarrow{*} H_{i+j-n}(\Lambda) \tag{1.1.1}$$

on the homology $H_*(\Lambda)$ of the free loop space Λ of M. In [CKS], it was shown that this product is a homotopy invariant of the underlying manifold M. In contrast, the closed geodesics on M depend on the choice of a Riemannian metric, which we now fix. In this article, we investigate the interaction beween the Chas-Sullivan product on Λ and the energy function, or rather, its square root (see §10.6),

$$F(\alpha) = \sqrt{E(\alpha)} = \Big(\int_0^1 |\alpha'(t)|^2\, dt\Big)^{1/2},$$

whose critical points are exactly the closed geodesics. For any a, $0 \le a \le \infty$, we denote by

$$\Lambda^{\le a}, \qquad \Lambda^{>a}, \qquad \Lambda^{=a}, \qquad \Lambda^{(a,b]} \tag{1.1.2}$$

those loops $\alpha \in \Lambda$ such that $F(\alpha) \le a$, $F(\alpha) > a$, $F(\alpha) = a$, $a < F(\alpha) \le b$, and so on. (We set $\Lambda^{<\infty} = \Lambda^{\le\infty} = \Lambda$ and $\Lambda^{\le 0} = \Lambda_0$.) In this article, we use homology

$H_*(\Lambda^{\le a}; G)$ with coefficients in the ring $G = \mathbb{Z}$ if M is orientable and $G = \mathbb{Z}/(2)$ otherwise. The following result is proven in §5.

THEOREM 1.2
*The Chas-Sullivan product extends to a family of products**

$$\check{H}_i(\Lambda^{\le a}) \times \check{H}_j(\Lambda^{\le b}) \xrightarrow{*} \check{H}_{i+j-n}(\Lambda^{\le a+b}), \tag{1.2.1}$$

$$\check{H}_i(\Lambda^{\le a}, \Lambda^{\le a'}) \times \check{H}_j(\Lambda^{\le b}, \Lambda^{\le b'}) \xrightarrow{*} \check{H}_{i+j-n}(\Lambda^{\le a+b}, \Lambda^{\le \max(a+b', a'+b)}), \tag{1.2.2}$$

$$\check{H}_i(\Lambda^{\le a}, \Lambda^{< a}) \times \check{H}_j(\Lambda^{\le b}, \Lambda^{< b}) \xrightarrow{*} \check{H}_{i+j-n}(\Lambda^{\le a+b}, \Lambda^{< a+b}), \tag{1.2.3}$$

whenever $0 \le a' < a \le \infty$ *and* $0 \le b' < b \le \infty$. *These products are compatible with respect to the natural inclusions* $\Lambda^{\le c'} \to \Lambda^{\le c}$ *whenever* $c' \le c$.

We refer to $\check{H}_i(\Lambda^{\le a}, \Lambda^{< a})$ as the *level homology* group, or the *homology at level* a, also called the *Morse group* in [SMT]. This is the group that is addressed by Morse theory which measures the “change” in the homology of $\Lambda^{\le v}$ as v passes through the value $v = a$ (cf. §1.6). It is zero unless a is a critical value of the function F. The product (1.2.3) is called the *level homology product*.

1.3
In [Su1] and [Su2], D. Sullivan constructed coproducts $\vee_t$ and $\vee$ on the group of transverse chains of Λ (cf. §8). If the Euler characteristic of M is zero, Sullivan showed (see [Su1], [Su2]) that $\vee_t$ vanishes on homology, and $\vee$ descends to a coproduct on homology (cf. §8.4; see also [CG], [G]). Coproducts on homology correspond to products on cohomology. In §9.1, we show that the cohomology product corresponding to $\vee_t$ vanishes identically except possibly in low degrees, but the cohomology product corresponding to $\vee$ is new and interesting and is well defined on relative cohomology, whether or not the Euler characteristic of M is zero. (However, the resulting ring does not have a unit **1**.)

THEOREM 1.4
Let $0 \le a' < a \le \infty$*, and let* $0 \le b' < b < \infty$. *Sullivan's operation* $\vee$ *determines a family of products*

$$H^i(\Lambda, \Lambda_0) \times H^j(\Lambda, \Lambda_0) \xrightarrow{\circledast} H^{i+j+n-1}(\Lambda, \Lambda_0), \tag{1.4.1}$$

$$\check{H}^i(\Lambda^{\le a}, \Lambda^{\le a'}) \times \check{H}^j(\Lambda^{\le b}, \Lambda^{\le b'}) \xrightarrow{\circledast} \check{H}^{i+j+n-1}(\Lambda^{\le \min(a+b', a'+b)}, \Lambda^{\le a'+b'}), \tag{1.4.2}$$

$$\check{H}^i(\Lambda^{\le a}, \Lambda^{< a}) \times \check{H}^j(\Lambda^{\le b}, \Lambda^{< b}) \xrightarrow{\circledast} \check{H}^{i+j+n-1}(\Lambda^{\le a+b}, \Lambda^{< a+b}), \tag{1.4.3}$$

*Here, $\check{H}_i(\Lambda^{\le a})$ denotes Čech homology. In Lemma A.4, we show that the singular and Čech homology agree if $0 \le a \le \infty$ is a regular value or if it is a nondegenerate critical value in the sense of Bott.

which are associative and (sign-)commutative, and are compatible with the homomorphisms induced by the inclusions $\Lambda^{\le c'} \to \Lambda^{\le c}$ *whenever* $c' < c$. *If* $i > n$, *then* $H^i(\Lambda, \Lambda_0) \cong H^i(\Lambda)$ *so equation (1.4.1) becomes a product on absolute cohomology. The product (1.4.1) is independent of the Riemannian metric. If* M *admits a metric with all geodesics closed, then the ring* $(H^*(\Lambda, \Lambda_0), \circledast)$ *is finitely generated.*

The sphere S^n and the projective spaces $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, and CaP^2 all admit such metrics. For these spaces, the cohomology has infinite dimension as a G-module, so the $\circledast$-product is highly nontrivial. Moreover, just the existence of a product $\circledast$, satisfying Theorem 1.4 such that the ring $(H^*(\Lambda, \Lambda_0), \circledast)$ is finitely generated, is already enough to answer a geometric question of Y. Eliashberg (cf. §10.4): there exists a constant C, independent of the metric, so that

$$d(t_1 + t_2) \le d(t_1) + d(t_2) + C,$$

where $d(t) = \max\{k : \operatorname{Image}(H_k(\Lambda^{\le t}) \to H_k(\Lambda)) \ne 0\}$.

1.5

Theorem 1.2, which concerns the Chas-Sullivan product on homology, does not come as a surprise to experts. It is not hard to see that the Pontrjagin product (on the homology of the based loop space) has the same properties. So Theorem 1.4, which concerns the $\circledast$-product on cohomology, might also be anticipated. However, the analogous statement for the cup product on cohomology is false (see §§10.3, 16.1).

1.6

In this subsection, assume that the critical values $\mathsf{cr}(F)$ are discrete. Then the functional $F : \Lambda \to \mathbb{R}$ determines a filtration I of the chain complex for Λ and of its homology $H_*(\Lambda)$. The E^1-page of the resulting spectral sequence is the "total" level homology

$$E^1 = \bigoplus_{p,q} E^1_{p,q} \cong \bigoplus_{a \in \mathsf{cr}(F)} \check{H}_*(\Lambda^{\le a}, \Lambda^{< a}).$$

This spectral sequence converges to $H_*(\Lambda)$, and it determines an isomorphism of groups

$$E^\infty \cong \operatorname{Gr}_I H_*(\Lambda). \tag{1.6.1}$$

The level homology product (1.2.3) determines a ring structure on E^1, and also on all the other pages E^k (including E^∞) of the spectral sequence. However, according to Theorem 1.2, the Chas-Sullivan (C-S) product is compatible with this filtration, so it passes to a product on $\operatorname{Gr}_I H_*(\Lambda)$. (The product of two classes at levels $a, a' \in \mathsf{cr}(F)$

is zero unless $a + a'$ is also a critical value of F.) The isomorphism (1.6.1) is then an isomorphism of rings.

If the functional F is a *perfect Morse function*, then $E^1 \cong E^\infty$ as rings, which gives an isomorphism of rings between the level homology and $\mathrm{Gr}_I H_*(\Lambda)$. This is what happens when all geodesics are closed (cf. §§1.10, 13). If we use field coefficients, then there is a further, noncanonical isomorphism $\mathrm{Gr}_I H_*(\Lambda) \cong H_*(\Lambda)$ of groups. The product on $\mathrm{Gr}_I H_*(\Lambda)$ may be thought of as the leading term of the product on $H_*(\Lambda)$.

Similar comments apply to the cohomology $H^*(\Lambda, \Lambda_0)$ with its cohomology product $\circledast$. The cohomology cup product on $H^*(\Lambda, \Lambda_0)$ is not compatible with the filtration I, so the analogous statements involving the cup product are false.

1.7

The *critical level* (see §4) of a homology class $0 \neq x \in H_i(\Lambda)$ is defined to be

$$\mathsf{cr}(x) = \inf \left\{ a \in \mathbb{R} : \ x \text{ is supported on } \Lambda^{\leq a} \right\}. \tag{1.7.1}$$

The critical level of a cohomology class $0 \neq X \in H^*(\Lambda, \Lambda_0)$ is defined to be

$$\mathsf{cr}(X) = \sup \left\{ a \in \mathbb{R} : \ X \text{ is supported on } \Lambda^{\geq a} \right\}.$$

These are critical values of F. In Propositions 5.3 and 9.5 (cf. Corollary 10.1), we show that the products $*$ and $\circledast$ satisfy the following relations:

$$\mathsf{cr}(x * y) \leq \mathsf{cr}(x) + \mathsf{cr}(y) \quad \text{for all } x, y \in H_*(\Lambda), \tag{1.7.2}$$

$$\mathsf{cr}(X \circledast Y) \geq \mathsf{cr}(X) + \mathsf{cr}(Y) \quad \text{for all } X, Y \in H^*(\Lambda, \Lambda_0). \tag{1.7.3}$$

For appropriate x, y, X, Y, both inequalities are sharp (i.e., they are equalities) when M is a sphere or projective space with the standard metric (cf. §§13, 14, 15). The inequality (1.7.2) is also sharp when all closed geodesics are nondegenerate and the index growth (cf. §6) is minimal (cf. §12). The inequality (1.7.3) is sharp when all closed geodesics are nondegenerate and the index growth is maximal.

1.8

A homology class $\eta \in H_*(\Lambda)$ is said to be *level-nilpotent* if $\mathsf{cr}(\eta^{*N}) < N\mathsf{cr}(\eta)$ for some $N > 1$, where $\eta^{*N} = \eta * \eta * \cdots * \eta$ (N times). A cohomology class $\alpha \in H^*(\Lambda, \Lambda_0)$ is level-nilpotent if $\mathsf{cr}(\alpha^{\circledast N}) > N\mathsf{cr}(\alpha)$ for some $N > 1$. There are analogous notions in level homology and cohomology. A homology (resp., cohomology) class η in $\check{H}(\Lambda^{\leq a}, \Lambda^{<a})$ (where $\check{H}$ denotes homology, resp., cohomology) is said to be level-nilpotent if some power vanishes: $\eta^{*N} = 0$ (resp., $\eta^{\circledast N} = 0$) in $\check{H}(\Lambda^{\leq Na}, \Lambda^{<Na})$. In §§7 and 11, we prove the following.

THEOREM 1.9
If all closed geodesics on M are nondegenerate, then every homology class in $H_*(\Lambda)$, every cohomology class in $H^*(\Lambda, \Lambda_0)$, and every level homology class and every level cohomology class in $H(\Lambda^{\le a}, \Lambda^{<a})$ is level-nilpotent (for all $a \in \mathbb{R}$).*

1.10
On the other hand, nonnilpotent classes exist when all geodesics are closed. Suppose that E is the energy function of a metric in which all geodesics on M are closed, simply periodic, and have the same prime length ℓ, as defined in §13.1. The critical points of $F = \sqrt{E}$ are the closed geodesics, and the critical values of F are the (nonnegative) integer multiples of ℓ. The set of critical points with critical value $r\ell$ ($r \ge 1$) form a (Morse-Bott) nondegenerate critical submanifold $\Sigma_r \subset \Lambda$ which is diffeomorphic to the unit sphere bundle SM by the mapping $\alpha \mapsto \alpha'(0)/r\ell$. (For any point $x \in M$ and any unit tangent vector $v \in T_x M$, there is, up to reparametrization, a unique geodesic α that starts from x and moves in the direction v. If all geodesics are closed, then this is a geodesic loop, so taking the speed to be $|\alpha'(0)| = r\ell$ gives an element of Λ.)

Let λ_r be the Morse index of any geodesic of length $r\ell$. Let $h = \lambda_1 + 2n - 1$, where $n = \dim(M)$. Then $H_i(\Lambda^{\le \ell}) = 0$ for $i > h$. Let

$$\Theta \in H_h(\Lambda^{\le \ell}; G) \cong G$$

be a generator of the top-degree homology group. In §13 and Corollary 13.7, we prove the following.

THEOREM 1.11
The r-fold Chas-Sullivan product

$$\Theta^{*r} \in H_{\lambda_r+2n-1}(\Lambda^{\le r\ell}, \Lambda^{<r\ell}; G) \cong G$$

generates the top-degree homology at the level $r\ell$, and more generally, the Chas-Sullivan product with Θ induces an isomorphism

$$H_i(\Lambda^{\le a}, \Lambda^{<a}) \to H_{i+h-n}(\Lambda^{\le a+\ell}, \Lambda^{<a+\ell})$$

for all degrees i and for all level values a. The energy E determines a filtration $0 = I_0 \subset I_1 \subset \cdots \subset H_*(\Lambda, \Lambda_0)$ such that $I_j * I_k \subset I_{j+k}$. The associated graded ring is isomorphic (with degree shifts) to the ring

$$\mathrm{Gr}_I H_*(\Lambda, \Lambda_0) \cong H_*(SM)[T]_{\ge 1} \tag{1.11.1}$$

*See previous footnote.

of polynomials whose constant term is zero, where $H_*(SM)$ *denotes the homology (intersection) ring of* SM.

The full Chas-Sullivan ring $H_*(\Lambda)$ was computed by R. Cohen, J. Jones, and J. Yan [CJY] for spheres and projective spaces. The relatively simple formula (1.11.1) is compatible with their computation. It seems likely that there may be other results along these lines when the Riemannian metric has large sets of closed geodesics.

1.12

In §14 and Corollary 14.8, we prove the analogous result for the new product $\circledast$ in cohomology. Suppose that all geodesics on M are closed, simply periodic, and have the same prime length ℓ. Then $H^i(\Lambda^{\leq \ell}, \Lambda_0) = 0$ for $i < \lambda_1$. Let

$$\omega \in H^{\lambda_1}(\Lambda^{\leq \ell}, \Lambda_0; G) \cong H^{\lambda_1}(\Lambda^{\leq \ell}, \Lambda^{<\ell}; G) \cong G$$

be a generator of the lowest-degree cohomology group $G = \mathbb{Z}$ or $\mathbb{Z}/(2)$.

THEOREM 1.13

Multiplication by ω *is an injective mapping. The* r*-fold product*

$$\omega^{\circledast r} \in H^{\lambda_r}(\Lambda^{\leq r\ell}, \Lambda^{<r\ell}) \cong G$$

generates the lowest-degree cohomology class at level $r\ell$*, and more generally, the product with* ω *induces an isomorphism*

$$H^i(\Lambda^{\leq a}, \Lambda^{<a}) \to H^{i+h-n}(\Lambda^{\leq a+\ell}, \Lambda^{<a+\ell})$$

for all degrees i *and all level values* a*. Moreover, the energy induces a filtration*

$$H^i(\Lambda, \Lambda_0) = I^0 \supset I^1 \supset I^2 \supset \cdots$$

by ideals such that $I^j \circledast I^k \subset I^{j+k}$*. The associated graded ring* $\mathrm{Gr}^I H^*(\Lambda, \Lambda_0)$ *is isomorphic (with degree shifts) to the ring,*

$$H^*(SM)[T]_{\geq 1},$$

where $H^*(SM)$ *denotes the cohomology ring of* SM.

1.14. Cohomology ring of ΩS^n *and* ΛS^n

The full cohomology ring is computed for spheres in §15. There is a natural homomorphism $h_1 : H^*(SM) \to H^*(\Lambda, \Lambda_0)$ which takes the cup product on $H^*(SM)$ into "level" products on $H^*(\Lambda, \Lambda_0)$, that is, $\mathsf{cr}(h_1(a) \circledast h_1(b)) = \mathsf{cr}(h_1(a)) + \mathsf{cr}(h_1(b))$.

It turns out that there are additional "above level" products (i.e., classes u, v such that $\mathsf{cr}(u \circledast v) > \mathsf{cr}(u) + \mathsf{cr}(v)$) that are not detected by the theorems in §14.

1.15. Counting closed geodesics

By [VS], if M is a compact, simply connected Riemannian manifold whose cohomology algebra $H^*(M;\mathbb{Q})$ cannot be generated by a single element, then the Betti numbers of Λ form an unbounded sequence, whence by [GrM], the manifold M admits infinitely many prime closed geodesics. This result leaves open the case of spheres and projective spaces (among others).

It is known (see [Ba], [F], [Hi2]) that any Riemannian metric on S^2 has infinitely many prime closed geodesics, and it is conjectured that the same holds for any Riemannian sphere or projective space of dimension $n > 2$. (But see [K], [Z2] for examples of Finsler metrics on S^2 with finitely many prime closed geodesics, all of which are nondegenerate.) It should, in principle, be possible to count the number of closed geodesics using Morse theory on the free loop space Λ, but each prime geodesic γ is associated with infinitely many critical points, corresponding to the iterates γ^m. So it would be useful to have an operation on $H_*(\Lambda)$ which corresponds to the iteration of closed geodesics.

If λ_1 is the Morse index of a prime closed geodesic γ of length ℓ, then by [Bo1] (cf. Proposition 6.1), the Morse index λ_m of the iterate γ^m can be anywhere between $m\lambda_1 - (m-1)(n-1)$ and $m\lambda_1 + (m-1)(n-1)$. For nondegenerate critical points, the Chas-Sullivan product $[\overline{\gamma}] * \cdots * [\overline{\gamma}]$ is nonzero exactly when (cf. Theorem 12.3) the index growth is minimal (i.e., when $\lambda_m = m\lambda_1 - (m-1)(n-1)$). Here, $[\overline{\gamma}] \in H_{\lambda_1+1}(\Lambda^{\le \ell}, \Lambda^{<\ell})$ is the level homology class represented by the S^1-saturation of γ. The Pontrjagin product (on the level homology of the based loop space) is zero unless $\lambda_m = m\lambda_1$. The level cohomology product $\circledast$ is nonzero when the index growth is maximal (cf. Proposition 6.1).

1.16. Geometric motivation

The Chas-Sullivan product "detects" closed geodesics with minimal index growth. The second author was led by symmetry (cf. Lemma 6.4) to search for a cohomology product that would detect closed geodesics with maximal index growth. It is clear from the geometry that such a product should have degree $n-1$. Assume that the critical values of the energy are isolated. In a neighborhood of a critical orbit, the subspace of the finite-dimensional approximation $\mathscr{M}_N$ (cf. §3), consisting of loops with N geodesic pieces all of the same length, is a smooth finite-dimensional manifold. Using Poincaré duality in this manifold and the Chas-Sullivan product on relative "upside-down" chains (or *Morse cochains*) gives a product on the level cohomology as in equation (1.4.3). This product stabilizes as $N \to \infty$, and it extends to a globally defined product $\circledast$ which turns out to be dual to Sullivan's operation $\vee$ (see [Su1], [Su2]).

1.17. Related products
As mentioned in [Su2], the operation $\vee$ also gives (cf. §9.3) a (possibly noncommutative) product $\circledast$ on the cohomology of the based loop space Ω such that $i^*(a \circledast b) = i^*(a) \circledast i^*(b)$, where $a, b \in H^*(\Lambda)$ and $i : \Omega \to \Lambda$ denotes the inclusion. In §§14.9 and 15.5, we calculate some nonzero examples of this product.

In [CS], Chas and Sullivan also defined a Lie algebra product $\{\alpha, \beta\}$ on the homology $H_*(\Lambda)$ of the free loop space. In §17, we combine their ideas with the construction of the cohomology product $\circledast$ to produce a Lie algebra product on the cohomology $H^*(\Lambda, \Lambda_0)$. In §17.3, we use the calculations described in §1.15 to show that these products are sometimes nonzero. Also, following [CS], we construct products on the $(T = S^1)$-equivariant cohomology $H^*_T(\Lambda, \Lambda_0)$.

There is a well-known isomorphism between the Floer homology of the cotangent bundle of M and the homology of the free loop space of M, which transforms the pair-of-pants product into the Chas-Sullivan product on homology (see [AS1], [AS2], [SW], [Vi], [CHV]). The cohomology product $\circledast$ should therefore correspond to some geometrically defined product on the Floer *cohomology*; it would be interesting to see an explicit construction of this product. (Presumably, a candidate would be some 1-parameter variation of the coproduct on chains given by the upside-down pair of pants.)

1.18
Several of the proofs in this article require technical results that are well known to experts (in different fields) but are difficult to find in the literature. These technical tools are described in the appendices, as are the (tedious) proofs of Proposition 9.2 and Theorem 17.2. The collection of products and their definitions can be rather confusing, so in each case we have created a boxed $\boxed{\text{diagram}}$ which gives a concise way to think about the product.

2. The free loop space

2.1
Throughout this article, M denotes an n-dimensional smooth connected compact Riemannian manifold. Let $\alpha : [a, b] \to M$ be a piecewise smooth curve. Its *length* and *energy* are given by

$$L(\alpha) = \int_a^b |\alpha'(t)|\,dt \qquad \text{and} \qquad E(\alpha) = \int_a^b |\alpha'(t)|^2\,dt.$$

The Cauchy-Schwarz inequality says that $L(\alpha)^2 \le (b-a)E(\alpha)$. The formulas work out most simply if we use the Morse function $F(\alpha) = \sqrt{E(\alpha)}$.

The *free loop space* Λ consists of H^1-mappings $\alpha : [0,1] \to M$ such that $\alpha(0) = \alpha(1)$. It admits the structure of a Hilbert manifold (see [Kl], [Cha]), so it is a complete metric space and hence paracompact and Hausdorff. The loop space Λ is homotopy equivalent to the Frechet manifold of smooth loops $\beta : S^1 \to M$. Denote by $\Lambda_0 = \Lambda^{\le 0} \cong M$ the space of constant loops.

The energy of a loop depends on its parametrization; the length does not. Thus, $L(\alpha) \le F(\alpha)$ for all $\alpha \in \Lambda$, with equality if and only if the loop is *parametrized proportionally to arc length* (PPAL), meaning that $|\alpha'(t)|$ is constant. Every geodesic is, by definition, parametrized proportionally to arclength. A loop $\alpha \in \Lambda$ is a critical point of F if and only if α is a closed geodesic. Let $\Sigma \subset \Lambda$ be the set of critical points of F, and set $\Sigma^{=a} = \Sigma \cap \Lambda^{=a}$.

The index and nullity of the critical points of F coincide with those of E. Recall (e.g., from [Kl, p. 57]) that the index of a closed geodesic γ is the dimension of a maximal subspace of $T_\gamma(\Lambda)$ on which the Hessian $d^2F(\gamma)$ is negative definite, and the nullity of γ is $\dim(T_\gamma^0\Lambda) - 1$, where $T_\gamma^0\Lambda$ is the null space of the Hessian $d^2F(\gamma)$. The -1 is incorporated to account for the fact that every closed geodesic γ occurs in an $O(2)$-orbit of closed geodesics. The critical point γ is *nondegenerate* if this single orbit is a nondegenerate Morse-Bott critical submanifold or, equivalently, if the nullity is zero. A number $a \in \mathbb{R}$ is a *nondegenerate critical value* if the critical set $\Sigma^{=a}$ consists of nondegenerate critical orbits. In this case, there are finitely many critical orbits in $\Sigma^{=a}$, and the number $a \in \mathbb{R}$ is an isolated critical value. In §13, we encounter a critical set $\Sigma^{=a}$ of dimension greater than 1 (consisting of geodesics with nullity greater than 0), which is nondegenerate in the sense of Bott. In this case, we say that the critical value $a \in \mathbb{R}$ is *nondegenerate in the sense of Bott*. To distinguish *nondegenerate* from *nondegenerate in the sense of Bott*, we sometimes refer to the former case with the phrase *isolated nondegenerate critical orbit*.

Denote by $\mathscr{A} \subset \Lambda$ the subspace of loops parametrized proportionally to arc length (PPAL). Then $F(\alpha) = L(\alpha)$ for all $\alpha \in \mathscr{A}$. We write $\mathscr{A}^{\le a}$ (etc.) for those $\alpha \in \mathscr{A}$ such that $F(\alpha) \le a$ (cf. equation (1.1.2)). The following result is due to Anosov [A].

PROPOSITION 2.2
For all $a \le \infty$, the inclusion $\mathscr{A}^{\le a} \to \Lambda^{\le a}$ is a homotopy equivalence. A homotopy inverse is the mapping $A : \Lambda^{\le a} \to \mathscr{A}^{\le a}$ which associates to any path α the same path parametrized proportionally to arclength, with the same basepoint. It follows that the set of loops of length $\le a$ *also has the homotopy type of $\Lambda^{\le a}$.*

2.3

The *evaluation mapping* $\mathbf{ev}_s : \Lambda \to M$ is given by $\mathbf{ev}_s(\alpha) = \alpha(s)$. The *figure-eight space* $\mathscr{F} = \Lambda \times_M \Lambda$ is the pullback of the diagonal under the mapping

$$\mathbf{ev}_0 \times \mathbf{ev}_0 : \Lambda \times \Lambda \to M \times M. \tag{2.3.1}$$

It consists of composable pairs of loops. Denote by $\phi_s : \mathscr{F} \to \Lambda$ the mapping that joins the two loops at time s, that is,

$$\phi_s(\alpha, \beta)(t) = \begin{cases} \alpha\left(\frac{t}{s}\right) & \text{for } t \le s, \\ \beta\left(\frac{t-s}{1-s}\right) & \text{for } s \le t \le 1. \end{cases}$$

The mapping ϕ_s is one to one. The energy of the composed loop $\phi_s(\alpha, \beta)$ is

$$E\big(\phi_s(\alpha, \beta)\big) = \frac{E(\alpha)}{s} + \frac{E(\beta)}{1-s},$$

which is minimized when

$$s = \sqrt{E(\alpha)}/(\sqrt{E(\alpha)} + \sqrt{E(\beta)}). \tag{2.3.2}$$

LEMMA 2.4

Consider $M = \Lambda_0 \times_M \Lambda_0$ to be a subspace of $\mathscr{F} = \Lambda \times_M \Lambda$. Then the mapping $\phi_{\min} : \mathscr{F} - M \to \Lambda$ defined by $\phi_{\min}(\alpha, \beta) = \phi_s(\alpha, \beta)$ for

$$s = \frac{F(\alpha)}{F(\alpha) + F(\beta)}$$

extends continuously across M, giving a mapping $\phi_{\min} : \Lambda \times_M \Lambda \to \Lambda$ which is homotopic to the embedding $\phi_s : \mathscr{F} \to \Lambda$ for any $s \in (0, 1)$, and which satisfies

$$F\big(\phi_{\min}(\alpha, \beta)\big) = F(\alpha) + F(\beta). \tag{2.4.1}$$

If α and β are PPAL, then so is $\phi_{\min}(\alpha, \beta)$.

If $A, B \subset \Lambda$, write $A \times_M B = (A \times B) \cap (\Lambda \times_M \Lambda)$ and define $A * B = \phi_{\min}(A \times_M B)$ to be the subset consisting of all composable loops, glued together at the energy-minimizing time. Then $\Lambda^{\le a} * \Lambda^{\le b} \subset \Lambda^{\le a+b}$.

2.5

By [Cha, Proposition 2.2.3] or [BO, Proposition 1.17], the figure-eight space $\mathscr{F} = \Lambda \times_M \Lambda$ has an n-dimensional normal bundle $\nu_{\mathscr{F}}$ and tubular neighborhood N in $\Lambda \times \Lambda$ (see §B.1) because the mapping (2.3.1) is a submersion whose domain is a

Hilbert manifold. Similarly, for any $a, b \in \mathbb{R}$, the space

$$\mathscr{F}^{<a,<b} = \big\{(\alpha, \beta) \in \Lambda^{<a} \times \Lambda^{<b} : \ \alpha(0) = \beta(0)\big\}$$

has a normal bundle and tubular neighborhood in $\Lambda^{<a} \times \Lambda^{<b}$, and the image $\phi_s(\mathscr{F})$ has a normal bundle and tubular neighborhood in Λ because it is the preimage of the diagonal $\Delta \subset M \times M$ under the submersion

$$(\mathbf{ev}_0, \mathbf{ev}_s) : \Lambda \longrightarrow M \times M.$$

The normal bundle ν_Δ of Δ in $M \times M$ is noncanonically isomorphic to the tangent bundle TM, and $\nu_{\mathscr{F}}$ is the pullback of ν_Δ. Consequently, if M is orientable, then so are the normal bundles ν_Δ and $\nu_{\mathscr{F}}$. Throughout this article, we make the following.

Convention on orientations and coefficients
The symbol G always denotes the coefficient ring for homology or cohomology. The symbol $\check{H}_i(\cdot; G)$ denotes Čech homology (cf. Appendix A for further details). The symbol $H_i(\cdot, G)$ denotes singular homology. If M is not orientable, we take $G = \mathbb{Z}/(2)$. If M is orientable, then we fix orientations for M and for ν_Δ (hence also for $\nu_{\mathscr{F}}$) and we take $G = \mathbb{Z}$. We sometimes suppress mention of the coefficient ring G.

3. The finite-dimensional approximation of Morse

3.1
In this section, we recall some standard facts concerning the finite-dimensional approximation $\mathscr{M}$ to the free loop space Λ of a smooth compact Riemannian manifold M. This finite-dimensional approximation was described by Morse [Mo1], but his description is rather difficult to interpret by modern standards. It was clarified by Bott [Bo2] and further described by Milnor [Mi]. Related finite-dimensional models are discussed in [BC].

Fix $\rho > 0$ less than one half the injectivity radius of M. For points $x, y \in M$ which lie at a distance less than ρ, we write $|x - y|$ for this distance.

LEMMA 3.2
Fix $N \geq 1$. Let $x = (x_0, x_1, \ldots, x_N) \in M^{N+1}$. Let $\alpha : [0, 1] \to M$ be any piecewise smooth curve such that $\alpha(i/N) = x_i$. If $F(\alpha) \leq \rho\sqrt{N}$, then $|x_i - x_{i-1}| \leq \rho$ for each $i = 1, 2, \ldots, N$, and hence, for each i, there is a unique geodesic segment from x_{i-1} to x_i. If $\gamma = \gamma(x)$ denotes the path obtained by patching these geodesic segments together with $\gamma(i/N) = x_i$, then

$$F\big(\gamma(x)\big) = \sqrt{N \textstyle\sum_{i=1}^{N} |x_i - x_{i-1}|^2}.$$

Proof
Let $\alpha_i : [(i-1)/N, i/N] \to M$ denote the ith segment of the path. Then $L(\alpha_i)^2 \le E(\alpha_i)/N \le \rho^2$. Therefore $|x_i - x_{i-1}| \le \rho$. The energy of the resulting piecewise geodesic path γ is therefore $E(\gamma) = \Sigma_{i=1}^N E(\gamma_i) = N\Sigma_{i=1}^N |x_i - x_{i-1}|^2$. $\square$

For $N \ge 1$ and $a \in \mathbb{R}$, let

$$\mathscr{M}_N^{\le a} = \big\{(x_0, x_1, \dots, x_N) \in M^{N+1} :\ x_0 = x_N \text{ and } F(\gamma(x)) \le a\big\}.$$

According to Lemma 3.2, if $a \le \sqrt{N}\rho$, then we have a well-defined mapping

$$\gamma : \mathscr{M}_N^{\le a} \hookrightarrow \Lambda. \tag{3.2.1}$$

PROPOSITION 3.3
Suppose that $a \le \sqrt{N}\rho$. Then the mapping $F \circ \gamma : \mathscr{M}_N^{\le a} \to \mathbb{R}$ is smooth and proper. The restrictions $\gamma : \mathscr{M}_N^{\le a} \hookrightarrow \Lambda^{\le a}$ and $\gamma : \mathscr{M}_N^{<a} \hookrightarrow \Lambda^{<a}$ are homotopy equivalences. The mapping γ identifies the critical points (with values less than or equal to a) of $F \circ \gamma$ with the critical points (with values less than or equal to a) of F. The Morse index and nullity of each critical point are preserved under this identification. If, in addition, a is a regular value of F or if a is a nondegenerate critical value of F in the sense of Bott (cf. §2.1), then the spaces $\mathscr{M}_N^{\le a}$ and $\Lambda^{\le a}$ have the homotopy types of finite simplicial complexes.

Proof
There is a homotopy inverse $h : \Lambda^{\le a} \to \mathscr{M}_N^{\le a}$ which assigns to any loop $\alpha : [0,1] \to M$ the element $x = (x_0, \dots, x_N)$, where $x_i = \alpha(i/N)$ for $0 \le i \le N$. Since $F(\alpha) \le a$, Lemma 3.2 implies that $F \circ \gamma(h(\alpha)) \le a$. The composition $h \circ \gamma$ is the identity. The composition $\gamma \circ h : \Lambda^{\le a} \to \Lambda^{\le a}$ is homotopic to the identity: we describe a homotopy H_T from $\alpha \in \Lambda^{\le a}$ to $\gamma h(\alpha)$. Given $T \in [0,1]$, there exists i such that $(i-1)/N \le T \le i/N$. The homotopy $H_T(\alpha)(t)$ coincides with $\alpha(t)$ for $t \le (i-1)/N$. It coincides with the piecewise geodesic path $\gamma(\alpha)(t)$ for $t \ge (i/N)$. For t in the interval $[(i-1)/N, i/N]$, the path $H_T(\alpha)(t)$ agrees with α for $t \le T$, and it is geodesic on $[T, i/N]$. Replacing part of the curve α with a geodesic segment between the same two points does not increase its energy, so $H_T : \Lambda^{\le a} \times [0,1] \to \Lambda^{\le a}$ is the desired homotopy.

If $a \in \mathbb{R}$ is a regular value, then $\mathscr{M}_N^{\le a}$ is a smooth compact manifold with boundary, so it can be triangulated; hence $\Lambda^{\le a}$ is homotopy equivalent to a simplicial complex.

If the critical value a of F is nondegenerate in the sense of Bott, then a is also a (Bott)-nondegenerate critical value of $F \circ \gamma$. It is then possible to Whitney stratify

$\mathscr{M}_N^{\leq a}$ so that $\mathscr{M}_N^{=a}$ is a closed union of strata. The complete argument is standard but technically messy; here is an outline. Each connected component of the singular set S of $F \circ \gamma$ is a stratum. The set $\mathscr{M}_N^{=a} - S$ is another stratum; it is a manifold because it contains no critical points of $F \circ \gamma$. Finally, $\mathscr{M}_N^{<a}$ is the open stratum. According to the generalized Morse lemma, there exist local coordinates near each point x in the critical set, with respect to which the function $F \circ \gamma$ has the form $F(\gamma(x)) + \Sigma_{i=1}^r x_i^2 - \Sigma_{i=r+1}^s x_i^2$ (with the last $n - s$ coordinates not appearing in the formula). Using this, it is possible to see that the above stratification satisfies the Whitney conditions.

Every Whitney stratified space can be triangulated (see [Gor], [J]), so it follows that $\mathscr{M}_N^{\leq a}$ is homeomorphic to a finite simplicial complex; hence $\Lambda^{\leq a}$ is homotopy equivalent to a finite simplicial complex. □

4. Support, critical values, and level homology

4.1

Continue with the notation M, Λ, F, Σ of §2 and the conventions of §2.5. In this section, we describe a modern version of the Birkhoff-Hestenes minimax principle (see [BH1], [BH2]). A class $\alpha \in \check{H}_i(\Lambda; G)$ is *supported* on a closed set $A \subset \Lambda$ if there is a class $\alpha' \in \check{H}_i(A; G)$ such that $\alpha = i_*(\alpha')$, where $i : A \to \Lambda$ is the inclusion. This implies that $\alpha \mapsto 0 \in \check{H}_i(\Lambda, A; G)$, but the converse does not necessarily hold.* Define the *critical level* $\mathsf{cr}(\alpha)$ to be the infimum

$$\begin{aligned} \mathsf{cr}(\alpha) &= \inf\left\{a \in \mathbb{R} :\ \alpha \in \text{Image}\left(\check{H}_i(\Lambda^{\leq a}; G) \to H_i(\Lambda; G)\right)\right\} \\ &= \inf\left\{a \in \mathbb{R} :\ \alpha \text{ is supported on } \Lambda^{\leq a}\right\}. \end{aligned} \tag{4.1.1}$$

A nonzero homology class α normally gives rise to a nonzero class β in level homology at the level $\mathsf{cr}(\alpha)$. Let us say that two classes $\alpha \in H_i(\Lambda; G)$ and $\beta \in \check{H}_i(\Lambda^{\leq a}, \Lambda^{<a}; G)$ are *associated* if there exists an *associating class* $\omega \in \check{H}_i(\Lambda^{\leq a}; G)$ with

$$\begin{tikzcd} \omega \arrow[r] \arrow[d] & \alpha \\ \beta & \end{tikzcd} \quad \text{in} \quad \begin{tikzcd} \check{H}_i(\Lambda^{\leq a}) \arrow[r] \arrow[d] & \check{H}_i(\Lambda) = H_i(\Lambda) \\ \check{H}_i(\Lambda^{\leq a}, \Lambda^{<a}) & \end{tikzcd} \tag{4.1.2}$$

LEMMA 4.2
Let $\alpha \in H_i(\Lambda; G)$*,* $\alpha \neq 0$*. Then the following statements hold:*
(1) $\mathsf{cr}(\alpha)$ *is a critical value of* F*;*
(2) $\mathsf{cr}(\alpha)$ *is independent of the homology theory (Čech or singular) used in the definition;*

*This is because Čech homology does not satisfy the exactness axiom; see the "solenoid" in [ES, Chapter 10, §4].

(3) $\mathsf{cr}(\alpha) = \inf\{a \in \mathbb{R} : \alpha \in \ker(H_i(\Lambda) \to \check{H}_i(\Lambda, \Lambda^{\le a}; G))\}$;

If $a \in \mathbb{R}$ is a nondegenerate critical value in the sense of Bott, or if G is a field, then

(4) $\mathsf{cr}(\alpha) = a$ *if and only if there exists* $0 \neq \beta \in H_i(\Lambda^{\le a}, \Lambda^{<a}; G)$ *associated with* α*;*

(5) $\mathsf{cr}(\alpha) < a$ *if and only if* α *is associated to the zero class* $0 = \beta \in H_i(\Lambda^{\le a}, \Lambda^{<a}; G)$.

Proof

For part (1), if $\mathsf{cr}(\alpha)$ were a regular value, then the flow of $-\mathrm{grad}(F)$ would reduce the support of α below this value. Now, let $b_n \downarrow \mathsf{cr}(\alpha)$ be a convergent sequence of regular values (which exists because the regular values of F are dense in $\mathbb{R}$). By Lemma A.4, the Čech homology and singular homology of $\Lambda^{\le b_n}$ coincide, which proves (2); and the homology sequence for the pair $(\Lambda, \Lambda^{\le b_n})$ is exact, which proves (3). If a is a nondegenerate critical value (in the sense of Bott; cf. §2.1), then the set $\Lambda^{\le a}$ is a deformation retract of some open set $U \subset \Lambda$ which therefore contains $\Lambda^{\le b_n}$ for sufficiently large n. Hence α is the image of some

$$\omega \in H_i(\Lambda^{\le b_n}) \to H_i(U) \cong H_i(\Lambda^{\le a}).$$

Moreover, the homology sequence for $(\Lambda^{\le a}, \Lambda^{<a})$ is exact, which proves (4) and (5). If G is a field, see Lemma A.5. □

4.3

Similarly, a cohomology class $\alpha \in H^j(\Lambda, \Lambda_0; G)$ is *supported* on a closed set $B \subset \Lambda - \Lambda_0$ if it maps to zero in $\check{H}^j(\Lambda - B, \Lambda_0; G)$ or, equivalently, if it comes from a class in $\check{H}^j(\Lambda, \Lambda - B; G)$. Define the *critical level*

$$\begin{aligned}\mathsf{cr}(\alpha) &= \sup\left\{b : \alpha \in \ker\left(H^j(\Lambda, \Lambda_0; G) \to \check{H}^j(\Lambda^{<b}, \Lambda_0; G)\right)\right\}\\ &= \sup\left\{b : \alpha \text{ is supported on } \Lambda^{\ge b}\right\}.\end{aligned}$$

Let us say that the classes $\alpha \in H^j(\Lambda, \Lambda_0; G)$ and $\beta \in \check{H}^j(\Lambda^{\le b}, \Lambda^{<b}; G)$ are *associated* if there exists $\omega \in \check{H}^j(\Lambda, \Lambda^{<b}; G)$ with

$$\begin{tikzcd} \omega \arrow[r] \arrow[d] & \alpha \\ \beta & \end{tikzcd} \quad \text{in} \quad \begin{tikzcd} \check{H}^j(\Lambda, \Lambda^{<b}) \arrow[r] \arrow[d] & H^j(\Lambda, \Lambda_0) \\ \check{H}^j(\Lambda^{\le b}, \Lambda^{<b}) & \end{tikzcd} \tag{4.3.1}$$

LEMMA 4.4

Let $\alpha \in H^j(\Lambda, \Lambda_0; G)$, $\alpha \neq 0$. Then the following statements hold:

(1) $\mathsf{cr}(\alpha)$ *is a critical value of* F*;*

(2) $\mathsf{cr}(\alpha)$ *is independent of the homology theory used in its definition.*

If $a \in \mathbb{R}$ is a nondegenerate critical value in the sense of Bott, or if G is a field, then

(3) $\mathsf{cr}(\alpha) = b$ *if and only if there exits* $0 \neq \beta \in H^j(\Lambda^{\leq b}, \Lambda^{<b}; G)$ *associated with* α*;*

(4) $\mathsf{cr}(\alpha) < b$ *if and only if* α *is associated with the zero class* $0 \in H^j(\Lambda^{\leq b}, \Lambda^{<b}; G)$.

Proof

The proof is the same as in Lemma 4.2. □

5. The Chas-Sullivan product

5.1

Throughout this section, M denotes a compact connected Riemannian manifold. We continue with the conventions of §2.5. In [CS], a product $* : H_i(\Lambda) \times H_j(\Lambda) \to H_{i+j-n}(\Lambda)$ on the homology of the free loop space Λ was defined. It has since been reinterpreted in a number of different contexts (see [CKS], [CJ], [Co]). Recall, for example, from [CJ] or [AS1], that it can be constructed as the following composition:

$$\begin{tikzcd} H_i(\Lambda) \times H_j(\Lambda) \arrow[d, "\epsilon\times"] & & & \\ H_{i+j}(\Lambda \times \Lambda) \arrow[r, "s"] & H_{i+j}(\Lambda \times \Lambda, \Lambda \times \Lambda - \mathscr{F}) \arrow[r, "\tau"] & H_{i+j-n}(\mathscr{F}) \arrow[r, "\phi_*"] & H_{i+j-n}(\Lambda) \end{tikzcd}$$

In this diagram, $\epsilon = (-1)^{n(n-j)}$ (cf. [D, Chapter 8, §13.3], [Cha]), and $\times$ denotes the homology cross product. The map τ is the Thom isomorphism (B.1.2) for the normal bundle $\nu_{\mathscr{F}}$ of $\mathscr{F} = \Lambda \times_M \Lambda$ in $\Lambda \times \Lambda$ (see [Cha, Proposition 2.2.3], [BO, Proposition 1.17]). The composition $\tau \circ s$ is a Gysin homomorphism (B.2.4). The map $\phi = \phi_{1/2}$ composes the two loops at time $t = 1/2$. If M is orientable, then $\nu_{\mathscr{F}}$ is also orientable (see §2.4). We often substitute the homotopic mapping $\phi_{\min}$ of Lemma 2.4 for $\phi_{1/2}$. The construction may be summarized as passing from the left to the right in the following diagram:

$$\boxed{\Lambda \times \Lambda \longleftarrow \mathscr{F} \longrightarrow \Lambda} \tag{5.1.1}$$

It is well known that the Chas-Sullivan product is (graded) commutative, but this is not entirely obvious since it involves reversing the order of composition of loops, and the fundamental group of M may be noncommutative. We include the short proof because the same method is used in §9.

PROPOSITION 5.2 ([CS, Theorem 3.3])

If $a \in H_i(\Lambda)$ and $b \in H_j(\Lambda)$, then $b * a = (-1)^{(i-n)(j-n)} a * b$.

Proof
The map $\sigma : \Lambda\times\Lambda \to \Lambda\times\Lambda$ that switches factors satisfies $\sigma_*(a\times b) = (-1)^{ij}(b\times a)$. It restricts to an involution $\sigma : \mathscr{F} \to \mathscr{F}$. Identifying $S^1 = \mathbb{R}/\mathbb{Z}$, define $\chi_r : S^1 \to S^1$ by $\chi_r(t) = t + r$. The (usual) action, $\widehat{\chi} : S_1 \times \Lambda \to \Lambda$ of S^1 on Λ is given by

$$\widehat{\chi}_r(\gamma) = \gamma \circ \chi_r$$

for $r \in S^1$. The action of $\widehat{\chi}_{1/2}$ preserves $\phi_{1/2}(\mathscr{F})$, and in fact,

$$\sigma(\gamma) = \widehat{\chi}_{1/2}(\gamma) = \beta\cdot\alpha \tag{5.2.1}$$

for any $\gamma = \alpha\cdot\beta \in \phi_{1/2}(\mathscr{F})$ which is a composition of two loops α, β glued at time $1/2$. Let $\mu_{\mathscr{F}} \in H^n(\Lambda\times\Lambda, \Lambda\times\Lambda - \mathscr{F})$ be the Thom class of the normal bundle ν_F. Then $\sigma^*(\mu_{\mathscr{F}}) = (-1)^n\mu_{\mathscr{F}}$. Since $\chi_r : \Lambda \to \Lambda$ is homotopic to the identity, we have

$$\begin{aligned}
(-1)^{n(n-i)} b * a &= \phi_*\big(\mu_{\mathscr{F}} \cap (b\times a)\big)\\
&= (-1)^{ij}\phi_*\big(\mu_{\mathscr{F}} \cap \sigma_*(a\times b)\big)\\
&= (-1)^{ij}\phi_*\sigma_*\big(\sigma^*(\mu_{\mathscr{F}}) \cap (a\times b)\big)\\
&= (-1)^{ij}(-1)^n\widehat{\chi}_{1/2}\phi_*\big(\mu_{\mathscr{F}} \cap (a\times b)\big)\\
&= (-1)^{ij}(-1)^n(-1)^{n(n-j)} a * b.
\end{aligned}$$

□

PROPOSITION 5.3
Let $\alpha, \beta \in H_*(\Lambda; G)$ be homology classes supported on closed sets $E, F \subset \Lambda$, respectively. Then $\alpha * \beta$ is supported on the closed set $E * F = \phi_{\min}(E \times_M F)$. In particular,

$$\mathsf{cr}(\alpha * \beta) \le \mathsf{cr}(\alpha) + \mathsf{cr}(\beta). \tag{5.3.1}$$

For any a, b with $0 \le a, b \le \infty$, the Chas-Sullivan product extends to a family of products

$$\check{H}_i(\Lambda^{\le a}; G) \times \check{H}_j(\Lambda^{\le b}; G) \to \check{H}_{i+j-n}(\Lambda^{\le a+b}; G) \tag{5.3.2}$$

and, for any $0 \le a' < a \le \infty$ and $0 \le b' < b \le \infty$, to products

$$\check{H}_i(\Lambda^{\le a}, \Lambda^{\le a'}; G) \times \check{H}_j(\Lambda^{\le b}, \Lambda^{\le b'}; G) \to \check{H}_{i+j-n}(\Lambda^{\le a+b}, \Lambda^{\le \max(a+b', a'+b)}; G), \tag{5.3.3}$$

$$\check{H}_i(\Lambda^{\le a}, \Lambda^{< a}; G) \times \check{H}_j(\Lambda^{\le b}, \Lambda^{< b}; G) \to \check{H}_{i+j-n}(\Lambda^{\le a+b}, \Lambda^{< a+b}; G). \tag{5.3.4}$$

These products are compatible under the mappings induced by inclusion. If the set

$$
\begin{tikzcd}
H_i(A,A')\times H_i(B,B') \arrow[r, "\epsilon\times"] & H_{i+j}(A\times B, A'\times B\cup A\times B') \arrow[d, "="] \\
& H_{i+j}\big(A\times B, A\times B-(A-A')\times(B-B')\big) \arrow[d] \\
& H_{i+j}\big(A\times B, A\times B-(A-A')\times_M(B-B')\big) \arrow[d, "\tau", "\text{Thom isomorphism §B.2}"'] \\
& H_{i+j-n}\big(A\times_M B, A\times_M B-(A-A')\times_M(B-B')\big) \arrow[d, "="] \\
H_{i+j-n}(A*B, A'*B\cup A*B') & H_{i+j-n}(A\times_M B, A'\times_M B\cup A\times_M B') \arrow[l, "(\phi_{\min})_*"']
\end{tikzcd}
$$

Figure 1. Relative Chas-Sullivan product

$\mathsf{cr}\subset\mathbb{R}$ *of critical values is discrete, then we obtain a ring structure on the* level homology

$$\bigoplus_{a\in\mathsf{cr}}\check{H}_*(\Lambda^{\le a}, \Lambda^{<a}; R). \tag{5.3.5}$$

Proof

First, we construct, for any open sets $A'\subset A\subset\Lambda$ and $B'\subset B\subset\Lambda$, a product

$$H_i(A, A'; G)\times H_j(B, B'; G)\to H_{i+j-n}(A*B, A*B'\cup A'*B) \tag{5.3.6}$$

on singular homology, as shown in Figure 1.

In Figure 1, $\times$ denotes the homology cross product, $\epsilon=(-1)^{n(n-j)}$, and the Thom isomorphism τ of Proposition B.2 is applied to the triple

$$(A-A')\times_M(B-B')\subset A\times_M B\subset A\times B.$$

The hypotheses of Proposition B.2 are satisfied because $A\times B$ is a Hilbert manifold, so $A\times_M B$ has a normal bundle and tubular neighborhood in $A\times B$, and because the subspace $(A-A')\times_M(B-B')$ is closed in $A\times_M B$.

It is easy to see that this product is compatible with the product in §5.1 since each line in Figure 1 is pulled back from the corresponding line in §5.1. Consequently, the following diagram commutes:

$$
\begin{tikzcd}
H_i(\Lambda)\times H_j(\Lambda) \arrow[d] & H_i(A)\times H_j(B) \arrow[l] \arrow[r] \arrow[d] & H_i(A,A')\times H_j(B,B') \arrow[d] \\
H_{i+j-n}(\Lambda) & H_{i+j-n}(A*B) \arrow[l] \arrow[r] & H_{i+j-n}(A*B, A'*B\cup A*B')
\end{tikzcd}
$$

Now, the other products may be obtained by a limiting procedure using Lemma A.5. The statement about the support of $\alpha * \beta$ follows by taking a sequence of open neighborhoods $A_n \downarrow E$ and $B_n \downarrow F$. To construct the product (5.3.4), for example, start with

$$H_i(\Lambda^{<a+\epsilon}, \Lambda^{<a'-\delta}) \times H_j(\Lambda^{<b+\epsilon}, \Lambda^{<b'-\delta}) \\ \to H_{i+j-n}(\Lambda^{<a+b+\epsilon-\delta}, \Lambda^{<\max(a'+b+\epsilon-\delta, a+b'+\epsilon-\delta)}),$$

where $\delta > \epsilon$. Taking the (inverse) limit as $\epsilon \downarrow 0$ gives a pairing

$$\check{H}_i(\Lambda^{\le a}, \Lambda^{<a-\delta}) \times \check{H}_j(\Lambda^{\le b}, \Lambda^{<b-\delta}) \to \check{H}_{i+j-n}(\Lambda^{\le a+b}, \Lambda^{<\max(a'+b-\delta, a+b'-\delta)}).$$

Taking the direct limit as $\delta \downarrow 0$ (and recalling from §A.2 that homology commutes with direct limits) gives the pairing (5.3.4). The other products are similarly constructed. (The Chas-Sullivan product can even be constructed this way; cf. [Cha].) This completes the proof of Proposition 5.3. $\square$

We remark that singular homology is not so well behaved with respect to supports and intersections (see, e.g., Bredon's example [Br1, p. 373]).

5.4

If $\alpha = [A, \partial A]$ and $\beta = [B, \partial B]$ are the fundamental classes of manifolds $(A, \partial A) \subset (\Lambda^{\le a}, \Lambda^{\le a'})$ and $(B, \partial B) \subset (\Lambda^{\le b}, \Lambda^{\le b'})$ which are transverse over M (meaning that the mappings $\mathbf{ev}_0 : A \to M$ and $\mathbf{ev}_0 : B \to M$ are transverse), then the C-S product $[\alpha] * [\beta]$ is represented by the fundamental class of the manifold

$$\phi_{\min}(A \times_M B, A \times_M \partial B \cup \partial A \times_M B) = \big(A * B, \partial(A * B)\big).$$

For equations (12.5.2) and (13.5.2), we need a similar fact about homology classes α, β which are supported on $(A, \partial A)$ and $(B, \partial B)$ but which are not necessarily the fundamental classes. The following proposition is more or less the original definition of the product $*$ from [CS].

PROPOSITION 5.5

Let $(A, \partial A)$ and $(B, \partial B)$ be smooth manifolds with boundary. Let $(A, \partial A) \to (\Lambda^{<\alpha}, \Lambda^{<\alpha'})$ and $(B, \partial B) \to (\Lambda^{<\beta}, \Lambda^{<\beta'})$ be smooth embeddings, where $\alpha' < \alpha$ and $\beta' < \beta$. Assume that the mappings $\mathbf{ev}_0 : A \to M$ and $\mathbf{ev}_0 : B \to M$ are transverse. If M is oriented, then assume that A, B are orientable and oriented. Let $\alpha \in H_i(A, \partial A)$ and $\beta \in H_i(B, \partial B)$. Denote their images in the homology of Λ by $[\alpha] \in H_i(\Lambda^{<a}, \Lambda^{<a'})$ and $[\beta] \in H_j(\Lambda^{<b}, \Lambda^{<b'})$. Define

$$\alpha*\beta \in H_{i+j-n}(A\times_M B, A\times_M \partial B \cup \partial A\times_M B) \cong H_{i+j-n}(A\times_M B, A\times_M B - A'\times_M B')$$

to be the image of (α, β) *under the following composition:*

$$\begin{array}{ccc} H_i(A, \partial A) \times H_j(B, \partial B) & \xrightarrow{\epsilon \times} & H_{i+j}(A \times B, A \times B - A' \times B') \\ & & \downarrow \\ H_{i+j-n}(A \times_M B, A \times_M B - A' \times_M B') & \xleftarrow[\text{(B.2)}]{\tau} & H_{i+j}(A \times B, A \times B - A' \times_M B') \end{array} \tag{5.5.1}$$

Then $[\alpha] * [\beta] = (\phi_{\min})_*(\alpha * \beta) \in H_{i+j-n}(\Lambda^{<a+b}, \Lambda^{\max(a+b', a'+b)})$.

Proof
The transversality assumption is equivalent to the statement that the mapping

$$(\mathbf{ev}_0, \mathbf{ev}_0) : A \times B \to M \times M$$

is transverse to the diagonal Δ. By [Cha, Proposition 2.2.3] or [BO, Proposition 1.17], the intersection $A \times_M B = A \times B \cap \mathscr{F}$ has a tubular neighborhood in $A \times B$ (with normal bundle $\nu_{\mathscr{F}} | (A \times_M B)$. As in the proof of Proposition 5.3, this makes it possible to apply the Thom isomorphism (B.2). Then the diagram (5.5.1) maps, term by term, to the diagram in the proof of Proposition 5.3 where the relative C-S product is defined. The proposition amounts to the statement that these mappings commute, which they obviously do. □

6. Index growth

Continue with the notation M, Λ, F, Σ of §2 and the conventions of §2.5.

PROPOSITION 6.1
Let γ *be a closed geodesic with index* λ *and nullity* ν. *Let* λ_m *and* ν_m *denote the index and nullity of the* m*-fold iterate* γ^m. *Then* $\nu_m \le 2(n-1)$ *for all* m *and*

$$|\lambda_m - m\lambda| \le (m-1)(n-1), \tag{6.1.1}$$

$$|\lambda_m + \nu_m - m(\lambda + \nu)| \le (m-1)(n-1). \tag{6.1.2}$$

The average index

$$\lambda_{av} = \lim_{m \to \infty} \frac{\lambda_m}{m}$$

exists and

$$|\lambda - \lambda_{av}| \le n - 1 \qquad \textit{and} \qquad |\lambda + \nu - \lambda_{av}| \le n - 1. \tag{6.1.3}$$

Now, assume that γ and γ^2 are nondegenerate critical points (i.e., they lie on isolated nondegenerate critical orbits). Then the inequalities in (6.1.3) are strict, and for any $k \geq 0$, there exists M so that if $m \geq M$, then

$$|\lambda_m - m\lambda| \leq (m-1)(n-1) - k, \tag{6.1.4}$$

$$|\lambda_m + \nu_m - m(\lambda + \nu)| \leq (m-1)(n-1) - k. \tag{6.1.5}$$

Moreover, if we let

$$\lambda_m^{\min} = m\lambda_1 - (m-1)(n-1),$$
$$\lambda_m^{\max} = m\lambda_1 + (m-1)(n-1)$$

be the greatest and smallest possible values for λ_m that are compatible with (6.1.1), then

$$\lambda_m > \lambda_m^{\min} \Longrightarrow \lambda_j > \lambda_j^{\min} \quad \text{for all } j > m, \tag{6.1.6}$$

$$\lambda_m < \lambda_m^{\max} \Longrightarrow \lambda_j < \lambda_j^{\max} \quad \text{for all } j > m. \tag{6.1.7}$$

SCHOLIUM
Let r, k be positive integers. If the index growth for γ is maximal (resp., minimal) up to level rk in the sense that for all $m \leq rk$,

$$\lambda_m = m\lambda_1 \pm (m-1)(n-1),$$

then the iterate γ^r has maximal (resp., minimal) index growth up to level k.

Much of this is standard and is well known to experts (see the references at the beginning of Appendix D), but for completeness we include a proof based on the following well-known results of Bott [Bo1] (cf. [Kl, §3.2.9]).

6.2. Bott's index formula

Let M be an n-dimensional Riemannian manifold. Let γ be a closed geodesic. The Poincaré map P (linearization of the geodesic flow at a periodic point) is in $\mathrm{Sp}(2(n-1), \mathbb{R})$ and is defined up to conjugation. The index formula of Bott is

$$\lambda_m = \text{index}(\gamma^m) = \sum_{\omega^m=1} \Omega_\gamma(\omega), \tag{6.2.1}$$

where the ω-index Ω_γ is a nonnegative integer-valued function defined on the unit circle with $\Omega_\gamma(\omega) = \Omega_\gamma(\overline{\omega})$ (see also Lemma 6.5). The function Ω_γ is constant except at the eigenvalues of P. Its jump at each eigenvalue is determined by the *splitting numbers* $S_P^{\pm}(\omega)\epsilon\mathbb{Z}$:

$$\lim_{\varepsilon\to 0^+}\Omega_\gamma(\omega e^{\pm i\varepsilon}) = \Omega_\gamma(\omega) + S_P^{\pm}(\omega), \tag{6.2.2}$$

which depend only upon the conjugacy class of P. The nullity satisfies

$$\nu_m = \text{ nullity}(\gamma^m) = \sum_{\omega^m=1}\mathcal{N}_P(\omega), \tag{6.2.3}$$

where $\mathcal{N}_P(\omega) := \dim\ker(P-\omega I)$. The numbers $S_P^{\pm}(\omega)$ and $\mathcal{N}_P(\omega)$ are additive on indecomposable symplectic blocks, and on each block

$$S_P^{\pm}(\omega)\epsilon\{0,1\}, \tag{6.2.4}$$

$$\mathcal{N}_P(\omega) - S_P^{\pm}(\omega)\epsilon\{0,1\}. \tag{6.2.5}$$

6.3. Proof of Proposition 6.1

Assuming the above facts (6.2.1) – (6.2.5), the index-plus-nullity satisfies

$$\lambda_m + \nu_m = \text{ index-plus-nullity}(\gamma^m) = \sum_{\omega^m=1}\Upsilon_\gamma(\omega), \tag{6.3.1}$$

$$\lim_{\varepsilon\to 0^+}\Upsilon_\gamma(\omega e^{\pm i\varepsilon}) = \Upsilon_\gamma(\omega) - T_P^{\pm}(\omega), \tag{6.3.2}$$

where

$$\Upsilon_\gamma(\omega) := \Omega_\gamma(\omega) + \mathcal{N}_P(\omega),$$
$$T_P^{\pm}(\omega) := \mathcal{N}_P(\omega) - S_P^{\pm}(\omega).$$

Thus $T_P^{\pm}(\omega)$ is additive and takes values in $\{0,1\}$ on each indecomposable block, and $-(\lambda_m+\nu_m)$ and $-\Upsilon_\gamma$ have the same formal properties (6.2.1) – (6.2.4) as λ_m and Ω_γ. Thus a proof of the statements about λ_m using only these three properties also serves as a proof of the statements about $\lambda_m+\nu_m$.

As a consequence of equations (6.2.1) – (6.2.4) we have

$$|\Omega_\gamma(\omega) - \Omega_\gamma(\tau)| \le n-1 \quad \text{ for all } \omega,\tau. \tag{6.3.3}$$

Moreover, if $|\Omega_\gamma(\omega) - \Omega_\gamma(\tau)| = n-1$ with $\operatorname{Re}\omega < \operatorname{Re}\tau$, then all the eigenvalues of P lie in the unit circle with real part in $[\operatorname{Re}\omega, \operatorname{Re}\tau]$. (To see this, note that each indecomposable block has dimension at least 2.) Equation (6.1.1) and the first half of (6.1.3) follow. Moreover,

$$\lambda_{av} = \frac{1}{2\pi}\int_0^{2\pi}\Omega_\gamma(e^{it})\,dt. \tag{6.3.4}$$

Equality in (6.1.1) implies that $\lambda_m = \lambda_m^{\max}$ or $\lambda_m = \lambda_m^{\min}$. Equality in the first half of (6.1.3), together with (6.3.3) and (6.3.4), implies that $|\Omega_\gamma(\omega) - \Omega_\gamma(1)| = n - 1$ almost everywhere on the circle, that 1 is the only eigenvalue of P, and thus that $\lambda_m = \lambda_m^{\max}$ for all m or $\lambda_m = \lambda_m^{\min}$ for all m.

If γ and γ^2 are nondegenerate, then neither -1 nor 1 is an eigenvalue of P. Let $S^1 \subset \mathbb{C}$ denote the unit circle, and let

$$r_0 = \min\big\{\mathrm{Re}(\omega)\big|\omega \in S^1 \text{ is an eigenvalue of } P\big\}.$$

Then $-1 < r_0 < 1$, and for any $\omega \in S^1$,

$$|\Omega_\gamma(\omega) - \Omega_\gamma(1)| \leq \begin{cases} n-2 & \text{if } \mathrm{Re}(\omega) > r_0, \\ n-1 & \text{if } \mathrm{Re}(\omega) \leq r_0, \\ 0 & \text{if } \omega = 1. \end{cases}$$

Summing over the mth roots of unity gives

$$|\lambda_m - m\lambda| \leq (m-1)(n-1) - h_m, \tag{6.3.5}$$

where h_m denotes the number of mth roots of unity $\omega \neq 1$ such that $\mathrm{Re}(\omega) > r_0$. Thus, choosing $m \geq M = k\pi/\arccos(r_0)$ gives equation (6.1.4), which also implies that the inequalities in (6.1.3) are strict. Equations (6.1.6) and (6.1.7) follow. The scholium is self-contained and straightforward. This concludes the proof of Proposition 6.1. $\square$

The following lemma is used in the proof of Theorem 12.3.

LEMMA 6.4

Fix a basepoint $x_0 \in M$, and let $\Omega = \Omega_{x_0} = \mathbf{ev}_0^{-1}(x_0)$ be the Hilbert manifold of loops that are based at x_0. Let $\gamma \in \Omega$ be a closed geodesic, all of whose iterates are nondegenerate. For any $r \geq 1$, let λ_r be the Morse index of the iterate γ^r, and let λ_r^Ω be the index of γ^r in the based loop space.

(i) *Let $r \geq 1$, and suppose that the index growth is maximal up to level rn, that is,*

$$\lambda_{rn} = rn\lambda_1 + (rn-1)(n-1).$$

Then $\lambda_r^\Omega = \lambda_r$.

(ii) *Let $r \geq 1$, and suppose that the index growth is minimal up to level rn, that is,*

$$\lambda_{rn} = rn\lambda_1 - (rn-1)(n-1).$$

Then the difference between λ_r^Ω and λ_r is maximal, that is, $\lambda_r^\Omega = \lambda_r - (n-1)$.

Proof
Let $T^{\perp}_{\gamma(0)}M$ be the subspace that is orthogonal to the tangent vector $\gamma'(0)$. Let $T^{\perp}_{\gamma}\Lambda$ (resp., $T^{\perp}_{\gamma}\Omega$) be the subspace of vector fields $V(t)$ along γ with $V(t) \perp \gamma'(t)$ for all t. By a standard argument, for all $s \geq 1$, λ_s (resp., λ^{Ω}_s) is the dimension of a maximal negative subspace of $T^{\perp}_{\gamma^s}\Lambda$ (resp., of $T^{\perp}_{\gamma^s}(\Omega)$). Let $W_s \subset T^{\perp}_{\gamma^s}\Lambda$ be a maximal negative subspace. Let K_s be the kernel of the map

$$W_s \xrightarrow{\phi} T^{\perp}_{\gamma(0)}M \times T^{\perp}_{\gamma(0)}M \times \cdots \times T^{\perp}_{\gamma(0)}M,$$
$$V \mapsto \left(V(0), V\Big(\frac{1}{s}\Big), \ldots, V\Big(\frac{s-1}{s}\Big)\right).$$

The dimension of the image of this map is at most $s(n-1)$, so $\dim(K_s) \geq \lambda_s - s(n-1)$. On the other hand, the kernel of ϕ on $T^{\perp}_{\gamma^s}\Lambda$ is the direct sum of s copies of $T^{\perp}_{\gamma}\Omega$ in a way that is compatible with the Hessian of F, so the index of F on this kernel is $s\lambda^{\Omega}_1$. Thus, for all $s \geq 1$, we have

$$s\lambda^{\Omega}_1 \geq \dim(K_s) \geq \lambda_s - s(n-1). \tag{6.4.1}$$

Now, consider part (i). Clearly, $\lambda^{\Omega}_r \leq \lambda_r$, so we need to verify the opposite inequality. Consider the geodesic γ^r. By the scholium of Proposition 6.1, the index growth for γ^r is maximal up to level n. Apply equation (6.4.1) to the geodesic γ^r (and take $s = n$) to obtain

$$n\lambda^{\Omega}_r \geq \lambda_n(\gamma^r) - n(n-1) = n\lambda_r - (n-1)$$

or $\lambda^{\Omega}_r \geq \lambda_r - ((n-1)/n)$ as claimed.

Now, consider part (ii). According to the scholium of Proposition 6.1, the geodesic γ^r has minimal index growth up to level n. So it suffices to show the following: if γ is a geodesic with minimal index growth up to level n, then $\lambda_1 = \lambda^{\Omega}_1 + n - 1$. Taking $s = 1$ in (6.4.1) gives $\lambda_1 \leq \lambda^{\Omega}_1 + n - 1$. For the reverse inequality, note that $\lambda_n \geq \lambda^{\Omega}_n \geq n\lambda^{\Omega}_1$ because we can concatenate n negative vector fields along γ to obtain a negative vector field along γ^n. But $\lambda_n = n\lambda_1 - (n-1)^2$, which gives

$$\lambda_1 - \frac{(n-1)^2}{n} = \lambda_1 - (n-1) + \frac{n-1}{n} \geq \lambda^{\Omega}_1. \qquad \square$$

The following lemma, due to [R] (see also [W]) is used in §§12.3 and 13.2.

LEMMA 6.5
Let $\gamma \subset \Lambda$ be a prime geodesic. Let γ^r be its r-fold iterate, and let $\Sigma_r \subset \Lambda$ be the S^1-saturation of γ^r. Let $\Gamma_r \to \Sigma_r$ be the negative bundle. If r is odd, then $\Gamma_r \to \Sigma_r$ is orientable and $\lambda_r \equiv \lambda_1 \pmod 2$. If r is even, then $\Gamma_r \to \Sigma_r$ is orientable if and only if $\Omega_\gamma(-1) = \lambda_2 - \lambda_1$ is even, in which case $\lambda_r \equiv \lambda_1 \pmod 2$.

Proof
Following [R, §§1.1, 2.2], the S^1-action on Λ induces an S^1-action on the bundle $\Gamma_r \to \Sigma_r$. The subgroup $\mathbb{Z}/(r) \subset S^1$ fixes the base space,

$$\Sigma_r \cong S^1/\big(\mathbb{Z}/(r)\big).$$

Let E denote the fiber of Γ_r over the basepoint γ^r. The bundle $\Gamma_r \to \Sigma_r$ is orientable if and only if the $\mathbb{Z}/(r)$-action on the fiber E preserves the orientation. If $T : E \to E$ denotes a generator of $\mathbb{Z}/(r)$, then $T^r = I$ so $\det(T)^r = 1$. Hence $\det(T) = \pm 1$ with $\det(T) = +1$ if and only if Γ_r is orientable. If r is odd, then $\det(T)$ cannot be -1 so it is $+1$. If r is even, then the eigenvalues of T that are not equal to ± 1 come in complex conjugate pairs, so the sign of $\det(T)$ is determined by the dimension of the -1 eigenspace. But according to [Bo1] (see also [Kl, §§3.2.9, 4.1.5, pp. 128, 129]), if ω is an rth root of unity, then $\Omega_\gamma(\omega)$ is the dimension of the ω-eigenspace of the action of T on E. Hence Γ_r is orientable if and only if $\Omega_\gamma(-1)$ is even.

Similarly, $\Omega_\gamma(\omega) = \Omega_\gamma(\overline{\omega})$ implies that $\lambda_r \equiv \Omega_\gamma(1) \pmod 2$ if r is odd, and $\lambda_r \equiv \Omega_\gamma(1) + \Omega_\gamma(-1)$ if r is even. □

7. Level nilpotence

7.1
Continue with the notation M, Λ, F, Σ of §2 and the conventions of §2.5. Let $a \in \mathbb{R}$, and let $\beta \in \check{H}_i(\Lambda^{\le a}, \Lambda^{<a}; G)$. We say that β is *level-nilpotent* if there exists m such that the Chas-Sullivan product in level homology vanishes:

$$0 = \beta^{*m} = \beta * \beta * \cdots * \beta \in \check{H}_{mi+(m-1)n}(\Lambda^{\le ma}, \Lambda^{<ma}; G).$$

Let $\alpha \in \check{H}_i(\Lambda; G)$. We say that α is *level-nilpotent* if there exists m such that $\mathsf{cr}(\alpha^{*m}) < m\,\mathsf{cr}(\alpha)$.

LEMMA 7.2
Let $\alpha \in \check{H}_i(\Lambda; G)$, $\alpha \ne 0$, and let $a = \mathsf{cr}(\alpha)$. Let $\beta \in \check{H}_i(\Lambda^{\le a}, \Lambda^{<a}; G)$ be an associated class. (Such a nonzero class β exists when a is a nondegenerate critical value in the sense of Bott; cf. Lemma 4.2.) If β is level-nilpotent, then α is also level-nilpotent.

Proof
Let $\omega \in H_i(\Lambda^{\le a}; G)$ be a class that associates α and β. Then $\omega^{*m} \in \check{H}_b(\Lambda^{\le ma}; G)$ associates α^{*m} and β^{*m}, where $b = mi + (m-1)n$. But $\beta^{*m} = 0$ if m is sufficiently large, which implies by Lemma 4.2 that $\mathsf{cr}(\alpha^{*m}) < ma$. □

THEOREM 7.3
Let $F : \Lambda \to \mathbb{R}$ as above. Suppose that all the critical orbits of F are isolated and nondegenerate. Then every homology class $\alpha \in H_i(\Lambda; G)$ is level-nilpotent, and for every $a \in \mathbb{R}$, every level homology class $\beta \in H_i(\Lambda^{\le a}, \Lambda^{<a}; G)$ is level-nilpotent.

Proof
By Lemma 7.2, it suffices to prove that $\beta \in H_i(\Lambda^{\le a}, \Lambda^{<a}; G)$ is level-nilpotent, where $a \in \mathbb{R}$ is a nondegenerate critical value. We suppress mention of the coefficient ring G. The critical set $\Sigma^{=a} = \Sigma(F) \cap F^{-1}(a)$ consists of the S^1-orbits of finitely many closed geodesics, say, $\gamma_1, \ldots, \gamma_r$. Let $\overline{\gamma}$ denote the S^1-orbit of γ. For $1 \le j \le r$, let $U_j \subset \Lambda$ be a neighborhood of $\overline{\gamma}_j$, chosen so that $U_j \cap U_k = \phi$ whenever $j \ne k$. We may choose the ordering so that there exists $s \le r$ such that $H_i(\Lambda^{\le a} \cap U_j, \Lambda^{<a} \cap U_j) \ne 0$ if and only if $1 \le j \le s$. The γ_j with $1 \le j \le s$ are the critical points that are relevant to β, and Theorem D.2 implies that the index of γ_j $(1 \le j \le s)$ is either i or $i-1$.

Set $\Sigma_0^{=a} = \bigcup_{j=1}^{s} \overline{\gamma}_j$. Using Proposition A.6, the level homology group is a direct sum

$$H_i(\Lambda^{\le a}, \Lambda^{<a}) \cong \bigoplus_{j=1}^{r} H_i(\Lambda^{\le a} \cap U_j, \Lambda^{<a} \cap U_j).$$

Since these factors vanish for $j > s$, we have canonical isomorphisms

$$\begin{aligned} H_i(\Lambda^{\le a}, \Lambda^{<a}) &\cong \bigoplus_{j=1}^{s} H_i(U_j^{<a} \cup \overline{\gamma}_j, U_j^{<a}) \\ &\cong H_i(\Lambda^{<a} \cup \Sigma_0^{=a}, \Lambda^{<a}) \end{aligned}$$

using excision and homotopy equivalences. By comparing the long exact sequence for the pair $(\Lambda^{\le a}, \Lambda^{<a})$ with the long exact sequence for the pair $(\Lambda^{<a} \cup \Sigma_0^{=a}, \Lambda^{<a})$ and using the five lemma, we conclude that the inclusion induces an isomorphism

$$H_i(\Lambda^{<a} \cup \Sigma_0^{=a}) \cong H_i(\Lambda^{\le a}).$$

Therefore β is supported on $\Lambda^{<a} \cup \Sigma_0^{=a}$ so for any $m \ge 1$, β^{*m} is supported on

$$(\Lambda^{<a} \cup \Sigma_0^{=a})^{*m} \subset \Lambda^{<ma} \cup (\Sigma_0^{=a})^{*m}.$$

The only critical points in $(\Sigma_0^{=a})^{*m}$ are the m-fold iterates of the geodesics in $\overline{\gamma}_j$ (with $1 \le j \le s$). Thus, using an arbitrarily brief flow along the trajectories of $-\nabla F$, we obtain an isomorphism

$$H_b\big(\Lambda^{<ma} \cup (\Sigma_0^{=a})^{*m}, \Lambda^{<ma}\big) \cong \bigoplus_{j=1}^{s} H_b(U_{j,m}^{<ma} \cup \overline{\gamma_j^m}, U_{j,m}^{<ma}) \ni \beta^{*m}, \tag{7.3.1}$$

where the $U_{j,m}$ are disjoint neighborhoods of the $\overline{\gamma_j^m}$ containing no other critical points, and where

$$b = mi - (m-1)n.$$

(There may be other critical points at level ma, but the support of β^{*m} does not contain such points.)

We now show that each of the summands on the right-hand side of equation (7.3.1) vanishes if m is sufficiently large. Fix j with $1 \le j \le s$, and let λ_m denote the index of γ_j^m. According to Theorem D.2, $\lambda_1 \in \{i-1, i\}$. If the jth summand in (7.3.1) is not zero, then $\lambda_m \in \{b-1, b\}$, so $\lambda_m \le mi - (m-1)n$. If $\lambda_1 = i$ and $m \ge 2$, this contradicts equation (6.1.1). If $\lambda_1 = i-1$, this contradicts equation (6.1.4) with $k = 2$, namely,

$$\lambda_m \ge m\lambda_1 - \big((m-1)(n-1) - 2\big) = mi - (m-1)n + 1,$$

which holds for m sufficiently large. □

8. Coproducts in homology

8.1

In [Su1] and [Su2], D. Sullivan constructs operations $\vee$ and $\vee_t$ on the group of transversal chains of certain path spaces. His constructions give rise to coproducts in homology with coefficients in $\mathbb{Q}$ if M is orientable and $\mathbb{Z}/(2)$ otherwise,

$$\wedge_t : H_*(\Lambda) \xrightarrow{u} H_*(\Lambda \times \Lambda) \cong H_*(\Lambda) \otimes H_*(\Lambda),$$

$$\wedge : H_*(\Lambda, \Lambda_0) \xrightarrow{v} H_*\big((\Lambda, \Lambda_0) \times (\Lambda, \Lambda_0)\big) \cong H_*(\Lambda, \Lambda_0) \otimes H_*(\Lambda, \Lambda_0)$$

of degrees $-n$ and $-n+1$, respectively, which we now describe. The Künneth isomorphisms in the above display require field coefficients. But the mappings u, v can be defined over $\mathbb{Z}$ if M is orientable. So we may continue to use the conventions of §2.5.

For fixed $t \in (0,1)$, the evaluation mapping $(\mathbf{ev}_0, \mathbf{ev}_t) : \Lambda \to M \times M$ (given by $\alpha \mapsto (\alpha(0), \alpha(t))$) is a submersion and hence transverse to the diagonal D so its preimage $\mathscr{F}_t \subset \Lambda$ has a tubular neighborhood and normal bundle in Λ (see [Cha, Proposition 2.2.3], [BO, Proposition 1.17]). By Proposition B.2, there is a Thom isomorphism $\tau : H_i(\Lambda, \Lambda - \mathscr{F}_t) \to H_{i-n}(\mathscr{F}_t)$. The space $\mathscr{F}_t$ consists of loops α with a self-intersection $\alpha(0) = \alpha(t)$ at time t. Let $c_t : \mathscr{F}_t \to \Lambda \times \Lambda$ be the map that "cuts" such a loop $\alpha \in \mathscr{F}_t$ at the time t, giving the two loops

$$c_t(\alpha)(s) = \big(\alpha(st), \alpha(t + s(1-t))\big).$$

The coproduct $\vee_t$ is obtained from the composition

$$H_i(\Lambda) \longrightarrow H_i(\Lambda, \Lambda - \mathscr{F}_t) \xrightarrow[\tau]{} H_{i-n}(\mathscr{F}_t) \xrightarrow[c_*]{} H_{i-n}(\Lambda \times \Lambda).$$

Sullivan comments (see [Su1], [Su2]) that this coproduct vanishes if the Euler characteristic of M vanishes. In §9.1, we show that this coproduct *always* vanishes (on homology) if $i > 3n$; in fact, the associated relative coproduct $H_i(\Lambda) \to H_{i-n}((\Lambda, \Lambda_0) \times (\Lambda, \Lambda_0))$ vanishes for all i.

8.2
The nontrivial homology coproduct $\vee$ arises from the following diagram:

$$\begin{tikzcd} & \Lambda \times I \arrow[r, "(\mathbf{ev}_0, \mathbf{ev})"] & M \times M \\ \Lambda \times \Lambda & \mathscr{F}_{[0,1]} \arrow[l, "c"'] \arrow[u, hook] \arrow[r] & D \arrow[u, hook] \end{tikzcd} \tag{8.2.1}$$

Here, $(\mathbf{ev}_0, \mathbf{ev})$ is the evaluation: $(\alpha, t) \mapsto (\alpha(0), \alpha(t))$ and $\mathscr{F}_{[0,1]} = (\mathbf{ev}_0, \mathbf{ev})^{-1}(D)$. The mapping $c(\alpha, t) = c_t(\alpha)$ is the "cutting map."

Set $I = [0, 1]$, and set $\partial I = \{0, 1\}$. For any $A \subset I$, let $\mathscr{F}_A \subset \Lambda \times I$ denote the part of $(\mathbf{ev}_0, \mathbf{ev})^{-1}(D)$ that lies over A. (In particular, $\mathscr{F}_0 \cup \mathscr{F}_1 = \Lambda \times \partial I$.) The restriction $(\mathbf{ev}_0, \mathbf{ev})|\Lambda \times (0, 1)$ is transverse to the diagonal D (in fact, it is a submersion), so $\mathscr{F}_{(0,1)}$ has a tubular neighborhood and normal bundle in $\Lambda \times (0, 1)$. Unfortunately, these properties do not extend to the endpoints $t = 0, 1$. Nevertheless, we have the following result, whose proof appears below.

LEMMA 8.3
The Thom isomorphism

$$\tau : H_j\big((\Lambda \times (0, 1)), \Lambda \times (0, 1) - \mathscr{F}_{(0,1)}\big) \cong H_{j-n}(\mathscr{F}_{(0,1)})$$

extends to a (relative) Gysin homomorphism

$$\tau : H_j\,(\Lambda \times I, \Lambda \times \partial I \cup \Lambda_0 \times I) \to H_{j-n}\left(\mathscr{F}_{[0,1]}, \mathscr{F}_0 \cup \mathscr{F}_1 \cup (\Lambda_0 \times I)\right).$$

8.4
Using Lemma 8.3, Sullivan's coproduct $\vee$ becomes well defined on *relative* homology. It can be obtained from the Künneth formula and the composition

$$\begin{tikzcd} H_i(\Lambda,\Lambda_0)\cong H_{i+1}\big((\Lambda,\Lambda_0)\times(I,\partial I)\big) \arrow[r, "\tau"] & H_{i+1-n}\left(\mathscr{F}_{[0,1]},\mathscr{F}_0\cup\mathscr{F}_1\cup(\Lambda_0\times I)\right) \arrow[d, "c_*"] \\ & H_{i+1-n}\big(\Lambda\times\Lambda,(\Lambda\times\Lambda_0)\cup(\Lambda_0\times\Lambda)\big) \end{tikzcd}$$

One would like to have a coproduct on the absolute homology of Λ. If $j > n$, then $H_j(\Lambda,\Lambda_0)\cong H_j(\Lambda)$ so $\vee$ becomes a coproduct on absolute homology for $i > 3n$. What if $i \le 3n$? As mentioned in [Su1], [Su2], if the Euler characteristic $\chi(M)$ of M vanishes, then the diagonal D lies in an isotopy $D_s \subset M\times M$ with $D_0 = D$ and $D_s\cap D = \phi$ for $s \ne 0$. Consequently, the mapping (8.2.1) *is* transverse to $D' = D_1$, and its preimage $\mathscr{F}'\subset\Lambda\times[0,1]$ has $\mathscr{F}'_0 = \mathscr{F}'_1 = \phi$. We therefore obtain a coproduct on absolute homology $H_i(\Lambda)$ (for all $i\ge 0$) using the composition

$$\begin{tikzcd} H_i(\Lambda)\cong H_{i+1}\big(\Lambda\times(I,\partial I)\big) \arrow[r] & H_{i+1-n}(\mathscr{F}'_{[0,1]},\phi) \arrow[d, "c'"] \\ & H_{i+1-n}(\Lambda\times\Lambda) \end{tikzcd}$$

(The cutting map c' can be defined using the isotopy and letting $s\to 0$.) As mentioned in [Su1] and [Su2], when $\chi(M) = 0$, this coproduct satisfies

$$\vee(x * y) = (x\otimes 1) * \vee(y) + \vee(x) * (1\otimes y). \tag{8.4.1}$$

Unfortunately, this coproduct depends on the choice of isotopy. It is easy to see this for $M = S^1$, in which case Λ is the disjoint union

$$\Lambda = \coprod_{m\in\mathbb{Z}} \Lambda_{(m)},$$

where $\Lambda_{(m)}$ consists of loops $\alpha : S^1\to S^1$ of degree m. For each m, the evaluation map $\Lambda_{(m)}\to S^1$ is a homotopy equivalence. Let $[m]\in H_0(\Lambda_{(m)})$ be the generator corresponding to the degree m mapping $\gamma^m : S^1\to S^1$, where $\gamma^m(t) = mt$ (mod 1). To calculate $\wedge[m]$, consider the deformation of the diagonal and the resulting deformation of $\mathscr{F}_{[0,1]}$,

$$D_s = \big\{(u,v)\in M\times M :\ v = u+s\big\},$$
$$\mathscr{F}^s_{[0,1]} = (\mathbf{ev}_0,\mathbf{ev})^{-1}(D_s) = \big\{(\alpha,t)\in\Lambda\times[0,1] :\ (\alpha(0),\alpha(t))\in D_s\big\}.$$

It follows that $\tau_s([m]) \in H_0(\mathscr{F}^s_{[0,1]}, \phi)$ is the homology class represented by the cycle

$$(\gamma^m \times [0,1]) \cap \mathscr{F}^s_{[0,1]} = \big\{(\gamma^m, t) \in \Lambda \times [0,1] : \ (\gamma^m(0), \gamma^m(t)) \in D_s\big\}.$$

This intersection consists of exactly m points, corresponding to m values of $t \in [0,1]$, one of which is $t = s/m$ and is close to zero. After cutting at these times and letting $s \to 0$, we find

$$\wedge([m]) = [0] \otimes [m] + [1] \otimes [m-1] + \cdots + [m-1] \otimes [1] \quad \text{if } s > 0,$$
$$\wedge([m]) = [1] \otimes [m-1] + \cdots + [m-1] \otimes [1] + [m] \otimes [0] \quad \text{if } s < 0.$$

We remark that the coproduct becomes well defined if we work modulo $\Lambda_0 \subset \Lambda_{(0)}$ (see also §16.2).

8.5. Proof of Lemma 8.3

For $\epsilon > 0$ sufficiently small, let $I_\epsilon = [0,\epsilon) \cup (1-\epsilon, 1]$ be a neighborhood of the endpoints of the unit interval, and set

$$U_\epsilon = (\Lambda^{<\epsilon} \times I) \cup (\Lambda \times I_\epsilon) \subset \Lambda \times I,$$

where $\Lambda^{<\epsilon}$ denotes loops with energy less than ϵ. Similarly, let $I^o = (0,1)$, and set $I^o_\epsilon = I^o \cap I_\epsilon$ and $U^o_\epsilon = (\Lambda^{<\epsilon} \times I^o) \cup \Lambda \times I^o_\epsilon$. The Gysin homomorphism of equation (B.2.4), together with excision, gives mappings

$$\begin{array}{ccc} H_j(\Lambda \times I^o, U^o_\epsilon) & \longrightarrow & H_{j-n}(\mathscr{F}_{(0,1)}, \mathscr{F}_{(0,1)} \cap U^o_\epsilon) \\ \downarrow \cong & & \downarrow \cong \\ H_j(\Lambda \times I, U_\epsilon) & & H_{j-n}(\mathscr{F}_{[0,1]}, \mathscr{F}_{[0,1]} \cap U_\epsilon) \end{array}$$

which gives a mapping

$$\lim_{\epsilon \to 0} H_j(\Lambda \times I, U_\epsilon) \ \to \ \lim_{\epsilon \to 0} H_{j-n}(\mathscr{F}_{[0,1]}, \mathscr{F}_{[0,1]} \cap U_\epsilon).$$

Using the finite-dimensional approximation (§3) and the properties of Čech homology described in Appendix A, it is easy to see that the singular and Čech homology coincide in these cases, so the limits may be identified with

$$H_j\big(\Lambda \times I, (\Lambda \times \partial I) \cup (\Lambda_0 \times I)\big) \quad \text{and} \quad H_{j-n}\big(\mathscr{F}_{[0,1]}, (\Lambda \times \partial I) \cup (\Lambda_0 \times I)\big). \quad \square$$

8.6. Sullivan's path coproduct

Let P be a smooth compact manifold, and let L_a, L_b, L_c be compact submanifolds. Let Ω_{ab} (etc.) denote the Hilbert manifold of H^1-paths $\alpha : [0,1] \to P$ that start on

L_a and end on L_b. Let $\Omega_{a\cap b}$ be the constant paths in $L_a \cap L_b$. In [Su1] and [Su2], Sullivan constructs a coproduct on transverse chains which gives rise to a coproduct in homology,

$$H_i(\Omega_{ac}, \Omega_{a\cap c}) \to H_{i+1-\mathrm{cod}(L_b)}\big((\Omega_{ab}, \Omega_{a\cap b}) \times (\Omega_{bc}, \Omega_{b\cap c})\big).$$

Its construction is parallel to that of $\vee$ in the preceding paragraphs, using the following diagram in place of diagram (8.2.1):

$$\begin{tikzcd} & \Omega_{ac} \times I \arrow[r, "\mathbf{ev}"] & P \\ \Omega_{ab} \times \Omega_{bc} & \mathscr{K} \arrow[l, "c"'] \arrow[r] \arrow[u, hook] & L_b \arrow[u, hook] \end{tikzcd}$$

Here, $\mathbf{ev}$ is the evaluation map, so that $\mathscr{K} = \mathbf{ev}^{-1}(L_b)$ consists of pairs (α, t) with $\alpha(t) \in L_b$, and c is the "cutting" map. As discussed in [Su1] and [Su2], taking $P = M \times M$ and $L_a = L_b = L_c = D$ gives the above case of the free loop space.

9. Cohomology products

9.1

We continue with the conventions of §2.5. To simplify the notation, we suppress further mention of the coefficient ring G. Sullivan's homology coproducts $\vee_t$ and $\vee$ of §8 can be converted into products on cohomology by reversing all the arrows. The product corresponding to $\vee_t$ $(t = 1/2)$ is the composition

$$\begin{aligned} H^i(\Lambda) \times H^j(\Lambda) &\to H^{i+j}(\Lambda \times \Lambda) \\ &\to H^{i+j}(\mathscr{F}) \xrightarrow{\cong} H^{i+j+n}\big(\Lambda, \Lambda - \phi_{1/2}(\mathscr{F})\big) \to H^{i+j+n}(\Lambda) \end{aligned} \tag{9.1.1}$$

(with $\phi_{1/2} : \mathscr{F} \to \Lambda$ as in §2.3) using the Thom isomorphism for the normal bundle of the figure-eight space $\phi_{1/2}(\mathscr{F}) \subset \Lambda$. We show below that this product is zero when $i, j > n$. In the remainder of this article, we study the cohomology product

$$H^i(\Lambda, \Lambda_0) \times H^j(\Lambda, \Lambda_0) \xrightarrow{\circledast} H^{i+j+n-1}(\Lambda, \Lambda_0) \tag{9.1.2}$$

that arises from Sullivan's operation $\vee$, that is, with respect to the Kronecker pairing,

$$\langle x \circledast y, a\rangle = \langle x \otimes y, \vee a\rangle \tag{9.1.3}$$

for all $x, y \in H^*(\Lambda, \Lambda_0)$ and $a \in H_*(\Lambda, \Lambda_0)$. In order to avoid the technical difficulties involved with Lemma 8.3 (which arise because the evaluation mapping in (8.2.1)

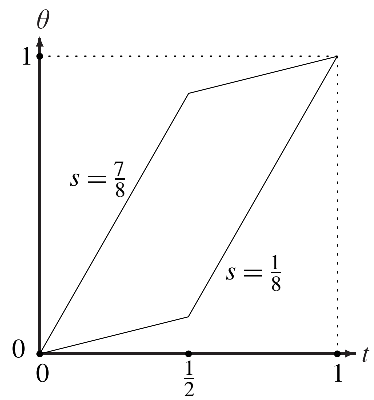

Figure 2. Graph of $\theta_{1/2\to s}$

fails to be transverse at $t = 0, 1$), we modify the construction slightly by using the following mapping:

$$J : \Lambda \times [0, 1] \to \Lambda \quad \text{given by } J(\alpha, s) = \alpha \circ \theta_{1/2\to s}.$$

Here, $\theta = \theta_{1/2\to s} : [0, 1] \to [0, 1]$ is the reparametrization function that is linear on $[0, 1/2]$, linear on $[1/2, 1]$, has $\theta(0) = 0$, $\theta(1) = 1$, and $\theta(1/2) = s$ (see Figure 2 and §9.8).

Let $\mathscr{F}^{>0,>0} = (\Lambda - \Lambda_0) \times_M (\Lambda - \Lambda_0)$ denote the set of composable pairs (α, β) such that $F(\alpha) > 0$ and $F(\beta) > 0$. The relative Thom isomorphism (B.2.1) for cohomology gives

$$H^m(\mathscr{F}, \mathscr{F} - \mathscr{F}^{\geq\epsilon,\geq\epsilon}) \cong H^{m+n}\big(\Lambda, \Lambda - \phi_{1/2}(\mathscr{F}^{\geq\epsilon,\geq\epsilon})\big)$$

and, by taking a limit as $\epsilon \to 0$, an isomorphism

$$H^m(\mathscr{F}, \mathscr{F} - \mathscr{F}^{>0,>0}) \cong H^{m+n}\big(\Lambda, \Lambda - \phi_{1/2}(\mathscr{F}^{>0,>0})\big). \tag{9.1.4}$$

The cohomology product is then the composition down the left-hand column of Figure 3. Here, $\omega = (-1)^{j(n-1)}$, $I = [0, 1]$, $\mathscr{F}^{\bullet,0} = \Lambda \times_M \Lambda_0$, and $\mathscr{F}^{0,\bullet} = \Lambda_0 \times_M \Lambda$. The mapping τ is the Thom isomorphism given by the cup product with the Thom class $\mu_{\mathscr{F}}$ of the normal bundle of $\phi_{1/2}\mathscr{F}$ in Λ, and κ is given by the Künneth theorem. It uses the fact that $J(\Lambda_0 \times [0, 1])$ and $J(\Lambda \times \{0, 1\})$ are disjoint from $\phi_{1/2}(\mathscr{F}^{>0,>0})$.

Denote the cohomology product of two classes $\alpha, \beta \in H^*(\Lambda, \Lambda_0)$ by $\alpha \circledast \beta$. The construction may be summarized as passing from the left to the right in the following

$$
\begin{tikzcd}
H^i(\Lambda, \Lambda_0) \times H^j(\Lambda, \Lambda_0) \arrow[r] \arrow[d, "\omega\times"] & H^i(\Lambda) \times H^j(\Lambda) \arrow[d, "\omega\times"] \\
H^{i+j}(\Lambda \times \Lambda, \Lambda \times \Lambda_0 \cup \Lambda_0 \times \Lambda) \arrow[r] \arrow[d] & H^{i+j}(\Lambda \times \Lambda) \arrow[d] \\
H^{i+j}(\mathscr{F}, \mathscr{F}^{\bullet,0} \cup \mathscr{F}^{\bullet,0}) \arrow[r] \arrow[d, equal] & H^{i+j}(\mathscr{F}) \arrow[d, equal] \\
H^{i+j}(\mathscr{F}, \mathscr{F} - \mathscr{F}^{>0,>0}) \arrow[r] \arrow[d, "\tau"', "(B.2.1)"] & H^{i+j}(\mathscr{F}) \arrow[d, "\tau"', "(B.2.3)"] \\
H^{i+j+n}\big(\Lambda \times \Lambda - \phi_{1/2}(\mathscr{F}^{>0,>0})\big) \arrow[r] \arrow[d, "J^*"'] & H^{i+j+n}\big(\Lambda, \Lambda - \phi_{1/2}(\mathscr{F})\big) \arrow[d, "J^*"'] \\
H^{i+j+n}(\Lambda \times I, \Lambda \times \partial I \cup \Lambda_0 \times I) \arrow[r, "\eta"] \arrow[d, "\kappa"'] & H^{i+j+n}(\Lambda \times I) = H^{i+j+n}(\Lambda) \\
H^{i+j+n-1}(\Lambda, \Lambda_0) &
\end{tikzcd}
$$

Figure 3. Definition of $\circledast$

diagram:

$$\boxed{\Lambda \times \Lambda \longleftarrow \mathscr{F} \longrightarrow \Lambda \xleftarrow{J} \Lambda \times I}$$

In Figure 3, the first horizontal mapping is an isomorphism if $i, j > n$ since $\Lambda_0 \cong M$ has dimension n. The mapping η is part of the long exact sequence for the pair $(\Lambda, \Lambda_0) \times (I, \partial I)$, so it is zero. It follows that the cohomology product (9.1.1), which is the composition down the right-hand column, vanishes if $i, j > n$.

PROPOSITION 9.2
The cohomology product $\circledast$ is associative and commutative up to a sign: if $x \in H^i(\Lambda, \Lambda_0)$ and $y \in H^j(\Lambda, \Lambda_0)$, then

$$y \circledast x = (-1)^{(i+n-1)(j+n-1)} x \circledast y. \tag{9.2.1}$$

Proof
The proof of associativity can be found in Appendix F. In this paragraph, we prove (9.2.1). As in §5.2, let $\sigma : \Lambda \times \Lambda \to \Lambda \times \Lambda$ switch the two factors, and let $\widehat{\chi}_r : \Lambda \to \Lambda$

be the action of $r \in S^1$ given by $\widehat{\chi}_r(\gamma) = \gamma \circ \chi_r$, where $\chi_r(t) = r + t \pmod 1$. Then

$$\chi_{1-s} \circ \theta_{1/2\to s} = \theta_{1/2\to(1-s)} \circ \chi_{1/2} \tag{9.2.2}$$

(which may be seen from a direct calculation). For $0 \le r \le 1$, define $J_r : \Lambda \times I \to \Lambda$ by

$$J_r(\gamma, s) = \gamma \circ \chi_{r(1-s)} \circ \theta_{1/2\to s}.$$

Then $J_r\left((\Lambda \times \partial I) \cup (\Lambda_0 \times I)\right) \subset \Lambda - \phi_{1/2}(\mathscr{F}^{>0,>0})$ because when $s = 0$ or $s = 1$ the loop $J_r(\gamma, s)$ stays fixed on either $[0, 1/2]$ or $[1/2, 1]$. Therefore the mapping J^* in Figure 3 may be replaced by J_r^* for any $r \in [0, 1]$. However, $J_0 = J$ and

$$\begin{aligned} J_1(\gamma, s) &= \gamma \circ \chi_{1-s} \circ \theta_{1/2\to s} = \gamma \circ \theta_{1/2\to(1-s)} \circ \chi_{1/2} \\ &= \widehat{\chi}_{1/2}\big(J(\gamma, 1-s)\big). \end{aligned}$$

So J_1 reverses the $s \in I$ coordinate, and it switches the front and back half of any figure-eight loop $\gamma \in \mathscr{F}$ (cf. equation (5.2.1)). Let $i : \mathscr{F} \to \Lambda \times \Lambda$ denote the inclusion; then

$$\begin{aligned} (-1)^{i(n-1)} y \circledast x &= \kappa J_0^*\big(\mu_{\mathscr{F}} \cup i^*(y \times x)\big) \\ &= (-1)^{ij} \kappa J_0^*\big(\mu_{\mathscr{F}} \cup i^*\sigma^*(x \times y)\big) \\ &= (-1)^{ij}(-1)^n \kappa J_0^*\sigma^*\big(\mu_{\mathscr{F}} \cup i^*(x \times y)\big) \\ &= (-1)^{ij}(-1)^n \kappa J_1^*\sigma^*\big(\mu_{\mathscr{F}} \cup i^*(x \times y)\big) \\ &= (-1)^{ij}(-1)^n(-1) \kappa J_0^*\big(\mu_{\mathscr{F}} \cup i^*(x \times y)\big) \\ &= (-1)^{ij+n+1}(-1)^{j(n-1)} x \circledast y. \end{aligned}$$

□

We remark that just as in the case of the relative cup product, the ring $(H^*(\Lambda, \Lambda_0), \circledast)$ does not have a unit element **1** (see also §16.3).

9.3. Based loop space and Pontrjagin product

Fix a base point $x_0 \in M$, and let $\Omega = \Omega_{x_0} = \mathbf{ev}_0^{-1}(x_0)$ be the (based) loop space. It is a Hilbert submanifold of codimension $n = \dim(M)$ in Λ. The embedding $i : \Omega \to \Lambda$ has a trivial normal bundle and it induces Gysin homomorphisms $i^! : H_i(\Lambda) \to H_{i-n}(\Omega)$ and $i_! : H^i(\Omega, x_0) \to H^{i+n}(\Lambda, \Lambda_0)$ (cf. equations (B.2.4) and (C.1.1)). The Pontrjagin product

$$\Omega \times \Omega \xrightarrow{\bullet} \Omega$$

(speed up by a factor of 2 and concatenate at time $1/2$) is an embedding. Denote its image (which is the based loops analog of the figure-eight space) by $\mathscr{F}_\Omega = \Omega \bullet \Omega$. It is a Hilbert submanifold of Ω, with trivial n-dimensional normal bundle. We obtain a fiber (or "Cartesian") square,

$$\begin{tikzcd} \Omega \times \Omega \arrow[r, "i \times i"] \arrow[d, "\bullet"] & \Lambda \times_M \Lambda \arrow[d, "\phi_{1/2}"] \\ \Omega \arrow[r, "i"] & \Lambda \end{tikzcd}$$

from which it follows that the Pontrjagin product (which is not necessarily commutative) and the Chas-Sullivan product are related (see [CS, Proposition 3.4]) by

$$\begin{aligned} i^!(a * b) &= i^!(a) \bullet i^!(b), \\ \big(i_*(c)\big) * a &= i_*\big(c \bullet i^!(a)\big), \\ i_* i^!(a) &= \overline{x_0} * a, \end{aligned}$$

for all $a, b \in H_*(\Lambda)$ and $c \in H_*(\Omega)$, where $\overline{x_0}$ denotes the constant loop at the basepoint $x_0 \in M$.

As indicated in [Su1] and [Su2], Sullivan's $\vee$ operation may be applied to the chains on the based loop space. This gives rise to a cohomology product.

PROPOSITION 9.4
Replacing (Λ, Λ_0) by (Ω, x_0) in Figure 3, gives a product

$$H^i(\Omega, x_0) \times H^j(\Omega, x_0) \xrightarrow{\circledast} H^{i+j+n-1}(\Omega, x_0)$$

such that

$$i^*(a \circledast b) = i^*(a) \circledast i^*(b), \tag{9.4.1}$$

$$i_!(x) \circledast a = i_!\big(x \circledast i^*(a)\big), \tag{9.4.2}$$

for any $a, b \in H^*(\Lambda, \Lambda_0)$ and $x \in H_*(\Omega, x_0)$. This product is often nontrivial. Suppose that $X, Y \subset \Omega$ are smooth compact oriented submanifolds of dimension i, j, respectively, and suppose that $Z \subset \Omega$ is an oriented compact submanifold of dimension $i + j + n - 1$ such that the mapping $J : Z \times I \to \Omega$ is transverse to $\mathscr{F}_\Omega = \Omega \bullet \Omega$ and such that

$$J^{-1}(\mathscr{F}_\Omega) = (X \bullet Y) \times \left\{\frac{1}{2}\right\}.$$

Then

$$\langle a \circledast b, Z\rangle = \langle a, [X]\rangle \cdot \langle b, [Y]\rangle \tag{9.4.3}$$

for any $a \in H^i(\Omega, x_0)$ *and* $b \in H^j(\Omega, x_0)$.

The nontriviality of the product is taken up in §14.9. The rest of the proposition is proven in Appendix C.

PROPOSITION 9.5
Let $0 \leq a' < a \leq \infty$*, and let* $0 \leq b' < b < \infty$*. Then the cohomology product induces a family of compatible products*

$$H^i(\Lambda^{<a}, \Lambda^{<a'}) \times H^j(\Lambda^{<b}, \Lambda^{<b'}) \xrightarrow{\circledast} H^{i+j+n-1}(\Lambda^{<c}, \Lambda^{<c'}), \tag{9.5.1}$$

$$\check{H}^i(\Lambda^{\leq a}, \Lambda^{\leq a'}) \times \check{H}^j(\Lambda^{\leq b}, \Lambda^{\leq b'}) \xrightarrow{\circledast} \check{H}^{i+j+n-1}(\Lambda^{\leq c}, \Lambda^{\leq c'}), \tag{9.5.2}$$

$$\check{H}^i(\Lambda^{\leq a}, \Lambda^{<a}) \times \check{H}^j(\Lambda^{\leq b}, \Lambda^{<b}) \xrightarrow{\circledast} \check{H}^{i+j+n-1}(\Lambda^{\leq a+b}, \Lambda^{<a+b}), \tag{9.5.3}$$

where $c = \min(a + b', a' + b)$ *and* $c' = a' + b'$*. It is compatible with the homomorphisms induced by inclusions* $\Lambda^{<e} \to \Lambda^{<f}$ *whenever* $e \leq f$.

Proof
The construction of the product (9.5.1) is taken up in the next few sections. The existence of (9.5.2) and (9.5.3) follows from (9.5.1) and §A.3.

9.6
We do not see how to prove Proposition 9.5 using the construction of §9.1 because the mappings $\phi_{1/2}$ and J (which occur in the definition of $\circledast$) are poorly behaved with respect to the energy. We worked around this in Proposition 5.3 by using $\phi_{\min}$ instead of $\phi_{1/2}$. But $\phi_{\min}$ is not an embedding. This does not present a problem for the homology product, but for the cohomology product, such a substitution interferes with the definition of the (Gysin) mapping τ in Figure 3. Our solution is to replace the free loop space Λ by the space $\mathscr{A}$ of loops parametrized proportionally with respect to arc length. On this (homotopy equivalent) space, the effect of $\phi_{1/2}$ on the energy is easy to determine (see (9.8.3); but at the cost of having to deal with a more complicated Thom isomorphism, see (9.1.4) vs. (9.7.2)).

Recall (see §2.1) if $\alpha \in \mathscr{A}$ is a loop parametrized proportionally to arclength, then $F(\alpha) = \sqrt{E(\alpha)} = L(\alpha)$ is its length. Let $\mathscr{F}_{\mathscr{A}} = \mathscr{A} \times_M \mathscr{A}$ be the associated figure-eight space consisting of pairs of composable loops, each parametrized proportionally to arclength. Let $\mathscr{A}_{1/2}$ be the set of loops $\alpha \in \Lambda$ such that $\alpha|[0, 1/2]$ is PPAL and $\alpha|[1/2, 1]$ is PPAL. Similarly, let

$$\mathscr{F}_{\mathscr{A}}^{<u,<v} = \big\{(\alpha, \beta) \in \mathscr{A} \times_M \mathscr{A} : \ L(\alpha) < u \text{ and } \ L(\beta) < v\big\},$$
$$\mathscr{A}_{1/2}^{<u,<v} = \big\{\alpha \in \mathscr{A}_{1/2} : \ L(\alpha|[0, 1/2]) < u \text{ and } \ L(\alpha|[1/2, 1]) < v\big\},$$

and similarly for $\mathscr{A}_{1/2}^{=u,=v}$, and so on. Then the mapping $\phi_{1/2}$ restricts to a closed embedding

$$\phi_{1/2} : \mathscr{F}_{\mathscr{A}}^{<u,<v} \to \mathscr{A}_{1/2}^{<u,<v}. \tag{9.6.1}$$

LEMMA 9.7

The orientations chosen in §2.5 determine a Thom isomorphism

$$H^i(\mathscr{F}_{\mathscr{A}}^{<u,<v}) \cong H^{i+n}\big(\mathscr{A}_{1/2}^{<u,<v}, \mathscr{A}_{1/2}^{<u,<v} - \phi_{1/2}(\mathscr{F}_{\mathscr{A}}^{<u,<v})\big) \tag{9.7.1}$$

for the image $\phi_{1/2}(\mathscr{F}_{\mathscr{A}}^{<u,<v})$. If $Z \subset \mathscr{F}_{\mathscr{A}}^{<u,<v}$ is a closed subset, then this restricts to a relative Thom isomorphism,

$$H^i(\mathscr{F}_{\mathscr{A}}^{<u,<v}, \mathscr{F}_{\mathscr{A}}^{<u,<v} - Z) \cong H^{i+n}\big(\mathscr{A}_{1/2}^{<u,<v}, \mathscr{A}_{1/2}^{<u,<v} - \phi_{1/2}(Z)\big). \tag{9.7.2}$$

Proof

The space $\phi_{1/2}(\mathscr{F}_{\mathscr{A}}^{<u,<v})$ may be described as the preimage of the diagonal under the mapping

$$(\mathbf{ev}_0, \mathbf{ev}_{1/2}) : \mathscr{A}_{1/2}^{<u,<v} \to M \times M.$$

If $\mathscr{A}_{1/2}$ were a Hilbert manifold, this would imply the existence of a normal bundle and tubular neighborhood for $\phi_{1/2}(\mathscr{F}_{\mathscr{A}}^{<u,<v})$ in $\mathscr{A}_{1/2}^{<u,<v}$. Unfortunately, $\mathscr{A}_{1/2}$ is probably not a Hilbert manifold, and even though it may be a Banach manifold, we do not know of any standard reference for the existence of tubular neighborhoods which can be applied in this setting. However, we have the following commutative diagram, where the vertical maps are homotopy equivalences (cf. Proposition 2.2):

$$\begin{tikzcd}
\phi_{1/2}(\mathscr{F}^{<u,<v}) \arrow[r] \arrow[dd, shift right, "Q"'] & \Lambda_{1/2}^{<u,<v} \arrow[dr] \arrow[dd, shift right, "Q"'] & \\
 & & M \times M \\
\phi_{1/2}(\mathscr{F}_{\mathscr{A}}^{<u,<v}) \arrow[r] \arrow[uu, shift right] & \mathscr{A}_{1/2}^{<u,<v} \arrow[ur] \arrow[uu, shift right] &
\end{tikzcd}$$

Here, $\Lambda_{1/2}^{<u,<v}$ denotes the set of all $\alpha \in \Lambda$ such that $E(\alpha_1) < u^2$ and $E(\alpha_2) < v^2$, where $\alpha_1(t) = \alpha(2t)$ $(0 \le t \le 1/2)$ and $\alpha_2(t) = \alpha(2t-1)$ $(1/2 \le t \le 1)$. (In other words, if $\alpha|[0,1/2]$ and $\alpha|[1/2,1]$ are both expanded into paths defined on $[0,1]$, then their respective energies are bounded by u^2 and v^2.) Since Λ is a Hilbert manifold, the same holds for $\Lambda_{1/2}^{<u,<v}$; hence $\phi_{1/2}(\mathscr{F}^{<u,<v})$ has a tubular neighborhood and normal bundle in $\Lambda^{<u,<v}$, and we have a Thom isomorphism

$$H^i(\mathscr{F}^{<u,<v}) \cong H^{i+n}\big(\Lambda^{<u,<v}, \Lambda^{<u,<v} - \phi_{1/2}(\mathscr{F}^{<u,<v})\big).$$

The vertical homotopy equivalence in this diagram assigns to any $\alpha \in \Lambda_{1/2}^{<u,<v}$ the same curve but with $\alpha|[0,1/2]$ reparametrized proportionally to arclength and with $\alpha|[1/2,1]$ similarly reparametrized. It restricts to a homotopy equivalences $\phi_{1/2}(\mathscr{F}^{<u,<v}) \to \phi_{1/2}(\mathscr{F}_{\mathscr{A}}^{<u,<v})$ and also

$$\Lambda_{1/2}^{<u,<v} - \phi_{1/2}(\mathscr{F}^{<u,<v}) \to \mathscr{A}_{1/2}^{<u,<v} - \phi_{1/2}(\mathscr{F}_{\mathscr{A}}^{<u,<v})$$

because the points $\alpha(0), \alpha(1/2)$ are fixed. The Thom isomorphism (9.7.1) follows. The relative Thom isomorphism (9.7.2) follows as in Proposition B.2. $\square$

9.8

Let $I = [0,1]$ denote the unit interval. The mapping $J : \Lambda \times I \to \Lambda$ restricts to a mapping $J_{\mathscr{A}} : \mathscr{A} \times I \to \mathscr{A}_{1/2}$ which factors through the quotient $\mathscr{A} \times I/R$ under the equivalence relation R which, for each $m \in \mathscr{A}_0$, collapses the interval $\{m\} \times I$ to the point $\{m\}$. The resulting mapping

$$\mathscr{A} \times I/R \to \mathscr{A}_{1/2}$$

is a homeomorphism, and in fact, the inverse mapping can be described as follows. Let $\alpha \in \mathscr{A}_{1/2} - \mathscr{A}_0$. Let L_0 denote the length of the segment $\alpha|[0,1/2]$ (which is PPAL), and let L_1 denote the length of the segment $\alpha|[1/2,1]$. Set $s = L_0/(L_0+L_1)$. Assume for the moment that $0 < s < 1$. Let $\theta_{s\to 1/2} = (\theta_{1/2\to s})^{-1}$ be the inverse function to $\theta_{1/2\to s}$; it is linear on $[0,s]$, linear on $[s,1]$, and takes the values $\theta_{s\to 1/2}(0) = 0$, $\theta_{s\to 1/2}(s) = 1/2$, $\theta_{s\to 1/2}(1) = 1$. Then $\alpha \circ \theta_{s\to 1/2}$ is PPAL throughout the interval $[0,1]$ so we may set

$$J_{\mathscr{A}}^{-1}(\alpha) = (\alpha \circ \theta_{s\to 1/2}, s).$$

If $s = 0$ (resp., $s = 1$), then this formula still makes sense because in this case, the loop α is constant on $[0,1/2]$ (resp., on $[1/2,1]$). If $\alpha_0, \alpha_1 \in \mathscr{A}$ are composable loops, not both constant, then the composed loop $\phi_{1/2}(\alpha_0, \alpha_1) \in \mathscr{A}_{1/2}$ and

$$J_{\mathscr{A}}^{-1}\big(\phi_{1/2}(\alpha_0, \alpha_1)\big) = \big(\phi_s(\alpha_0, \alpha_1), s\big) = \big(\phi_{\min}(\alpha_0, \alpha_1), s\big), \tag{9.8.1}$$

where $s = L(\alpha_0)/(L(\alpha_0)+L(\alpha_1))$ is the unique energy-minimizing value (cf. Lemma 2.4).

It follows that *if $A, B \subset \mathscr{A}$, then the mapping $J_{\mathscr{A}} : \mathscr{A} \times [0, 1] \to \mathscr{A}_{1/2}$ takes*

$$(\mathscr{A} - A * B) \times [0, 1] \quad \text{into } \mathscr{A}_{1/2} - \phi_{1/2}(A \times_M B). \tag{9.8.2}$$

For $J(\mathscr{A}_0 \times [0, 1]) = \mathscr{A}_0$ which is contained in the right-hand side, so it suffices to check that $J(A * B \times [0, 1]) \supset \phi_{1/2}(A \times_M B)$, which follows from (9.8.1).

Similarly, the mapping $J_{\mathscr{A}}$ satisfies

$$J_{\mathscr{A}}(\alpha, s) \in \mathscr{A}_{1/2}^{=sL(\alpha),=(1-s)L(\alpha)} \tag{9.8.3}$$

which means the following: if we express $J_{\mathscr{A}}(\alpha, s) = \phi_{1/2}(\beta_1, \beta_2)$ as a composition of two (not necessarily closed) paths β_1, β_2, each PPAL, and joined at time $1/2$, then $L(\beta_1) = sL(\alpha)$ and $L(\beta_2) = (1 - s)L(\alpha)$.

9.9

Define the following subsets of $\mathscr{A} \times I$:

$$\begin{aligned} T^{<a,<b} &= J_{\mathscr{A}}^{-1}(\mathscr{A}_{1/2}^{<a,<b}) \\ &= \big\{(\alpha, s) \in \mathscr{A} \times I : sL(\alpha) < a \text{ and } (1 - s)L(\alpha) < b\big\}, \\ T^{[a',a),[b',b)} &= J_{\mathscr{A}}^{-1}(\phi_{1/2}\mathscr{F}_{\mathscr{A}}^{[a',a),[b',b)}) \\ &= \big\{(\alpha, s) \in \mathscr{A} \times I : a' \le sL(\alpha) < a,\ b' \le (1 - s)L(\alpha) < b, \\ &\qquad \alpha(0) = \alpha(s)\big\}. \end{aligned}$$

Figure 4 consists of three diagrams of $L = \sqrt{E}$ versus $s \in [0, 1]$ illustrating the curves $sL = a$ and $(1 - s)L = b$ which occur in the definition of $T^{<a,<b}$. These curves intersect at the point with coordinates $s = a/(a + b)$ and $L = a + b$. The diagrams on the right illustrate the corresponding regions for $T^{[a'.a),[b',b)}$, where $I' = [a'/(a' + b), a/(a + b')]$.

Then $\mathscr{A}^{<c} \times I' \subset T^{<a,<b}$ and

$$(\mathscr{A}^{<c} \times \partial I') \cup (\mathscr{A}^{<c'} \times I') \subset T^{<a,<b} - T^{[a',a),[b',b)},$$

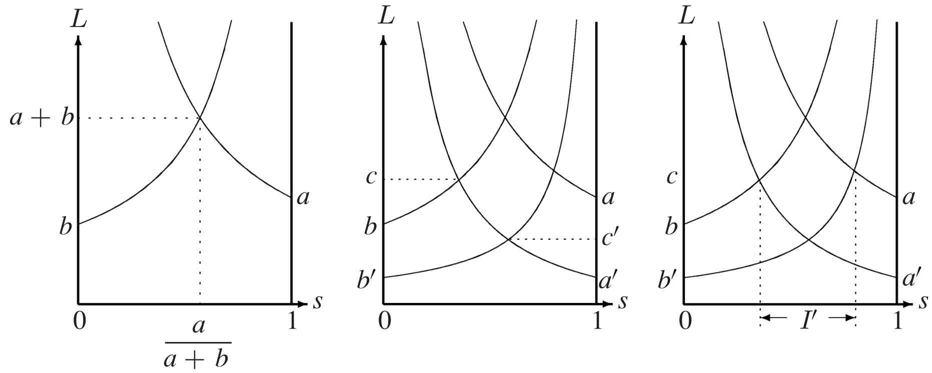

Figure 4. The regions $T^{<a,<b}$ and $T^{[a',a),[b',b)}$

where $c = \min(a + b', a' + b)$ and $c' = a' + b'$. In other words, we have a diagram of pairs,

$$\begin{array}{c} (\mathscr{A}^{<c}, \mathscr{A}^{<c'}) \times (I', \partial I') \\ \Big\downarrow \\ (T^{<a,<b}, T^{<a,<b} - T^{[a',a),[b',b)}) \\ \Big\downarrow {\scriptstyle J_{\mathscr{A}}} \\ \big(\mathscr{A}_{1/2}^{<a,<b}, \mathscr{A}_{1/2}^{<a,<b} - \phi_{1/2}(\mathscr{F}_{\mathscr{A}}^{[a',a),[b',b)})\big) \end{array} \tag{9.9.1}$$

9.10

The product (9.5.1) is constructed in several steps. First, use the cross product,

$$H^i(\mathscr{A}^{<a}, \mathscr{A}^{<a'}) \times H^j(\mathscr{A}^{<b}, \mathscr{A}^{<b'}) \to H^{i+j}(\mathscr{A}^{<a} \times \mathscr{A}^{<b}, \mathscr{A}^{<a} \times \mathscr{A}^{<b'} \cup \mathscr{A}^{<a'} \times \mathscr{A}^{<b});$$

then restrict to

$$H^{i+j}(\mathscr{F}_{\mathscr{A}}^{<a,<b}, \mathscr{F}_{\mathscr{A}}^{<a,<b'} \cup \mathscr{F}_{\mathscr{A}}^{<a',<b}) = H^{i+j}(\mathscr{F}_{A}^{<a,<b}, \mathscr{F}_{\mathscr{A}}^{<a,<b} - \mathscr{F}_{\mathscr{A}}^{[a',a),[b',b)}).$$

Using the Thom isomorphism (9.7.2), we arrive at

$$H^{i+j+n}\big(\mathscr{A}_{1/2}^{<a,<b}, \mathscr{A}_{1/2}^{<a,<b} - \phi_{1/2}(\mathscr{F}_{\mathscr{A}}^{[a',a),[b',b)})\big).$$

Pulling back under (9.9.1) gives a class in

$$H^{i+j+n}\left((\mathscr{A}^{<c}, \mathscr{A}^{<c'}) \times (I', \partial I')\right) \cong H^{i+j+n-1}(\mathscr{A}^{<c}, \mathscr{A}^{<c'}),$$

as claimed.

Taking $a, b = \infty$ and $a', b' = 0$ gives an equivalent construction of the cohomology product $\circledast$ using the space $\mathscr{A}$ rather than Λ, from which it is easy to see that the products of §9.5 are compatible with the product (9.1.2). This completes the proof of Proposition 9.5. □

10. Support and critical levels

10.1

We continue with the conventions of §2.5. To simplify the notation, we suppress further mention of the coefficient ring G. Proposition 9.5 gives the following.

COROLLARY

If $\alpha, \beta \in H^*(\Lambda, \Lambda_0)$, then

$$\mathsf{cr}(\alpha \circledast \beta) \geq \mathsf{cr}(\alpha) + \mathsf{cr}(\beta). \tag{10.1.1}$$

As in §§2.1 and 9.6, let $\mathscr{A}$ be the set of loops parametrized proportionally to arclength, let $\mathscr{A}_0 = \Lambda_0$ be the constant loops, and let $\mathscr{A}_{1/2} \subset \Lambda$ be the collection of those loops that are PPAL on $[0, 1/2]$ and are PPAL on $[1/2, 1]$. We have continuous mappings

$$J_{\mathscr{A}} : \mathscr{A} \times [0,1] \to \mathscr{A}_{1/2}, \qquad \phi_{1/2} : \mathscr{A} \times_M \mathscr{A} \to \mathscr{A}_{1/2},$$

and

$$\phi_{\min} : \mathscr{A} \times_M \mathscr{A} \to \mathscr{A} \subset \mathscr{A}_{1/2}.$$

PROPOSITION 10.2

Suppose that $\alpha \in H^i(\mathscr{A}, \mathscr{A}_0)$ is supported on a closed set $A \subset \mathscr{A} - \mathscr{A}_0$. Suppose that $\beta \in H^j(\mathscr{A}, \mathscr{A}_0)$ is supported on a closed set $B \subset \mathscr{A} - \mathscr{A}_0$. Then $\alpha \circledast \beta$ is supported on the closed set $A * B = \phi_{\min}(A \times_M B) \subset \mathscr{A} - \mathscr{A}_0$. The same holds for Čech cohomology.

Proof

Using (9.8.2), we obtain a cohomology product

$$H^i(\mathscr{A}, \mathscr{A} - A) \times H^j(\mathscr{A}, \mathscr{A} - B) \to H^{i+j+n-1}(\mathscr{A}, \mathscr{A} - A * B)$$

as the composition

$$\begin{tikzcd}
H^{i+j}(\mathscr{A} \times \mathscr{A}, \mathscr{A} \times \mathscr{A} - A \times B) \arrow[r] & H^{i+j}(\mathscr{F}_{\mathscr{A}}, \mathscr{F}_{\mathscr{A}} - A \times_M B) \arrow[d, "(9.7.2)"', "\phi_{1/2}"] \\
H^{i+j+n}\big((\mathscr{A}, \mathscr{A} - A * B) \times (I, \partial I)\big) & H^{i+j+n}\big(\mathscr{A}_{1/2}, \mathscr{A}_{1/2} - \phi_{1/2}(A \times_M B)\big) \arrow[l, "J_{\mathscr{A}}^*"']
\end{tikzcd}$$

where $I = [0, 1]$. □

10.3
We remark that the analogous fact for the cup product in cohomology is that $\alpha \smile \beta$ is supported on the intersection $A \cap B$. In particular, for any $0 \leq a' < a \leq \infty$ and $0 \leq b' < b \leq \infty$, the cup product gives mappings (with coefficients in $\mathbb{Z}$),

$$H^i(\Lambda^{\leq a}, \Lambda^{\leq a'}) \times H^j(\Lambda^{\leq b}, \Lambda^{\leq b'}) \to H^{i+j}(\Lambda^{\leq \min(a,b)}, \Lambda^{\leq \max(a',b')}).$$

The inequality $\mathsf{cr}(\alpha \smile \beta) \geq \mathsf{cr}(\alpha) + \mathsf{cr}(\beta)$ is false for $M = S^3$ with the standard metric (see §16.1).

10.4. On a question of Eliashberg
In a lecture at Princeton University in June 2007, Y. Eliashberg asked the following question. Let M be a smooth compact Riemannian manifold, and let Λ be its free loop space. Given $0 < t \in \mathbb{R}$, let $d(t)$ be the maximal degree of an essential homology class at level t, that is,

$$d(t) = \max \left\{k : \text{ Image} \left(H_k(\Lambda^{\leq t}, \mathbb{Q}) \to H_k(\Lambda; \mathbb{Q})\right) \neq 0\right\}. \tag{10.4.1}$$

Does there exist a constant $C \in \mathbb{R}$, independent of the metric, so that for all $t_1, t_2 \in \mathbb{R}^+$ the following holds:

$$d(t_1 + t_2) \leq d(t_1) + d(t_2) + C?$$

Inequalities in the opposite direction are known. If the cohomology ring $(H^*(\Lambda, \Lambda_0; \mathbb{Q}), \circledast)$ is finitely generated, then we are able to give an affirmative answer to this question. Moreover, in §14 we show the following: if M admits a metric in which all geodesics are closed, then the cohomology ring $(H^*(\Lambda), \circledast)$ (with $\mathbb{Q}$-coefficients if M is orientable; with $\mathbb{Z}/(2)$-coefficients otherwise) is finitely generated. This includes the case of spheres and projective spaces.

For the remainder of this section only, we modify the conventions of §2.5 and allow the coefficient ring G to be an arbitrary field.

THEOREM 10.5
Let M be a smooth Riemannian n-dimensional manifold. For $t \in \mathbb{R}^+$, define $d(t) = d(t; G)$ by (10.4.1) but replacing the coefficients $\mathbb{Q}$ with the field G. Assume that the cohomology ring $(H^*(\Lambda, \Lambda_0; G), \circledast)$ is finitely generated with all generators having degree at most g. If $t_1, t_2 \in \mathbb{R}^+$, then

$$d(t_1 + t_2) \leq d(t_1) + d(t_2) + 2n + g - 2. \tag{10.5.1}$$

Proof
Let $t_1, t_2, t_3 \in \mathbb{R}^+$, and let $d_i = d(t_i; G)$ for $i = 1, 2, 3$. We show that if

$$d_3 > d_1 + d_2 + 2n + g - 2, \tag{10.5.2}$$

then $t_3 > t_1 + t_2$. From the definition (10.4.1) and since $d_3 > n = \dim(\Lambda_0)$, there exist nonzero homology classes z' and z, with

$$\begin{array}{ccc} H_{d_3}(\Lambda^{\le t_3}, \Lambda_0) & \xrightarrow[i_*]{} & H_{d_3}(\Lambda, \Lambda_0), \\ z' & \longrightarrow & z = i_*(z'). \end{array}$$

Let $Z \in H^{d_3}(\Lambda, \Lambda_0)$ be a cohomology class with nonzero Kronecker product,

$$\langle Z, z \rangle \neq 0. \tag{10.5.3}$$

The class Z is a sum of products of generators of the ring $(H^*(\Lambda, \Lambda_0), \circledast)$, and at least one term in this sum has a nonzero Kronecker product with z. Replacing Z by this term, it may be expressed as a product of generators,

$$Z = U_1 \circledast U_2 \circledast \cdots \circledast U_q \tag{10.5.4}$$

with each $\deg(U_i) \le g$.

We claim there exist $X, Y \in H^*(\Lambda, \Lambda_0)$ such that $Z = X \circledast Y$ with $\deg(X) \ge d_1 + 1$ and $\deg(Y) \ge d_2 + 1$. To see this, choose p so that

$$\begin{aligned} \deg(U_1 \circledast U_2 \circledast \cdots \circledast U_{p-1}) &\le d_1, \\ \deg(U_1 \circledast U_2 \circledast \cdots \circledast U_p) &\ge d_1 + 1. \end{aligned}$$

Take $X = U_1 \circledast \cdots \circledast U_p$ and $Y = U_{p+1} \circledast \cdots \circledast U_q$. Then $\deg(U_p) \le g$, so $\deg(X) \le d_1 + g + n - 1$, while $\deg(X \circledast Y) = d_3 > d_1 + d_2 + 2n + g - 2$, so $\deg(Y) > d_2$.

Using this claim and (10.4.1), and setting $j = \deg(X)$ and $k = \deg(Y)$, there exists $\widehat{X}$ which maps to X in the following exact sequence:

$$\begin{array}{ccccc} H^j(\Lambda, \Lambda^{\le t_1}) & \longrightarrow & H^j(\Lambda, \Lambda_0) & \longrightarrow & H^j(\Lambda^{\le t_1}, \Lambda_0), \\ \widehat{X} & \longrightarrow & X. & & \end{array}$$

Similarly, the class Y has some lift $\widehat{Y} \in H^k(\Lambda, \Lambda^{\le t_2})$. Then $\widehat{X} \circledast \widehat{Y} \in H^{d_3}(\Lambda, \Lambda^{t_1+t_2})$ maps to Z. But this implies that $t_3 > t_1 + t_2$. Otherwise, $\Lambda^{\le t_3} \subset \Lambda^{\le t_1+t_2}$ so in the following diagram,

$$\begin{array}{ccccc} H^{d_3}(\Lambda, \Lambda^{\le t_3}) & \longrightarrow & H^{d_3}(\Lambda, \Lambda_0) & \xrightarrow[i^*]{} & H^{d_3}(\Lambda^{\le t_3}, \Lambda_0) \\ \uparrow & & \uparrow & & \uparrow \\ H^{d_3}(\Lambda, \Lambda^{\le t_1+t_2}) & \longrightarrow & H^{d_3}(\Lambda, \Lambda_0) & \longrightarrow & H^{d_3}(\Lambda^{\le t_1+t_2}, \Lambda_0) \\ \widehat{X} \circledast \widehat{Y} & \longrightarrow & Z, & & \end{array}$$

we would have $\langle Z, z\rangle = \langle Z, i_*(z')\rangle = \langle i^*(Z), z'\rangle = 0$, which contradicts (10.5.3). $\square$

10.6

We wish to remark on the importance of using the Morse function F rather than the length L (which lacks the nice analytical properties of E and F) or the energy E (which does not add when we compose loops). One could, of course, rewrite Propositions 5.3 and 9.5 and the definition (4.1.1) using $E = F^2$ instead of F. But the best *linear* formulas involving E are (with the obvious notation)

$$\check{H}_i(\Lambda^{E\le A}) \times \check{H}_j(\Lambda^{E\le B}) \xrightarrow{*} \check{H}_{i+j-n}(\Lambda^{E\le 2(A+B)}), \tag{10.6.1}$$

$$\check{H}^i(\Lambda, \Lambda^{E\le A}) \times \check{H}^j(\Lambda, \Lambda^{E\le B}) \xrightarrow{\circledast} \check{H}^{i+j+n-1}(\Lambda, \Lambda^{E\le(A+B)}), \tag{10.6.2}$$

and the energy-analog to (5.3.1) and (10.1.1) is

$$\mathsf{cr}_E(x * y) \le 2\big(\mathsf{cr}_E(x) + \mathsf{cr}_E(y)\big) \quad \text{for all } x, y \in H_*(\Lambda), \tag{10.6.3}$$

$$\mathsf{cr}_E(X \circledast Y) \ge \mathsf{cr}_E(X) + \mathsf{cr}_E(Y) \quad \text{for all } X, Y \in H^*(\Lambda, \Lambda_0). \tag{10.6.4}$$

These formulas follow from the corresponding formulas for F and the inequality

$$\sqrt{a^2+b^2} \le a + b \le \sqrt{2(a^2+b^2)} \tag{10.6.5}$$

when $a, b \ge 0$. Because of the slack in the second inequality in (10.6.5), the homology bounds (10.6.1) and (10.6.3) are weaker than (5.3.2) and (5.3.1), and they cannot be sharp unless $A = B$ (or $\mathsf{cr}_E(x) = \mathsf{cr}_E(y)$). Because of the slack in the first inequality in (10.6.5), the cohomology formulas (10.6.2) and (10.6.4) are weaker than (9.5.1) and (10.1.1) and are not sharp unless $A = 0$ or $B = 0$ (or unless $\mathsf{cr}_E(X) = 0$ or $\mathsf{cr}_E(Y) = 0$). Many of the results in this article depend essentially on the tautness of (5.3.2), (5.3.1), (9.5.1), and (10.1.1).

11. Level nilpotence for cohomology

11.1

We use the notation M, Λ, F, Σ of §2 and the conventions of §2.5. We say that a class $\alpha \in H^i(\Lambda, \Lambda_0)$ is *level-nilpotent* if there exists m so that $\mathsf{cr}(\alpha^{\circledast m}) > m\mathsf{cr}(\alpha)$. We say that a class $\beta \in \check{H}^i(\Lambda^{\le a}, \Lambda^{<a})$ is level-nilpotent if there exists m so that $\beta^{\circledast m} = 0$ in $\check{H}^{mi+(m-1)(n-1)}(\Lambda^{\le ma}, \Lambda^{<ma})$.

Let us say that two classes $\alpha \in H^i(\Lambda, \Lambda_0)$ and $\beta \in \check{H}^i(\Lambda^{\leq a}, \Lambda^{<a})$ are *associated* if there exists an *associating class* $\omega \in \check{H}^i(\Lambda, \Lambda^{<a})$ with

$$\begin{array}{ccc} \omega & \longrightarrow & \alpha \\ \downarrow & & \\ \beta & & \end{array} \quad \text{in} \quad \begin{array}{ccc} \check{H}^i(\Lambda, \Lambda^{<a}) & \longrightarrow & H_i(\Lambda, \Lambda_0) \\ \downarrow & & \\ \check{H}_i(\Lambda^{\leq a}, \Lambda^{<a}) & & \end{array}$$

Then $\mathsf{cr}(\alpha) > a$ if and only if α is associated to the zero class $\beta = 0 \in \check{H}^i(\Lambda^{\leq a}, \Lambda^{<a})$.

LEMMA 11.2
Suppose that $\alpha \in H^i(\Lambda, \Lambda_0)$ and $\beta \in \check{H}^i(\Lambda^{\leq a}, \Lambda^{<a})$ are associated, where $a = \mathsf{cr}(\alpha)$. If β is level-nilpotent, then α is also level-nilpotent.

Proof
The proof is exactly parallel to that of Lemma 7.2. □

THEOREM 11.3
Let M be a compact n-dimensional Riemannian manifold, and suppose that all critical points of the function $F = \sqrt{E} : \Lambda \to \mathbb{R}$ are nondegenerate (i.e., they lie on isolated nondegenerate critical orbits). If M is orientable, let $G = \mathbb{Z}$; otherwise, let $G = \mathbb{Z}/(2)$. Then every class $\alpha \in H^i(\Lambda, \Lambda_0; G)$ is level-nilpotent and every class $\beta \in H^i(\Lambda^{\leq a}, \Lambda^{<a}; G)$ is level-nilpotent (for any $i > 0$ and any $a \in \mathbb{R}$).

Proof
The proof is similar to that of Theorem 7.3. □

12. Level products in the nondegenerate case

12.1
Throughout this section, homology and cohomology are taken with coefficients in $G = \mathbb{Z}$. Let $\Sigma \subset \Lambda$ be a nondegenerate critical orbit of index λ, and let $U \subset \Lambda$ be a sufficiently small neighborhood of Σ. Assume that the negative bundle $\Gamma \to \Sigma$ is orientable. Then the (local, level) homology groups are

$$\check{H}_i(\Lambda^{<c} \cup \Sigma, \Lambda^{<c}) \cong H_i(\Lambda^{\leq c} \cap U, \Lambda^{<c} \cap U) \cong \begin{cases} \mathbb{Z} & \text{if } i = \lambda, \lambda + 1, \\ 0 & \text{otherwise,} \end{cases} \tag{12.1.1}$$

and the same holds for the cohomology groups $H^i(\Lambda^{\leq c} \cap U, \Lambda^{<c} \cap U)$.

12.2
Now, suppose that $\gamma \in \Lambda$ is a prime geodesic, all of whose iterates are nondegenerate. Let a be its length. Let γ^r denote the r-fold iterate, let λ_r denote its Morse index, and let $\Sigma_r \subset \Lambda$ be its S^1-saturation. Assume that the negative bundle Γ_r over Σ_r is orientable (see Lemma 6.5), and let $\sigma_r, \overline{\tau}_r, \overline{\sigma}_r, \tau_r$ be generators for the local level (co)homology classes, that is,

$$\sigma_r \in H_{\lambda_r}, \qquad \overline{\sigma}_r \in H_{\lambda_r+1}, \qquad \overline{\tau}_r \in H^{\lambda_r}, \qquad \tau_r \in H^{\lambda_r+1}. \tag{12.2.1}$$

As a consequence of the nilpotence results from §§7 and 11, the index λ_r can be neither minimal nor maximal for all r (in the language of Proposition 6.1), and the local level homology and cohomology rings

$$\Big(\bigoplus H_i(\Lambda^{<ar} \cup \Sigma_r, \Lambda^{<ar}; G), *\Big) \qquad \text{and} \qquad \Big(\bigoplus H^i(\Lambda^{<ar} \cup \Sigma_r, \Lambda^{<ar}; G), \circledast\Big)$$

are not finitely generated. However, if the index growth is minimal up to the nth iterate (with $\lambda_r = \lambda_r^{\min}$ for all $r \leq n$), then nontrivial (level) homology products exist; and if the index growth is maximal up to the $2n$th iterate (with $\lambda_r = \lambda_r^{\max}$ for all $r \leq 2n$), then nontrivial (level) cohomology products exist, as described in the following theorem.

THEOREM 12.3
Assume that the manifold M is orientable, $\gamma \in \Lambda$ is prime, all of its iterates are nondegenerate, and the negative bundle Γ_r is orientable for all r. Assume that $r \geq 2$. Then the following statements hold in the local level (co)homology group $H(\Lambda^{<ra} \cup \Sigma_r, \Lambda^{<ra}; \mathbb{Z})$:
(1) $(\sigma_1)^{*r} = 0$ *and* $(\tau_1)^{\circledast r} = 0$*;*
(2) *some further products are described in Figures 5 and 6;*
(3) *if $n - \lambda_1$ is even, then $(\overline{\sigma}_1)^{*r} = 0$;*
(4) *if $n - \lambda_1$ is even, then $(\overline{\tau}_1)^{\circledast r} = 0$.*

Proof
We begin with the parity statements (3) and (4). In general, if $\sigma \in H_k(\Lambda; \mathbb{Z})$ and $\tau \in H^k(\Lambda; \mathbb{Z})$, then $2\sigma * \sigma = 0$ if $n - k$ is odd and $2\tau \circledast \tau = 0$ if $n - k$ is even. This follows from Propositions 5.2 and 9.2, and it implies the vanishing of $(\overline{\sigma}_1)^{*r}$ and $(\overline{\tau}_1)^{\circledast r}$ if $n - \lambda_1$ is even.

Statement (1) follows from the fact that the homology class σ_1 is supported on a closed subset $A \subset \Lambda^{\leq a}$ such that $A \cap \Sigma_1$ consists of a single point. By Proposition 5.3, the product $\sigma_1 * \sigma_1$ is supported on the set $A * A'$, where $A' \subset \Lambda$ is a support set for σ_1 that intersects Σ_1 in a different point. Consequently, $A * A' \subset \Lambda^{<2a}$. Similarly, the cohomology class $\tau_1 \in H^{\lambda}(\Lambda^{\leq a+\epsilon}, \Lambda^{<a})$ is supported on a closed set $B \subset \Lambda^{\geq a}$ that intersects Σ_1 in a single point.

	$(\bar{\sigma}_1)^{*(r-1)}*\sigma_1$	$(\bar{\sigma}_1)^{*r}$
$\lambda_r = \lambda_r^{\min}$ and $\lambda_n = \lambda_n^{\min}$	σ_r	$\bar{\sigma}_r$
$\lambda_r \neq \lambda_r^{\min}$	0	0

Figure 5. Homology level products

	$(\bar{\tau}_1)^{\circledast(r-1)}\circledast\tau_1$	$(\bar{\tau}_1)^{\circledast r}$
$\lambda_{rn} = \lambda_{rn}^{\max}$	τ_r	$\bar{\tau}_r$
$\lambda_r \neq \lambda_r^{\max}$	0	0

Figure 6. Cohomology level products

The zeros in the second row of each of the tables (see Figures 5 and 6) are also easily explained. First, consider homology. By Lemma 6.5, the assumption that Γ_r is orientable for all r implies that all the λ_r have the same parity. We consider two cases.

(i) If $n-\lambda_1$ is even, then $\lambda_2^{\min}$ and λ_1 have opposite parity; hence $\lambda_r^{\min} < \lambda_r$ for all $r > 1$. In this case, $(\overline{\sigma}_1)^{*r} = 0$ for $r > 1$ as above, and $\overline{\sigma}_1 * \sigma_1 = 0$ because $\overline{\sigma}_1 * \sigma_1$ has degree $\lambda_2^{\min} < \lambda_2$.

(ii) If $n - \lambda_1$ is odd, then $\lambda_2^{\min}$ and λ_1 have the same parity. But all the λ_i have the same parity, so if λ_r does not attain its minimum value (i.e., if $\lambda_r \neq \lambda_r^{\min}$), then

$$\lambda_r^{\min} + 2 \leq \lambda_r.$$

In this case it follows from (12.1.1) that $(\overline{\sigma}_1)^{*r} = 0$ and $\overline{\sigma}_1 * \sigma_1 = 0$.

The cohomology calculations are similar. The remaining statements in Theorem 12.3 are proven in the next two sections.

12.4. Case of maximal index growth

In this section, we assume that $\lambda_i = \lambda_i^{\max}$ for $i \leq rn$. Choose $x_0 = \gamma(0)$ for the base point of M. By Lemma 6.4, the index λ_r equals the index λ_r^{Ω} of γ^r in the based loop space $\Omega = \Omega_{x_0}$, and it coincides with the index of γ^r in the spaces $T_{\gamma^r}^{\perp}\Lambda$ and $T_{\gamma^r}^{\perp}\Omega$ of vector fields $V(t)$ along γ^r such that $V(t) \perp \gamma'(t)$ for all t.

Let W_1 be a maximal negative subspace of $T_{\gamma}^{\perp}\Omega$ (so $\dim(W_1) = \lambda_1 = \lambda_1^{\Omega}$). Let $W_1^{\bullet r}$ be the $r\lambda_1$-dimensional negative subspace of $T_{\gamma^r}\Omega$ consisting of concatenations $V_1 \bullet V_2 \bullet \cdots \bullet V_r$ of vector fields $V_i \in W_1$. Then $W_1^{\bullet r}$ is a maximal negative subspace of the kernel of

$$\nu : T^{\perp}_{\gamma^r}\Omega \to T^{\perp}_{\gamma(0)}M \times \cdots \times T^{\perp}_{\gamma(0)}M,$$
$$V \mapsto \Big(V\Big(\frac{1}{r}\Big), \ldots, V\Big(\frac{r-1}{r}\Big)\Big).$$

Choose a maximal negative subspace $W_r \subset T^{\perp}_{\gamma^r}\Omega$ containing $W_1^{\bullet r}$. Then

$$\dim(W_r) = \lambda_r^{\Omega} = \lambda_r = \lambda_r^{\max} = r\lambda_1 + (r-1)(n-1).$$

It follows that *the restriction of* ν *to* W_r *is surjective* because its kernel has dimension $r\lambda_1$.

Let $\mathscr{C}_r$ be the r-leafed clover consisting of loops $\eta \in \Lambda$ such that $\eta(0) = \eta(i/r)$ for $i = 0, 1, 2, \ldots, r$. The exponential map $T_{\gamma^r}(\Lambda) \to \Lambda$ takes W_r to a relative cycle in $(\Lambda^{\le ra}, \Lambda^{\le ra-\epsilon})$ which we also denote by W_r, whose (relative) homology class is $[W_r] = \sigma_r$ (and $[W_1] = \sigma_1$).

The function $J : \Lambda \times [0,1] \to \Lambda$ extends in an obvious way to a family of reparametrizations,

$$J : W_r \times \Big[\frac{1}{r} - \epsilon, \frac{1}{r} + \epsilon\Big] \times \Big[\frac{2}{r} - \epsilon, \frac{2}{r} + \epsilon\Big] \times \cdots \times \Big[\frac{r-1}{r} - \epsilon, \frac{r-1}{r} + \epsilon\Big],$$

which is transverse* to $\mathscr{C}_r$ and such that

$$J\Big(W_r \times \Big[\frac{1}{r} - \epsilon, \frac{1}{r} + \epsilon\Big] \times \cdots \times \Big[\frac{r-1}{r} - \epsilon, \frac{r-1}{r} + \epsilon\Big]\Big) \cap \mathscr{C}_r = W_1 \bullet W_1 \bullet \cdots \bullet W_1$$

(Pontrjagin product). By (a relative version of) Proposition 9.4, we conclude that

$$\begin{aligned}\langle \overline{\tau}_1 \circledast \overline{\tau}_1 \circledast \cdots \circledast \overline{\tau}_1, \sigma_r\rangle &= \langle \overline{\tau}_1 \circledast \overline{\tau}_1 \circledast \cdots \circledast \overline{\tau}_1, [W_r]\rangle \\ &= \langle \overline{\tau}_1, [W_1]\rangle \cdot \langle \overline{\tau}_1, [W_1]\rangle \cdot \ldots \cdot \langle \overline{\tau}_1, [W_1]\rangle = 1.\end{aligned}$$

It follows that $\overline{\tau}_1^{\circledast r} = \overline{\tau_r}$. The calculation for $\overline{\tau}_1^{\circledast(r-1)} \circledast \tau_1$ is similar. The same technique, by explicitly displaying cycles, may be used to prove Theorem 14.2. □

12.5. Case of minimal index growth

If $(D\Gamma_r, S\Gamma_r)$ denote the ϵ-disk and sphere bundle of the negative bundle $\Gamma_r \to \Sigma_r$, then for sufficiently small $\epsilon > 0$, the exponential mapping $\exp : D\Gamma_r \to \Lambda$ is a smooth embedding whose image

$$(\Sigma_r^-, \partial\Sigma_r^-) = \big(\exp(D\Gamma_r), \exp(S\Gamma_r)\big)$$

*The restriction $J|W_r$ is transverse to $\mathscr{C}_r$ in the directions normal to $\gamma'(0)$ because $\nu|W_r$ is surjective. The intervals $[(i-1)/r, i/r]$ take care of the tangential directions.

is a smoothly embedded submanifold with boundary in Λ that "hangs down" from the critical set Σ_r. Its dimension is $\lambda_r + 1$ and its fundamental class is

$$\overline{\sigma}_r = [\Sigma_r^-, \partial \Sigma_r^-] \in H_{\lambda_r+1}(\Lambda^{<r\ell} \cup \Sigma_r, \Lambda^{<r\ell}) \cong \mathbb{Z},$$

where ℓ denotes the length of γ.

Now, assume that $\lambda_n = \lambda_n^{\min}$. By Lemma 6.4, this implies that the difference between λ_1^Ω and λ_1 is the *maximum* possible: $\lambda_1 = \lambda_1^\Omega + n - 1$. Let $W_1 \subset T_\gamma^\perp \Lambda$ be a maximal negative subspace. Then the mapping

$$\nu : W_1 \to T_{\gamma(0)}^\perp M, \qquad \nu(V) = V(0) \tag{12.5.1}$$

is surjective. Consequently,

$$\mathbf{ev}_0 : \Sigma_1^- \to M$$

is a submersion in a neighborhood of the closed geodesic γ. It follows that Σ_r^- and Σ_1^- are transverse over M (in some neighborhood of γ^r and γ), and

$$\Sigma_r^- \times_M \Sigma_1^- \xrightarrow[\phi_{\min}]{\cong} \Sigma_r^- * \Sigma_1^-$$

is a smooth submanifold of Λ in a neighborhood of

$$\Sigma_r \times_M \Sigma_1 \xrightarrow[\phi_{\min}]{\cong} \Sigma_r * \Sigma_1 = \Sigma_{r+1}$$

and is contained in $\Sigma_{r+1} \cup \Lambda^{<(r+1)\ell}$. Now, assume that the index growth is minimal up to level $r+1$ (i.e., $\lambda_{r+1} = (r+1)\lambda_1 - r(n-1)$), so that

$$\dim(\Sigma_r^- \times_M \Sigma_1^-) = \dim(\Sigma_{r+1}^-).$$

Then we may apply* Theorem D.2 to the embeddings

$$\Sigma_{r+1} \subset \Sigma_r^- \times_M \Sigma_1^- \xrightarrow[\phi_{\min}]{} \Lambda^{\leq (r+1)\ell}$$

to conclude that

$$[\Sigma_r^- * \Sigma_1^-, \partial(\Sigma_r^- * \Sigma_1^-)] = [\Sigma_{r+1}^-, \partial \Sigma_{r+1}^-] \in H_{\lambda_{r+1}+1}(\Sigma_{r+1} \cup \Lambda^{<(r+1)\ell}, \Lambda^{<(r+1)\ell}).$$

*The condition on the eigenvalues of the second derivative (in the hypotheses of Theorem D.2) is satisfied by the energy functional, as a consequence of [Kl, Theorem 2.4.2].

(In fact, we even obtain a local diffeomorphism $\tau : \Sigma_{r+1}^- \to \Sigma_r^- * \Sigma_1^-$ between the negative submanifolds by equation (D.3.1).) Using Proposition 5.5, we conclude that

$$[\Sigma_r^-, \partial\Sigma_r^-] * [\Sigma_1^-, \partial\Sigma_1^-] = [\Sigma_{r+1}^-, \partial\Sigma_{r+1}^-]$$

and, by induction, that

$$\overline{\sigma}_{r+1} = \overline{\sigma}_r * \overline{\sigma_1} = (\overline{\sigma}_1)^{*r} * \overline{\sigma}_1 = (\overline{\sigma}_1)^{*(r+1)}, \tag{12.5.2}$$

as claimed. The geometric calculation of the product $\overline{\sigma_r} * \sigma_1$ is similar. A similar procedure is used to prove Theorem 13.4. □

12.6. The nonnilpotent case

The case of isolated closed geodesics with slowest possible index growth was studied in [Hi2]; fastest possible index growth was studied in [Hi3] in a slightly different language because the $*$- and $\circledast$-products were not available at the time. The Chas-Sullivan homology product is modeled in the local geometry of an isolated closed geodesic with the slowest possible growth rate. The symmetry between the geometry in the case of slowest possible index growth (nonnilpotent level homology) and that of fastest possible index growth (nonnilpotent level cohomology) inspired the search for the cohomology product. We give statements here of two theorems on nonnilpotent products that are restatements of the “complementary theorem” ([Hi3, p. 3100]) and the theorem ([Hi3, p. 3099]).

THEOREM 12.7

Let γ be an isolated closed geodesic with nonnilpotent level homology. Let $L =$ length(γ). Then for any $\epsilon > 0$, if $m \in \mathbb{Z}$ is sufficiently large, there is a closed geodesic with length in the open interval $(mL, mL + \epsilon)$. It follows that M has infinitely many closed geodesics.

THEOREM 12.8

Let γ be an isolated closed geodesic with nonnilpotent level cohomology. Let $L =$ length(γ). Then for any $\epsilon > 0$, if $m \in \mathbb{Z}$ is sufficiently large, there exists a closed geodesic with length in the open interval $(mL - \epsilon, mL)$. It follows that M has infinitely many closed geodesics.

13. Homology product when all geodesics are closed

13.1

In this section, we continue with the notation M, Λ, F, Σ of §2 and the conventions of §2.5. Throughout this section, we assume that all geodesics γ are closed and simply periodic with the same prime length ℓ, meaning that $\gamma(0) = \gamma(1)$, $\gamma'(0) = \gamma'(1)$, γ is injective on $(0, 1)$, and $L(\gamma) = \ell$ if γ is prime.

For $r \geq 1$, denote by $\Sigma_r \subset \Lambda$ the critical set consisting of r-fold iterates of prime closed geodesics. There is a diffeomorphism $SM \cong \Sigma_r$ between the unit sphere bundle of M and Σ_r, which assigns to each unit tangent vector v the r-fold iterate of the prime geodesic with initial condition v. It follows that the nullity of each geodesic is at least $\dim(\Sigma_r) - 1 = 2n - 2$. Since this is the maximum nullity possible, we see that the nullity ν_r of every closed geodesic is $2n - 2$. In particular, each Σ_r is a nondegenerate critical submanifold (in the sense of Bott) with critical value $F(\Sigma_r) = r\ell$, and this accounts for all the critical points of the Morse function $F = \sqrt{E}$. Moreover, for any $c \in \mathbb{R}$, the singular and Čech homology $H_*(\Lambda^{\leq c})$ agree by Proposition 3.3. Every geodesic $\gamma \in \Sigma_r$ has the same index (see [Be, Theorem 7.23]), say, λ_r. By (6.1.1), $\lambda_r \leq r\lambda_1 + (r - 1)(n - 1)$. By (6.1.2), $\lambda_r \geq r\lambda_1 + (r - 1)(n - 1)$; hence *the index growth is maximal*,

$$\lambda_r = \lambda_r^{\max} = r\lambda_1 + (r - 1)(n - 1). \tag{13.1.1}$$

As in Theorem D.2, let $\Gamma_r \to \Sigma_r$ be the negative definite bundle. It is a real vector bundle whose rank is λ_r.

PROPOSITION 13.2

If M is orientable and all geodesics on M are closed with the same prime period, then for any r the negative bundle Γ_r is also orientable.

Proof

Fix r, and let $\gamma_0 \in \Sigma_r$ be a basepoint. Set $x_0 = \gamma_0(0) \in M$. Using the long exact sequence for the fibration $SM \to M$, we see that the projection $\Sigma_r \to M$ induces an isomorphism $\pi_1(\Sigma_r, \gamma_0) \cong \pi_1(M, x_0)$. If $\lambda_1 > 0$, then by [Be, Theorem 7.23] the manifold M is simply connected. So if $\dim(M) \geq 3$, the same is true of Σ_r; hence every vector bundle on Σ_r is orientable. If $\dim(M) = 2$, then $M = S^2$ is the 2-sphere and Γ_r is orientable by inspection.

So we may assume that $\lambda_1 = 0$. By [Be, Theorem 7.23], this implies that M is diffeomorphic to real projective space and $\pi_1(M, x_0) \cong \mathbb{Z}/(2)$. Since M is orientable, $n = \dim(M)$ is odd.

The bundle $\Gamma_r \to \Sigma_r$ is orientable if and only if its restriction to each loop in Σ_r is orientable, and it suffices to check this on any loop in the single nontrivial class in $\pi_1(\Sigma_r, \gamma_0)$. We may even take that loop to be the canonical lift $\widetilde{\gamma} : [0, 1] \to SM$,

$$\widetilde{\gamma}(t) = \big(\gamma(t), \gamma'(t)/\|\gamma'(t)\|\big)$$

of a periodic prime geodesic $\gamma : [0,1] \to M$ with $\gamma(0) = x_0$. (Since each geodesic is determined by its initial conditions, it follows that $\gamma_0 = \gamma^r$. This geodesic loop is contractible in M if and only if r is even.)

It now follows from Lemma 6.5 that the bundle Γ_r is orientable if r is odd, or if r is even and $\Omega_\gamma(-1)$ is even, where Ω_γ is Bott's index function (cf. §6.2). But $\Omega_\gamma(-1) = \lambda_2 - \lambda_1 = n - 1$, which is even. $\square$

It follows from Theorem D.2 that a choice of orientation for Γ_r determines an isomorphism

$$h_r : H_i(\Sigma_r; G) \cong H_{i+\lambda_r}(\Lambda^{\le r\ell}, \Lambda^{<r\ell}; G), \tag{13.2.1}$$

where $G = \mathbb{Z}$ if M is orientable and $G = \mathbb{Z}/(2)$ otherwise (see also [O]).

13.3. The nonnilpotent homology class

Continue with the notation M, Λ, F, Σ of §2 and the conventions of §2.5. For any $c \in \mathbb{R}$, the long exact sequence for the pair $(\Lambda^{\le c}, \Lambda_0)$ is canonically split by the evaluation mapping $\mathbf{ev}_0 : \Lambda^{\le c} \to \Lambda_0$ so we obtain a canonical isomorphism

$$H_i(\Lambda^{\le c}; G) \cong H_i(\Lambda_0; G) \oplus H_i(\Lambda^{\le c}, \Lambda_0; G). \tag{13.3.1}$$

Taking $c = \ell = F(\Sigma_1)$ and using Theorem D.2 gives a canonical isomorphism

$$H_i(\Lambda^{\le \ell}; G) \cong H_i(\Lambda_0; G) \oplus H_{i-\lambda_1}(\Sigma_1; G). \tag{13.3.2}$$

The manifold Σ_1 is orientable (whether or not M is) since $T\Sigma_1 \oplus \mathbf{1} \cong h^*(TM) \oplus h^*(TM)$. Choose an orientation of Σ_1 with a resulting fundamental class $[\Sigma_1] \in H_{2n-1}(\Sigma_1; G)$. Define

$$\Theta \in H_{2n-1+\lambda_1}(\Lambda^{\le \ell})$$

to be its image under the isomorphism (13.3.2). Set $b = \lambda_1 + n - 1$.

THEOREM 13.4

Let M be an n-dimensional compact Riemannian manifold, all of whose geodesics are simply periodic with the same prime length ℓ. Then the following statements hold:

(1) *the energy $E : M \to \mathbb{R}$ is a perfect Morse-Bott function for $H_*(\Lambda; G)$; that is, for each $r \ge 1$, every connecting homomorphism vanishes in the long exact sequence*

$$\longrightarrow H_i(\Lambda^{<r\ell}; G) \longrightarrow H_i(\Lambda^{\le r\ell}; G) \longrightarrow H_i(\Lambda^{\le r\ell}, \Lambda^{<r\ell}; G) \longrightarrow$$

(2) *the product* $*\Theta : H_i(\Lambda, \Lambda_0; G) \to H_{i+b}(\Lambda, \Lambda_0; G)$ *with the class* Θ *is injective; and*

(3) *for all* $r \geq 1$*, this product induces an isomorphism on level homology,*

$$w_r : H_i(\Lambda^{\leq r\ell}, \Lambda^{<r\ell}; G) \to H_{i+b}(\Lambda^{\leq (r+1)\ell}, \Lambda^{<(r+1)\ell}; G).$$

Proof

Assume, by induction on r, that $\beta_r : H_i(\Lambda^{\leq r\ell}, \Lambda_0; G) \to H_i(\Lambda^{\leq r\ell}, \Lambda^{<r\ell}; G)$ is surjective for all i. The case of $r = 1$ is handled by equation (13.3.2). Consider the following commutative diagram, where the vertical mappings are given by the Chas-Sullivan product $*\Theta$:

$$\begin{tikzcd}
H_i(\Lambda^{<r\ell}, \Lambda_0) \arrow[r, hook, "\alpha_r"'] \arrow[d, "u_r"'] & H_i(\Lambda^{\leq r\ell}, \Lambda_0) \arrow[r, two heads, "\beta_r"'] \arrow[d, "v_r"'] & H_i(\Lambda^{\leq r\ell}, \Lambda^{<r\ell}) \arrow[d, "w_r"'] \\
H_{i+b}(\Lambda^{<(r+1)\ell}, \Lambda_0) \arrow[r, "\alpha_{r+1}"'] & H_{i+b}(\Lambda^{\leq (r+1)\ell}, \Lambda_0) \arrow[r, "\beta_{r+1}"'] & H_{i+b}(\Lambda^{\leq (r+1)\ell}, \Lambda^{<(r+1)\ell})
\end{tikzcd}$$

We show below that the mapping w_r is an isomorphism. Assuming this for the moment, it follows that β_{r+1} is surjective in all degrees. Hence the horizontal sequences in this diagram split into short exact sequences (so the Morse function is perfect). Therefore u_r is injective if and only if v_r is injective. However, v_r may be identified with the mapping u_{r+1} under the isomorphism $H_i(\Lambda^{\leq r\ell}) \cong H_i(\Lambda^{<(r+1)\ell})$ so it is injective by induction. (The mapping u_1 is trivially injective.) The rest of §13 is devoted to proving that w_r is an isomorphism. $\square$

THEOREM 13.5

Let M *be an* n*-dimensional compact Riemannian manifold, all of whose geodesics are simply periodic with the same prime length* ℓ*, and continue with the conventions of §2.5. Then, after composing with the isomorphism*

$$h_r : H_*(SM) \to H_*(\Lambda^{\leq r\ell}, \Lambda^{<r\ell})$$

(where SM *denotes the unit sphere bundle of the tangent bundle to* M*), the Chas-Sullivan product becomes the intersection product on homology, which is to say that the following diagram commutes:*

$$\begin{tikzcd}
H_i(\Lambda^{\leq r\ell}, \Lambda^{<r\ell}) \times H_j(\Lambda^{\leq \ell}, \Lambda^{<\ell}) \arrow[r, "*"] & H_{i+j-n}(\Lambda^{\leq (r+1)\ell}, \Lambda^{<(r+1)\ell}) \arrow[d, "h_{r+1}"', "\cong"] \\
H_{i-\lambda_r}(SM) \times H_{j-\lambda_1}(SM) \arrow[r] \arrow[u, "h_r \times h_1", "\cong"'] & H_{i-\lambda_r+j-\lambda_1-2n+1}(SM)
\end{tikzcd}$$

where the bottom row denotes the intersection product in homology.

We remark that this immediately implies that w_r is an isomorphism because the mapping w_r is the C-S product with the unique top-dimensional class in $H_*(\Lambda^{\le \ell}, \Lambda^{<\ell})$ which becomes the fundamental class $[SM] \in H_{2n-1}(SM)$ under the vertical isomorphism in the above diagram. But the intersection with the fundamental class is the identity mapping $H_*(SM) \to H_*(SM)$.

Proof
The set $\Sigma_r * \Sigma_1 = \phi_{r/(r+1)}(\Sigma_r \times_M \Sigma_1)$ consists of pairs of composable loops; the first is an r-fold iterate of a prime geodesic, and the second is a single prime geodesic, all parametrized proportionally with respect to arclength. This set contains Σ_{r+1} as a submanifold of codimension $n-1$. In fact, the inclusion

$$\Sigma_{r+1} \to \Sigma_r * \Sigma_1 \to \Sigma_r \times \Sigma_1 \tag{13.5.1}$$

is the diagonal mapping $SM \to SM \times SM$.

Let $(\Sigma_r^-, \partial \Sigma_r^-) = (\exp(D\Gamma_r), \exp(\partial D\Gamma_r))$ be the negative submanifold that hangs down from Σ_r, as in Proposition D.2, where $D\Gamma_r$ denotes a sufficiently small disk bundle in the negative bundle $\Gamma_r \to \Sigma_r$ and where $\partial D\Gamma_r$ denotes its bounding sphere bundle. Then $\dim(\Sigma_r^- \times_M \Sigma_1^-) = \dim(\Sigma_{r+1}^-)$ so we can apply Theorem D.2 to the embeddings

$$\Sigma_{r+1} \subset \Sigma_r^- \times_M \Sigma_1^- \xrightarrow[\phi_{\min}]{} \Lambda^{\le (r+1)\ell}$$

followed by an arbitrarily brief flow under the vector field $-\nabla F$. The condition on the eigenvalues of the second derivative (in the hypotheses of Theorem D.2) is satisfied by the energy functional, as a consequence of [Kl, Theorem 2.4.2]. As in (D.3.1), we obtain a local (in a neighborhood of Σ_{r+1}) diffeomorphism

$$\tau : \Sigma_{r+1}^- \to \Sigma_r^- \times_M \Sigma_1^- \tag{13.5.2}$$

between the negative submanifolds (see Figure 7). By Proposition 5.5, the Chas-Sullivan product is given by the composition down the right side of Figure 7.

On the other hand, the composition down the left side of Figure 7 is the intersection pairing because the composition down the middle four rows is just the Gysin pullback for the (diagonal) embedding (13.5.1). This completes the proof of Theorem 13.5 and hence also of Theorem 13.4. □

13.6
Define the filtration $0 \subset I_0 \subset I_1 \subset \cdots \subset H_*(\Lambda, \Lambda_0; G)$ by

$$I_r = \mathrm{Image}\big(H_*(\Lambda^{\le r\ell}, \Lambda_0; G) \to H_*(\Lambda, \Lambda_0; G)\big).$$

$$
\begin{tikzcd}
 & H_i(\Lambda^{\le r\ell}, \Lambda^{<r\ell}) \otimes H_j(\Lambda^{\le \ell}, \Lambda^{<\ell}) \arrow[d, "\cong"] \\
H_{i-\lambda_r}(\Sigma_r) \otimes H_{j-\lambda_1}(\Sigma_1) \arrow[r, "\cong"] \arrow[d, "\times"'] & H_i(\Sigma_r^-, \Sigma_r^- - \Sigma_r) \otimes H_j(\Sigma_1^-, \Sigma_1^- - \Sigma_1) \arrow[d, "\times"] \\
H_{i-\lambda_r+j-\lambda_1}(\Sigma_r \times \Sigma_1) \arrow[r, "\cong"] \arrow[d, "(B.2.4)"'] & H_{i+j}(\Sigma_r^- \times \Sigma_1^-, \Sigma_r^- \times \Sigma_1^- - \Sigma_r \times \Sigma_1) \arrow[d] \\
H_{i-\lambda_r+j-\lambda_1-n}(\Sigma_r \times_M \Sigma_1) \arrow[r, "\cong"] \arrow[d] & H_{i+j}(\Sigma_r^- \times \Sigma_1^-, \Sigma_r^- \times \Sigma_1^- - \Sigma_r \times_M \Sigma_1) \arrow[d, "(B.2.2)"] \\
H_{i-\lambda_r+j-\lambda_1-n}(\Sigma_r \times_M \Sigma_1) \arrow[r, "\cong"] \arrow[d, "(B.2.4)"'] & H_{i+j-n}(\Sigma_r^- \times_M \Sigma_1^-, \Sigma_r^- \times_M \Sigma_1^- - \Sigma_r \times_M \Sigma_1) \arrow[d] \\
H_{i+j-n-\lambda_{r+1}}(\Sigma_{r+1}) \arrow[r, "\cong"] \arrow[d] & H_{i+j-n}(\Sigma_r^- \times_M \Sigma_1^-, \Sigma_r^- \times_M \Sigma_1^- - \Sigma_{r+1}) \arrow[d, "(13.5.2)"] \\
H_{i+j-n-\lambda_{r+1}}(\Sigma_{r+1}) \arrow[r, "\cong"] & H_{i+j-n}(\Sigma_{r+1}^-, \Sigma_{r+1}^- - \Sigma_{r+1}) \arrow[d, "\cong"] \\
 & H_{i+j-n}(\Lambda^{\le (r+1)\ell}, \Lambda^{<(r+1)\ell})
\end{tikzcd}
$$

Figure 7. The C-S product when all geodesics are closed

By Proposition 5.3, $I_r * I_s \subset I_{r+s}$ so the Chas-Sullivan product induces a product on the associated graded group,

$$\mathrm{Gr}_I H_*(\Lambda, \Lambda_0; G) = \bigoplus_{r=1}^{\infty} I^r/I^{r-1} \cong \bigoplus_{r=1}^{\infty} H_*(\Lambda^{\le r\ell}, \Lambda^{<r\ell}),$$

which therefore coincides with the level homology ring (5.3.5). Let $H_*(SM; G)$ be the homology (intersection) ring of the unit sphere bundle, and let $H_*(SM)[T]_{\ge 1} = T H_*(SM)[T]$ be the ideal of polynomials with zero constant term.

COROLLARY 13.7
The mapping

$$\Phi : H_*(SM; G)[T]_{\ge 1} \to \mathrm{Gr}_I H_*(\Lambda, \Lambda_0; G), \tag{13.7.1}$$

$$\Phi(aT^m) = h_1(a) * \Theta^{*(m-1)} \in H_{\deg(a)+\lambda_1+(m-1)b}(\Lambda^{\le m\ell}, \Lambda^{<m\ell}; G)$$

is an isomorphism of rings.

Proof
This follows immediately from Theorems 13.4 and 13.5. □

14. Cohomology products when all geodesics are closed

14.1
Continue with the notation M, Λ, F, Σ of §2 and the conventions of §2.5. As in §13.1, assume that M is compact n-dimensional and all geodesics on M are simply periodic with the same prime length, ℓ. Let $\Sigma_r \subset \Lambda$ denote the submanifold consisting of the r-fold iterates of prime geodesics. It is a nondegenerate critical submanifold, diffeomorphic to the unit sphere bundle SM, having index $\lambda_r = r\lambda_1 + (r-1)(n-1)$ and critical value $F(\Sigma_r) = r\ell$. Let $D\Gamma_r$, $S\Gamma_r$ be the unit disk bundle and unit sphere bundle of the negative bundle $\Gamma_r \to \Sigma_r$. Theorem D.2 then gives an isomorphism

$$h_r : H^i(\Sigma_r) \xrightarrow{\cong} H^{i+\lambda_r}(\Lambda^{\le r\ell}, \Lambda^{<r\ell}) \tag{14.1.1}$$

by identifying each with $H^{i+\lambda_r}(D\Gamma_r, S\Gamma_r)$. Thus, if $r \ge 2$ and $j < \lambda_2$, then $H^j(\Lambda^{\le r\ell}, \Lambda^{\le (r-1)\ell}) = 0$. Since F is perfect, we obtain isomorphisms

$$H^j(\Lambda, \Lambda_0) \cong H^j(\Lambda^{\le N\ell}, \Lambda_0) \cong \cdots \cong H^j(\Lambda^{\le \ell}, \Lambda_0) \cong H^{j-\lambda_1}(\Sigma_1) \tag{14.1.2}$$

for all $j < \lambda_2$ (if N is sufficiently large). Define

$$\omega \in H^{\lambda_1}(\Lambda, \Lambda_0) \cong H^0(\Sigma_1) \tag{14.1.3}$$

to be the image of 1.

THEOREM 14.2
Assume that M is compact n-dimensional, and assume that all geodesics on M are simply periodic with the same prime length ℓ. Then

(1) *the energy $E : \Lambda \to \mathbb{R}$ is a perfect Morse function for cohomology, meaning that for each $r \ge 1$ the connecting homomorphism vanishes in the long exact sequence*

$$\begin{tikzcd}
{} \arrow[r] & H^i(\Lambda^{\le (r+1)\ell}, \Lambda^{\le r\ell}; G) \arrow[r] & H^i(\Lambda^{\le (r+1)\ell}, \Lambda_0; G) \arrow[r] & H^i(\Lambda^{\le r\ell}, \Lambda^0; G) \arrow[r] & {} \\
& H^i(\Lambda^{\le (r+1)\ell}, \Lambda^{<(r+1)\ell}; G) \arrow[u, "\cong"'] & & &
\end{tikzcd}$$

(2) *the product $\circledast\omega : H^i(\Lambda, \Lambda_0) \to H^{i+b}(\Lambda, \Lambda_0)$ is injective; and*

(3) *this product induces an isomorphism*

$$w_r : H^i(\Lambda^{\le r\ell}, \Lambda^{<r\ell}) \to H^{i+b}(\Lambda^{\le (r+1)\ell}, \Lambda^{<(r+1)\ell})$$

for all $r \ge 1$ and all $i \ge 0$.

As in the proof of Theorem 13.4, part (3) implies parts (1) and (2). Part (3) is a consequence of the following stronger statement.

THEOREM 14.3
After composing with the isomorphism

$$h_r : H^*(SM) \to H^*(\Lambda^{\le r\ell}, \Lambda^{<r\ell}),$$

the cohomology product becomes the cup product on cohomology, which is to say that the following diagram commutes (recall that $\lambda_{r+1} = \lambda_1 + \lambda_r + n - 1$):

$$\begin{array}{ccc} H^a(\Sigma_r) \otimes H^b(\Sigma_1) & \xrightarrow[h_r \otimes h_1]{} & H^{a+\lambda_r}(\Lambda^{\le r\ell}, \Lambda^{<r\ell}) \otimes H^{b+\lambda_1}(\Lambda^{\le \ell}, \Lambda^{<\ell}) \\ \smile \downarrow & & \downarrow \circledast \\ H^{a+b}(\Sigma_{r+1}) & \xrightarrow[h_{r+1}]{} & H^{a+b+\lambda_{r+1}}(\Lambda^{\le (r+1)\ell}, \Lambda^{<(r+1)\ell}) \end{array}$$

The proof appears in the next few sections. In order to use Proposition 9.5 we need to work in the space $\mathscr{A}$ of PPAL loops.

14.4
Fix $r \ge 1$. Let $r\ell^+ = r\ell + 2\epsilon$, let $r\ell^- = r\ell - \epsilon$, let $\ell^+ = \ell + 2\epsilon$, and let $\ell^- = \ell - \epsilon$. Set $j = a + b + \lambda_r + \lambda_1$ so that $j + n - 1 = a + b + \lambda_{r+1}$. It is convenient to replace the gluing map $\phi_{1/2} : \mathscr{A} \times_M \mathscr{A} \to \mathscr{A}_{1/2}$ with the topologically equivalent embedding $\phi_{r/(r+1)} : \mathscr{A} \times_M \mathscr{A} \to \mathscr{A}_{r/(r+1)}$, which approximates $\phi_{\min}$ near $\Sigma_r \times \Sigma_1$; in fact,

$$\Sigma_r * \Sigma_1 = \phi_{\min}(\Sigma_r \times_M \Sigma_1) = \phi_{r/(r+1)}(\Sigma_r \times_M \Sigma_1).$$

We write

$$\mathscr{F}^{<a,<b}_{r/(r+1)} \text{ for } \phi_{r/(r+1)}(\mathscr{F}^{<a,<b}_{\mathscr{A}}).$$

Similarly, we replace mapping $J_{\mathscr{A}} : \mathscr{A} \times [0,1] \to \mathscr{A}_{1/2}$ with the mapping

$$J_r : \mathscr{A} \times [0,1] \to \mathscr{A}_{r/(r+1)}$$

given by $J_r(\alpha, s) = \alpha \circ \theta_{r/(r+1)\to s}$, where $\mathscr{A}_{r/(r+1)}$ and $\theta_{r/(r+1)\to s}$ are defined by replacing $1/2$ with $r/(r+1)$ in §9.1. The mapping $\mathscr{A}^{\le (r+1)\ell} \to \mathscr{A}^{\le r\ell, \le \ell}_{r/(r+1)}$ given by $\alpha \mapsto J_r(\alpha, r/(r+1))$ is a homotopy equivalence; its inverse assigns to a pair of joinable PPAL paths $\alpha, \beta \in \mathscr{A}_{r/(r+1)}$ (with $\alpha(1) = \beta(0)$ and $\beta(1) = \alpha(0)$) with lengths at most $r\ell$ and at most ℓ, respectively, the path $\phi_{\min}(\alpha, \beta)$ obtained by joining them at time $L(\alpha)/(L(\alpha) + L(\beta))$.

Let I' be the closed interval

$$I' = \left[\frac{r\ell - \epsilon}{(r+1)\ell + \epsilon}, \frac{r\ell + 2\epsilon}{(r+1)\ell + \epsilon}\right]$$

as in §9.10. Then

$$J_r(\mathscr{A}^{<(r+1)\ell+\epsilon} \times I') \subset \mathscr{A}^{<r\ell^+,<\ell^+}_{r/(r+1)},$$

and J_r takes both $\mathscr{A}^{<(r+1)\ell+\epsilon} \times \partial I'$ and $\mathscr{A}^{<(r+1)\ell-2\epsilon} \times I'$ into the subset

$$\mathscr{A}^{<r\ell^+,<\ell^+}_{r/(r+1)} - \phi_{r/(r+1)}(\mathscr{F}^{[r\ell^-,r\ell^+),[\ell^-,\ell^+)}_{\mathscr{A}}).$$

14.5
Recall from Proposition 13.2 that the negative bundle Γ_r over Σ_r is orientable if M is orientable, and that the exponential defines a diffeomorphism e_r of a sufficiently small disk bundle and its bounding sphere bundle, $(D\Gamma_r, \partial D\Gamma_r)$ onto a submanifold with boundary, $(\Sigma_r^-, \partial\Sigma_r)$ in $\Lambda^{\le r\ell}$, such that $e_r(D\Gamma_r - \Sigma_r) \subset \Lambda^{<r\ell}$ (where Σ_r is the zero section). Using the homotopy equivalence $Q : \Lambda^{\le r\ell} \to \mathscr{A}^{\le r\ell}$ of Proposition 2.2, we may assume that $\Sigma_r^- \subset \mathscr{A}^{\le r\ell}$ so we obtain isomorphisms that we also denote by

$$h_r : H^i(\Sigma_r) \cong H^{i+\lambda_r}(\Sigma_r^-, \partial\Sigma_r^-) \cong H^{i+\lambda_r}(\mathscr{A}^{\le r\ell}, \mathscr{A}^{<r\ell}).$$

Moreover, equation (13.5.2) gives a diffeomorphism $\tau : \Sigma_{r+1}^- \to \Sigma_r^- \times_M \Sigma_1^- \cong \Sigma_r^- * \Sigma_1^-$. The following diagram may help in sorting out these different spaces:

$$\begin{array}{ccccc}
\Sigma_{r+1} & \subset & \Sigma_{r+1}^- & \subset & \mathscr{A}^{\le(r+1)\ell} \\
\downarrow & & \downarrow & & \downarrow \eta \\
\Sigma_r * \Sigma_1 & \subset & \Sigma_r^- * \Sigma_1^- & \subset & \mathscr{A}^{\le r\ell,\le\ell}_{r/(r+1)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{F}^{[r\ell^-,r\ell^+),[\ell^-,\ell^+)}_{r/(r+1)} & \subset & \mathscr{F}^{<r\ell^+,<\ell^+}_{r/(r+1)} & \subset & \mathscr{A}^{<r\ell^+,<\ell^+}_{r/(r+1)}
\end{array}$$

In order to compact the notation, for the rest of the section we write

$$H^*(Y, \sim A) \quad \text{for} \quad H^*(Y, Y - A).$$

14.6
We are now in a position to expand the diagram in Theorem 14.3. This is accomplished in Figure 8. Here, $j = a + b + \lambda_r + \lambda_1$ so that $j + n - 1 = a + b + \lambda_{r+1}$. Each of the rectangles in this diagram is obviously commutative except possibly for the portion denoted $\boxed{1}$, which we now explain, as it involves the somewhat mysterious degree shift of 1, and its relationship to the mapping J_r.

$$
\begin{tikzcd}
H^{a}(\Sigma_r) \otimes H^{b}(\Sigma_1) \arrow[r] \arrow[d, "\times"] & H^{a+\lambda_r}(\Sigma_r^-, \sim \Sigma_r) \otimes H^{b+\lambda_1}(\Sigma_1, \sim \Sigma_1) \arrow[d, "\times"] & & H^{a+\lambda_r}(\mathscr{A}^{<r\ell^+}, \mathscr{A}^{<r\ell^-}) \otimes H^{b+\lambda_1}(\mathscr{A}^{<r\ell^+}, \mathscr{A}^{<r\ell^-}) \arrow[ll] \arrow[d, "\times"] \\
H^{a+b}(\Sigma_r \times \Sigma_1) \arrow[r, "\cong"'] \arrow[d] & H^{j}(\Sigma_r^- \times \Sigma_1^-, \sim \Sigma_r \times \Sigma_1) \arrow[d] & & H^{j}\big((\mathscr{A}^{<r\ell^+}, \mathscr{A}^{<r\ell^-}) \times (\mathscr{A}^{<\ell^+}, \mathscr{A}^{<\ell^-})\big) \arrow[ll] \arrow[d] \\
H^{a+b}(\Sigma_r \times_M \Sigma_1) \arrow[r, "\cong"'] \arrow[d, "\phi^*_{r/(r+1)}", "\cong"'] & H^{j}(\Sigma_r^- \times_M \Sigma_1^-, \sim \Sigma_r \times_M \Sigma_1) \arrow[d, "\phi^*_{r/(r+1)}", "\cong"'] & & H^{j}(\mathscr{F}_{\mathscr{A}}^{<r\ell^+, <r\ell^+}, \sim \mathscr{F}_{\mathscr{A}}^{[r\ell^-, r\ell^+), [\ell^-, \ell^+)}) \arrow[ll] \arrow[d, "\phi^*_{r/(r+1)}", "\cong"'] \\
H^{a+b}(\Sigma_r * \Sigma_1) \arrow[r, "\cong"'] \arrow[dd] & H^{j}(\Sigma_r^- * \Sigma_1^-, \sim \Sigma_r * \Sigma_1) \arrow[dd, "\smile \mu_{\Upsilon}", "\S B.4"'] & & H^{j}(\mathscr{F}_{r/(r+1)}^{<r\ell^+, <\ell^+}, \sim \mathscr{F}_{r/(r+1)}^{[r\ell^-, r\ell^+), [\ell^-, \ell^+)}) \arrow[ll] \arrow[d, "\smile \mu_{TM}", "(9.7.2)"'] \\
& & \boxed{1} & H^{j+n}(\mathscr{A}_{r/(r+1)}^{<r\ell^+, <\ell^+}, \sim \mathscr{F}_{r/(r+1)}^{[r\ell^-, r\ell^+), [\ell^-, \ell^+)}) \arrow[d, "J_r^*", "(9.9.1)"'] \\
H^{a+b}(\Sigma_{r+1}) \arrow[r, "\cong"'] \arrow[d, "\cong"'] & H^{j+n-1}(\Sigma_r^- * \Sigma_1^-, \sim \Sigma_{r+1}) \arrow[d, "\cong", "((13.5.2))"'] & & H^{j+n}\big((\mathscr{A}^{<(r+1)\ell^+}, \mathscr{A}^{<(r+1)\ell^-}) \times (I', \partial I')\big) \\
H^{a+b}(\Sigma_{r+1}) \arrow[r, "\cong"'] & H^{j+n-1}(\Sigma_{r+1}^-, \sim \Sigma_{r+1}) \arrow[rr, "\cong"'] & & H^{j+n-1}(\mathscr{A}^{\leq(r+1)\ell}, \sim \Sigma_{r+1}) \arrow[u, "\smile \mu_{\mathrm{O}}", "\cong"']
\end{tikzcd}
$$

Figure 8. ⊛-product

The figure-eight space $\mathscr{F}$ has a normal bundle in Λ which is isomorphic to the (pullback of the) normal bundle of the diagonal in $M \times M$ and hence to the tangent bundle TM of M. Its Thom class is denoted μ_{TM} and the Thom isomorphism (9.7.2) is given by the cup product with this Thom class. The normal bundle of Σ_{r+1} in $\Sigma_r \times_M \Sigma_1$ is denoted Υ. The Gysin mapping (labeled §B.4 in Figure 8) is given by the cup product with the Thom class μ_Υ. The Künneth isomorphism at the lower right corner of the diagram is given by the cup product with the generator of $H^1(I', \partial I')$ which may be identified with the Thom class $\mu_{\mathbf{O}}$ of the trivial one-dimensional bundle $\mathbf{O}$ on the interval I'.

So to prove that $\boxed{1}$ commutes, we need to compare the Thom class μ_{TM} with the product of Thom classes $\mu_\Upsilon \cup \mu_{\mathbf{O}}$. It suffices to construct a vector bundle isomorphism $J_r^*(TM) \cong \Upsilon \oplus \mathbf{O}$.

The critical set Σ_{r+1} is a submanifold of codimension $n-1$ in $\Sigma_r * \Sigma_1$. A point in the latter space is an r-fold iterate of a prime closed geodesic followed by a prime closed geodesic with the same basepoint, all parametrized proportionally with respect to arclength, so it is determined by a triple (p, u, v), where $p \in M$ and $u, v \in S_p$ are unit tangent vectors at p. This point lies in Σ_{r+1} if and only if $u = v$. It follows that the normal bundle Υ of Σ_{r+1} in $\Sigma_r * \Sigma_1$ may be naturally identified with the bundle $\ker(d\pi)$ of tangents to the fibers of the projection $\pi : SM \to M$. But there is another way to view this bundle.

Let ν be the tautological (trivial) bundle over SM whose fiber at the point $v \in S_p$ is the 1-dimensional span $\langle v \rangle \subset T_pM$. Then $\pi^*(TM) \cong \nu \oplus \nu^\perp$, where $\nu^\perp$ is the bundle whose fiber over $v \in S_p$ is $v^\perp$. For any $v \in S_p$, the inclusion of the unit tangent sphere $S_p \subset T_pM$ induces an injection $\ker(d\pi) \hookrightarrow T_pM$ whose image is $\nu^\perp$. In this way, we obtain a canonical isomorphism $\Upsilon \cong \nu^\perp$ and therefore an isomorphism $\pi^*(TM) \cong \Upsilon \oplus \nu$.

Consider the restriction $J_r : \Sigma_{r+1} \times I' \to \Lambda$, say, $\beta = J_r(\alpha, s)$. Then $\frac{\partial J_r}{\partial s}(\alpha, s)$ is a vector field along β that is a multiple of the tangent vector β' since the s-factor changes only the parametrization. This gives an isomorphism between $J_r^*(\nu)$ and the trivial 1-dimensional tangent bundle TI' on $\Sigma_{r+1} \times I'$. In summary, we have constructed an isomorphism $J_r^*(TM) \cong \Upsilon \oplus \mathbf{O}$. This completes the proof that the diagram in Theorem 14.3 commutes, so it completes the proof of Theorem 14.2. $\square$

14.7. Level cohomology ring

Continue with the assumption that all geodesics on M are simply periodic with the same prime length ℓ. Define the filtration $H^*(\Lambda, \Lambda_0; G) = I^0 \supset I^1 \supset \cdots$ by

$$I^r = \mathrm{Image}\big(H^*(\Lambda, \Lambda^{\le r\ell}; G) \to H^*(\Lambda, \Lambda_0; G)\big).$$

Each $I^r \subset H^*(\Lambda, \Lambda_0)$ is an ideal (with respect to the $\circledast$-product) and $I^r \circledast I^s \subset I^{r+s}$. Since the Morse function is perfect, it induces an isomorphism

$$\mathrm{Gr}^I H^*(\Lambda, \Lambda_0) \cong \bigoplus_{r\geq 1} H^*(\Lambda^{\leq r\ell}, \Lambda^{<r\ell})$$

between the associated graded ring and the level cohomology ring. Let $H^*(SM; G)$ denote the cohomology ring of the unit sphere bundle of M.

COROLLARY 14.8
The mapping (cf. §13.6)

$$\Psi : H^*(SM; G)[T]_{\geq 1} \to \mathrm{Gr}^I H^*(\Lambda, \Lambda_0; G) = \bigoplus_{r\geq 1} H^*(\Lambda^{\leq r\ell}, \Lambda^{<r\ell})$$

given by

$$\Psi(aT^m) = h_1(a) \circledast \omega^{\circledast(m-1)} \in H^{\deg(a)+\lambda_1+(m-1)b}(\Lambda^{\leq m\ell}, \Lambda^{<m\ell}; G)$$

is an isomorphism of rings. The ring $(H^*(\Lambda, \Lambda_0; G), \circledast)$ is (finitely) generated by the class $\omega \in H^{\lambda_1}(\Lambda, \Lambda_0)$ together with any lift to $H^*(\Lambda, \Lambda_0)$ of $h_1\left(H^*(\Sigma_1)\right)$.

Proof
Just as in Corollary 13.7, the $\circledast$-product on Σ_1 may be identified with the cup product because the diagram in §14.6 commutes, which proves the first statement. The second statement follows. $\square$

14.9. Based loop space
Let $\Omega = \Omega_{x_0} \subset \Lambda$ denote the space of loops in M that are based at x_0. Suppose as above that all geodesics on M are simply periodic with the same prime length ℓ. Since the index growth is maximal (cf. (13.1.1)), the index of each critical point in Ω is the same as that in Λ (cf. Lemma 6.4). The critical set $\Sigma_r^\Omega \subset \Omega$ at level $r\ell$ is parametrized by the unit sphere $S^{n-1} \subset T_{x_0}M$. The arguments of the preceding section may be applied to the based loop space with its product $\circledast$ (cf. Proposition 9.4), and we conclude that *the cohomology algebra $(H^*(\Omega_{x_0}, x_0), \circledast)$ is filtered by the energy and the associated graded algebra is isomorphic to the polynomial algebra $H^*(\Sigma_1^\Omega)[T]$, where $\deg(T) = b = \lambda_1 + n - 1$ and where $H^*(\Sigma_1^\Omega)$ is the cohomology algebra of the sphere S^{n-1}. The restriction mapping $H^*(\Lambda) \to H^*(\Omega)$ induces the mapping on the associated graded algebras*

$$H^*(SM)[T] \to H^*(S^{n-1})[T], \tag{14.9.1}$$

degree	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$H_*(\Lambda)$	$\mathbb{Z}$		$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
C-J-Y				u^0		u^1		u^2		u^3		u^4		u^5		u^6
	a		au		au^2		au^3		au^4		au^5		au^6		au^7	

Figure 9. Homology of ΛS^3

which is determined by the restriction homomorphism

$$H^*(\Sigma_1) \to H^*(\Sigma_1^{\Omega}). \tag{14.9.2}$$

15. Loop products on spheres

15.1

In this section, we explicitly work out the example of $M = S^n$. Throughout Section 15, the coefficient ring G denotes either $\mathbb{Z}$ or $\mathbb{Z}/(2)$. Fix $n \geq 3$, and let $M = S^n$ with the standard metric, having prime geodesics of length ℓ. The index λ_r can be found in [Mo1, equation (5.9), p. 324]; more recent references include [Z1], [B], or [Hi1, p. 103]:

$$\lambda_r = (2r - 1)(n - 1).$$

Set $\lambda := \lambda_1 = n - 1$. Let $i : \Omega \to \Lambda$ be the inclusion of the based loop space with basepoint $x_0 \in M$. The homology of the based loop space can be found, for example, in [Fu]: $H_q(\Omega; G) = G$ if $q = k\lambda$ (for any integer $k \geq 0$) and $H_q(\Omega; G) = 0$ otherwise.

Recall that Σ denotes the unit tangent bundle of $M = S^n$; it is naturally embedded, $\Sigma \to \Lambda$, as the submanifold Σ_1 of prime closed geodesics. Those that start at the basepoint x_0 constitute an $(n-1)$-sphere, $\mathscr{S} = \Sigma_1 \cap \Omega$.

15.2. Homology of $\Lambda(S^3)$

In [CJY] for n odd, the Chas-Sullivan ring of S^n was computed:

$$H_*(\Lambda S^n)[-n] \cong \bigwedge(a) \otimes \mathbb{Z}[u].$$

Here, $\deg(a) = -n$, and $\deg(u) = n - 1$, and $[-n]$ denotes a shift of degree by $-n$. This is illustrated, for $M = S^3$ and $G = \mathbb{Z}$, in Figure 9. The bottom row is the image $i_*(H_*(\Omega; \mathbb{Z}))$ of the homology of the based loop space.

The dark S-shaped boxes are copies of $H_*(\Sigma) = (\mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z})$, the homology of the unit sphere bundle, and they consist of homology classes a with a fixed critical value $\mathsf{cr}(a)$, which illustrates Corollary 13.7. The nonnilpotent class Θ, which comes from $[\Sigma_1]$, is the class $u^2 \in H_7(\Lambda)$.

15.3. Cohomology ring of ΩS^n and ΛS^n

The proof of the following result takes up the rest of Section 15.

THEOREM

For $M = S^n$, up to appropriate shifts in the gradings, the cohomology rings $(H^*(\Omega, x_0; G), \circledast)$ and $(H^*(\Lambda, \Lambda_0; G), \circledast)$ are described as follows.

(1) *For all n, $H^*(\Omega, x_0; G) \cong G[T]_{\geq 2}$.*

(2) *When n is odd,*

$$H^*(\Lambda, \Lambda_0; G) \cong \bigwedge(U) \otimes G[T]_{\geq 2}.$$

(3) *When n is even and $G = \mathbb{Z}/(2)$,*

$$H^*\big(\Lambda, \Lambda_0; \mathbb{Z}/(2)\big) \cong \bigwedge(U) \otimes \mathbb{Z}/(2)[T]_{\geq 2}.$$

(4) *When n is even and $G = \mathbb{Z}$,*

$$H^*(\Lambda, \Lambda_0; \mathbb{Z}) \cong H^*(\Sigma; \mathbb{Z})[T]_{\geq 1} \cong \mathrm{Gr}^I(\Lambda, \Lambda_0; \mathbb{Z}).$$

Here, $G[T]_{\geq r}$ denotes the ideal (T^r) in the ring of polynomials.

LEMMA 15.4

The isomorphism $h_1 : H^i(\Sigma; G) \to H^{i+\lambda}(\Lambda^{\leq \ell}, \Lambda_0; G)$ of equation (14.1.1) lifts to a canonical injection for $0 \leq i \leq 2\lambda + 1$,

$$h_1 : H^i(\Sigma; G) \to H^{i+\lambda}(\Lambda, \Lambda_0; G) \tag{15.4.1}$$

with critical values (or level), $\mathsf{cr}(h_1(a)) = \ell$ for all $a \in H^*(\Sigma; G)$.

Proof

The proof is an easy induction using the fact that the energy is a perfect Morse function, plus the fact that for every integer $q \geq 1$, the rank of $\bigoplus_{k=1}^{\infty} H^q(\Lambda^{\leq (k+1)\ell}, \Lambda^{\leq k\ell}; G)$ is less than or equal to one. □

As above, let $\omega = h_1(\mathbf{1}) \in H^\lambda(\Lambda, \Lambda_0; G)$ be the nonnilpotent class. By Theorem 14.2, the map $\omega \circledast \cdot$ (i.e., multiplication by ω) is injective, so every cohomology class is (a sum of classes) of the form $\omega^a \circledast h_1(g)$, where $g \in H^*(\Sigma; G)$. In order to compute the full cohomology product structure, it therefore suffices to determine all products $h_1(u) \circledast h_1(v)$ for $u, v \in H^*(\Sigma; G)$. We start with the based loop space ΩS^n.

15.5. Cohomology ring of ΩS^n

In this section we surpress mention of the map i^*, so we use the notation $\omega \in H^\lambda(\Omega, x_0; G)$ for the class $i^*(\omega)$. Let $\mathscr{S} \subset \Omega$ denote the set of prime closed geodesics that start at the basepoint; it is diffeomorphic to the unit sphere S^{n-1}. The mapping h_1 of (15.4.1) restricts to an injection

$$h_1^\Omega : H^i(\mathscr{S}; G) \to H^{i+\lambda}(\Omega, x_0; G).$$

Let $g \in H^{n-1}(\mathscr{S}; G)$ be a generator, and set

$$X = h_1^\Omega(g) \in H^\lambda(\Omega, x_0; G).$$

Then $H^*(\Omega, x_0; G)$ is a free G-module with generators $\omega^{\circledast a} \in H^{(2a-1)\lambda}(\Omega, x_0; G)$ for $a \geq 1$ and $\omega^{\circledast b} \circledast X \in H^{(2b+2)\lambda}(\Omega, x_0; G)$ for $b \geq 0$. It remains to compute $X \circledast X$ (see the bottom row in Figure 10).

LEMMA

In the cohomology ring $(H^*(\Omega, x_0; G), \circledast)$, we have

$$X \circledast X = \omega^{\circledast 3}. \tag{15.5.1}$$

Proof

Let $B \to \Omega$ be the space of circles, consisting of pairs (γ, V), where $\gamma \in \Omega$ starts at the basepoint $x_0 \in S^n$, is parametrized proportionally to arclength, and traces out the intersection of $S^n \subset \mathbb{R}^{n+1}$ with an affine 2-plane containing x_0; and where $V \in T_{x_0}S^n$ is a unit tangent vector such that $\gamma'(0) = \lambda V$ for some $\lambda \geq 0$. Fix a unit tangent vector $\mathbf{V} \in T_{x_0}S^n$, and let $A \subset B$ be the set of loops with initial tangent direction given by $\mathbf{V}$. As explained in [Hi4], the homology of Ω is generated over $\mathbb{Z}$ at level ℓ by the classes

$$[A] \in H_m(\Omega, x_0) \qquad \text{and} \qquad [B] \in H_{2m}(\Omega, x_0).$$

Then $[A]$ generates the Pontrjagin ring (see [BS], who essentially ascribe the result to Morse), $[A] \bullet [A] = [B]$, and the element $[A] \bullet [B]^{\bullet k} = [A \bullet B^{\bullet k}]$ generates $H_{(2k+1)m}(\Omega, x_0)$. In particular, $\langle \omega^{\circledast 3}, [A \bullet B \bullet B] \rangle = 1$. We claim that

$$\vee [A \bullet B \bullet B] = [A] \otimes [A \bullet B] + [A \bullet B] \otimes A + [B] \otimes [B]. \tag{15.5.2}$$

This proves equation (15.5.1) because, using (9.1.3),

$$\langle X \circledast X, [A \bullet B \bullet B] \rangle = \langle X \otimes X, \vee [A \bullet B \bullet B] \rangle = \langle X, B \rangle \cdot \langle X, B \rangle = 1.$$

We give a very brief outline of a geometric proof of (15.5.2). Let $\alpha \cdot \beta \cdot \gamma$ be a loop in $A \bullet B \bullet B$ such that none of α, β, γ is trivial. Then $\alpha \cdot \beta \cdot \gamma(t) = x_0$ if and only if $t \in \{0, 1/3, 2/3\}$, and $\alpha \cdot \beta \cdot \gamma$ has a "corner" (discontinuity in the derivative) at $t = 1/3$ if and only if β does not have initial tangent direction $\mathbf{V}$. Similarly, $\alpha \cdot \beta \cdot \gamma$ has a corner at $t = 2/3$ if and only if β and γ do not have the same initial tangent direction. If we cut off the corners where tangent directions do not match up, intersections at x_0 occur only when the tangent directions at $t = 1/3$ or $t = 2/3$ agree.

There is a diagram for the coproduct in $H_*(\Omega, x_0)$ that is analogous to the diagram (8.2.1) for the coproduct in $H_*(\Lambda, \Lambda_0)$. To calculate $\vee([A \bullet B \bullet B])$, we replace $\Lambda \times I$ in this diagram with $A \bullet B \bullet B$ and check that the required transversality holds, obtaining a diagram

$$\begin{tikzcd} & (A \bullet B \bullet B) \times I \arrow[r, "(\mathbf{ev}_0, \mathbf{ev})"] & \{x_0\} \times S^n \\ \Omega \times \Omega & T \arrow[l, "c"'] \arrow[u, hook] \arrow[r] & \{x_0\} \times \{x_0\} \arrow[u, hook] \end{tikzcd} \tag{15.5.3}$$

where $T = (A \bullet A \bullet B) \times \{1/3\} \cup (A \bullet W) \times \{2/3\}$ and where

$$W = \big\{\alpha \cdot \beta : \ \alpha'(0) = \beta'(0)\big\}.$$

The cycle W can be deformed in $\mathscr{F}_\Omega$ into $(A \bullet B) \cup (B \bullet A)$. This is because W may be realized as the preimage of the diagonal under the mapping $B \times B \to S_{x_0} \times S_{x_0}$ which forgets the paths but keeps the unit vectors, where S_{x_0} is the unit sphere in $T_{x_0} S^n$. The diagonal is cobordant within $S_{x_0} \times S_{x_0}$ to $(\{\mathbf{V}\} \times S_{x_0}) \cup (S_{x_0} \times \{\mathbf{V}\})$. Pulling back this cobordism gives a homology between W and $(B \bullet A) \cup (A \bullet B)$. In summary, cut $A \bullet A \bullet B$ at $t = 1/3$, and cut $(A \bullet B \bullet A) \cup (A \bullet A \bullet B)$ at $t = 2/3$ to obtain $[A] \otimes [A \bullet B] + [A \bullet B] \otimes [A] + [A \bullet A] \otimes [B]$. The last term equals $[B] \otimes [B]$. $\square$

As a consequence there are several ways to abstractly describe the ring $(H^*(\Omega, x_0; G), \circledast)$. The most obvious is the following: the ideal generated by (ω, X) in the polynomial ring $G[\omega, X]/(X^2 - \omega^3)$. However, another description is the ideal $G[T]_{\geq 2}$ generated by T^2 in the ring of polynomials. The isomorphism between these descriptions is obtained by setting $\omega = T^2$ and $X = T^3$. This completes the proof of item (1) in Theorem 15.3.

15.6. Odd-dimensional spheres; even spheres with $G = \mathbb{Z}/(2)$

If n is odd, set $G = \mathbb{Z}$ or $\mathbb{Z}/(2)$. If n is even, set $G = \mathbb{Z}/(2)$. Under these assumptions, the cohomology of the unit tangent bundle Σ is a free module over G with generators

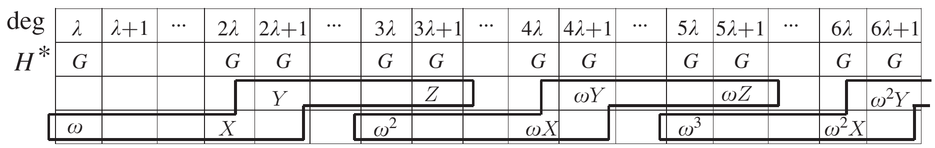

Figure 10. $H^*(\Lambda, \Lambda_0; G)$

$1 \in H^0$, $x \in H^\lambda$, $y \in H^{\lambda+1}$, and $z \in H^{2\lambda+1}$. Then

$$x \smile y = z, \tag{15.6.1}$$

and there are no other nontrivial cup products in $H^*(\Sigma; G)$. Applying the isomorphism h_1 of equation (15.4.1), we obtain classes $\omega \in H^\lambda$, $X \in H^{2\lambda}$, $Y \in H^{2\lambda+1}$, and $Z \in H^{3\lambda+1}$, respectively, in the cohomology $H^*(\Lambda, \Lambda_0; G)$ (see Figure 10). We need to compute all products among these four classes.

Figure 10 also illustrates the mappings (where $i_!$ is the Gysin map),

$$H^q(\Omega, x_0; G) \xleftarrow{i^*} H^q(\Lambda, \Lambda_0; G) \xleftarrow{i_!} H^{q-\lambda}(\Omega, x_0; G).$$

Using the spectral sequence for the map $\Lambda \to \Omega$, as described in [CJY] (see also [Bro], [S], [M]), we find the following: If $q = k\lambda$ ($k \geq 1$), then i^* is an isomorphism. Thus i^* kills the top row, and it maps the bottom row isomorphically to $H^*(\Omega, x_0; G)$. If $q = k\lambda + 1$ ($k \geq 2$), then $i_!$ is an isomorphism and its image is the top row. In particular,

$$i_!(i^*\omega) = Y \qquad \text{and} \qquad i_!(i^*X) = Z. \tag{15.6.2}$$

In the cohomology of the based loop space, we have

$$X \circledast X = \omega^{\circledast 3} \tag{15.6.3}$$

so the same holds in $H^*(\Lambda, \Lambda_0; G)$. Each dark S-shaped box in Figure 10 is a copy of $H^*(\Sigma; G)$. It consists of elements $u \in H^*(\Lambda, \Lambda_0; G)$ with the same critical value, $\mathsf{cr}(u)$. By Theorem 14.3 and equations (15.6.1) and (9.2.1),

$$X \circledast Y = Y \circledast X = \omega \circledast Z. \tag{15.6.4}$$

For $n \geq 4$, we also have

$$Z \circledast Z = Y \circledast Y = Y \circledast Z = 0 \tag{15.6.5}$$

since the cohomology is trivial in degrees congruent to 2 (mod λ). For $n = 3$, the same holds because, for example, $Y \circledast Y = a\omega \circledast \omega \circledast X$ for some $a \in G$. But $i^*(\omega \circledast \omega \circledast X) \neq 0$, while $i^*(Y \circledast Y) = i^*(Y) \circledast i^*(Y) = 0$. So $a = 0$.

Finally, use the projection formula (9.4.2) and equation (15.6.2) to obtain

$$\begin{aligned} X \circledast Z &= X \circledast i_!(X) = i_!\big(i^*(X \circledast X)\big) \\ &= i_!\big(i^*(\omega^{\circledast 3})\big) = \omega^{\circledast 2} \circledast Y. \end{aligned} \tag{15.6.6}$$

This gives parts (2) and (3) of Theorem 15.3 by setting $\omega = T^2$, $X = T^3$, $Y = U \otimes T^2$, and $Z = U \otimes T^3$. We remark that the products (15.6.3) and (15.6.6) are "supra-level"; that is, $\mathsf{cr}(X \circledast X) = 3\ell > 2\mathsf{cr}(X)$ and $\mathsf{cr}(X \circledast Z) = 3\ell > \mathsf{cr}(X) + \mathsf{cr}(Z)$.

15.7. Even-dimensional spheres, $\mathbb{Z}$-coefficients

Let $n \geq 4$ be even. The cohomology of the unit tangent bundle is

$$H^*(\Sigma; \mathbb{Z}) \cong \mathbb{Z}[0] \oplus \mathbb{Z}/(2)[-n] \oplus \mathbb{Z}[1-2n],$$

where $\mathbb{Z}[-r]$ denotes a copy of $\mathbb{Z}$ in degree r. This gives rise to classes $\omega \in H^\lambda(\Lambda, \Lambda_0)$, $Y \in H^{2\lambda+1}(\Lambda, \Lambda_0)$, and $Z \in H^{3\lambda+1}(\Lambda, \Lambda_0)$, respectively. For the same reasons as above,

$$Z \circledast Z = Y \circledast Y = Y \circledast Z = 0.$$

Part (4) of Theorem 15.3 follows; a more precise version is the following.

SCHOLIUM

The injection h_1 of (15.4.1) induces an isomorphism of graded rings

$$h : H^*(\Sigma; G)[T]_{\geq 1} \to H^*(\Lambda, \Lambda_0; G)[-m]$$

given by $h(aT^k) = h_1(a) \circledast \omega^{\circledast(k-1)}$, where $\deg(T) = 2m$. The associated critical value is $\mathsf{cr}(h(aT^k)) = k\ell$ for any $a \in H^*(\Sigma; G)$, and the product $\circledast$ is level-preserving; that is, if $u, v \in H^*(\Lambda, \Lambda_0; G)$ and $u \circledast v \neq 0$, then $\mathsf{cr}(u \circledast v) = \mathsf{cr}(u) + \mathsf{cr}(v)$.

This formula says, for example, that the element xT^2 has degree $m + 4m$, and it maps to $X \circledast \omega \in H^{4m}(\Lambda, \Lambda_0)$ which is the degree $5m$ part of $H^*(\Lambda, \Lambda_0)[-m]$.

15.8. Remark

The computations in this section were significantly simplified by the unique lift of $H^*(\Sigma) \cong H^*(\Lambda^{\leq \ell}, \Lambda_0)$ to $H^*(\Lambda, \Lambda_0)$ as provided by Lemma 15.4. But on a manifold with all geodesics closed, even without such a lift, the difference between two lifts of a given class in $H^*(\Lambda^{\leq \ell}, \Lambda_0)$ is a class $a \in H^*(\Lambda, \Lambda_0)$ such that $\mathsf{cr}(a) \geq 2\ell$. Thus the “leading term” of the product of two classes from $H^*(\Lambda^{\leq \ell}, \Lambda_0)$ is independent of the lift, with the result that the associated graded ring $\mathrm{Gr}^I H^*(\Lambda, \Lambda_0)$ has a canonical structure, which is described in Corollary 14.8.

16. Three counterexamples

16.1

The manifold S^3 illustrates the fact that the cup product on $H^*(\Lambda)$ is poorly behaved with respect to critical levels (cf. §10.3). From Figure 9, it follows that the inclusion $i : \Omega \to \Lambda$ of the based loop space induces a surjection $H^*(\Lambda, \Lambda_0) \to H^*(\Omega, \Omega_0)$ and an isomorphism

$$i^* : H^{2q}(\Lambda, \Lambda_0) \to H^{2q}(\Omega, \Omega_0) \cong \mathbb{Z}$$

for all $q \geq 0$. Denote by $\omega \in H^2(\Lambda, \Lambda_0)$ the nonnilpotent cohomology class; hence $\mathsf{cr}(\omega) = \lambda_1 = 2$. It is shown in [BT, p. 205] that the generator $i^*(\omega)$ of $H^2(\Omega, \Omega_0)$ satisfies

$$i^*(\omega) \smile i^*(\omega) \neq 0$$

from which it follows that $\omega \smile \omega \neq 0$. But since $\deg(\omega \smile \omega) = 4 < \lambda_2 = 6$, we may conclude that $\mathsf{cr}(\omega \smile \omega) = \lambda_1 = 2$. In particular, $\mathsf{cr}(\omega \smile \omega) \ngeq \mathsf{cr}(\omega) + \mathsf{cr}(\omega)$.

16.2

For $M = S^3$, the Chas-Sullivan product and Sullivan’s coproduct operation $\vee$ do *not* satisfy the following relation from [Su1, p. 349]:

$$\vee(x * y) = (x \otimes 1) * \vee(y) + \vee(x) * (1 \otimes y). \tag{16.2.1}$$

To see this, take $x = y = \Theta \in H_7(\Lambda)$ to be the nonnilpotent class. The standard representative for Θ is the space of all circles on S^3, and it consists of simple loops. This implies that $\vee\Theta = 0$. (One can also check that the cohomology class Z which is dual to Θ cannot be expressed as a nontrivial product of two classes.) Consequently, the right-hand side of (16.2.1) is zero.

On the other hand, the class $\omega \circledast Z$, which is dual to $\Theta * \Theta$, can be expressed as a nontrivial product in four ways:

$$\omega \circledast Z = Z \circledast \omega = X \circledast Y = Y \circledast X$$

in the notation of §15.2. The classes $a * u$, $a * u^{*2}$, u, and u^{*2} are dual to ω, X, Y, and Z, respectively. So the left-hand side of equation (16.2.1) is

$$\vee(\Theta * \Theta) = a * u \otimes u^{*2} + u^{*2} \otimes a * u + a * u^{*2} \otimes u + u \otimes a * u^{*2}.$$

16.3

The Pontrjagin product is not necessarily (sign-) commutative, although the Chas-Sullivan product is (sign-) commutative. Within the free loop space Λ, there is "enough room" to move two composed loops into their composition in the opposite order. The proof of [CS, Theorem 3.3] is reproduced in Proposition 5.2. A similar phenomenon occurs with the cohomology product $\circledast$. In fact, for $x \in H^i$ and $y \in H^j$, equation (9.2.1),

$$y \circledast x = (-1)^{(i+n-1)(j+n-1)} x \circledast y,$$

holds in $H^*(\Lambda, \Lambda_0)$ but fails in $H^*(\Omega, x_0)$ when $M = S^n$, n is even, and $x = y = X$, in the notation of §15.5.

17. Related products

17.1

Composing the Künneth isomorphism with the action $S^1 \times \Lambda \to \Lambda$ gives a map

$$\Delta_* : H_i(\Lambda; G) \to H_{i+1}(\Lambda; G) \qquad \text{and} \qquad \Delta^* : H^i(\Lambda; G) \to H^{i-1}(\Lambda; G)$$

for any coefficient group G. Then Chas and Sullivan [CS] define $H_i(\Lambda) \times H_j(\Lambda) \xrightarrow{\{\cdot,\cdot\}} H_{i+j-n+1}(\Lambda)$ such that

$$\{\sigma, \delta\} = (-1)^{|\sigma|} \Delta_*(\sigma * \delta) - (-1)^{|\sigma|} \Delta_*(\sigma) * \delta - \sigma * \Delta_*(\delta), \tag{17.1.1}$$

where $|\sigma| = i - n$ if $\sigma \in H_i(\Lambda)$. They prove that the bracket is (graded) anticommutative, it satisfies the (graded) Jacobi identity, and it is a derivation in each variable; that is,

(1) $\{\sigma, \tau\} = -(-1)^{(|\sigma|+1)(|\tau|+1)} \{\tau, \sigma\}$;

(2) $\{\sigma, \{\tau, \omega\}\} = \{\{\sigma, \tau\}, \omega\} + (-1)^{(|\sigma|+1)(|\tau|+1)} \{\tau, \{\sigma, \omega\}\}$;

(3) $\{\sigma, \tau * \omega\} = \{\sigma, \tau\} * \omega + (-1)^{|\tau|(|\sigma|+1)} \tau * \{\sigma, \omega\}$.

Since Δ_* preserves the energy, it follows that *the bracket operation is also defined on the relative homology groups $\check{H}_*(\Lambda^{\le a}, \Lambda^{\le a'}; G)$ and it satisfies the energy estimates of Proposition 5.3.*

Similarly, we may define $H^i(\Lambda, \Lambda_0) \times H^j(\Lambda, \Lambda_0) \xrightarrow{\{\cdot,\cdot\}} H^{i+j+n-2}(\Lambda, \Lambda_0)$ by

$$\{\tau, \omega\} = (-1)^{|\tau|} \Delta^*(\tau \circledast \omega) - (-1)^{|\tau|} \Delta^*(\tau) \circledast \omega - \tau \circledast \Delta^*(\omega), \tag{17.1.2}$$

where $|\tau| = i + n - 1$ if $\tau \in H^i(\Lambda)$.

THEOREM 17.2
The cohomology bracket satisfies the following for any $\sigma, \tau, \omega \in H^*(\Lambda, \Lambda_0)$:
(A) $\{\tau, \omega\} = -(-1)^{(|\tau|+1)(|\omega|+1)} \{\omega, \tau\}$;
(B) $\{\sigma, \{\tau, \omega\}\} = \{\{\sigma, \tau\}, \omega\} + (-1)^{(|\tau|+1)(|\omega|+1)} \{\tau, \{\sigma, \omega\}\}$;
(C) $\{\sigma, \tau \circledast \omega\} = \{\sigma, \tau\} \circledast \omega + (-1)^{|\tau|(|\sigma|+1)} \tau \circledast \{\sigma, \omega\}$.

Proof
Part (A) follows directly from the definition. The proof of parts (B) and (C) follows the ideas in [CS] (translated to our context) and appear in Appendix E. □

17.3. Nondegenerate case
As in Theorem 12.3, assume that the manifold M is orientable, γ is a closed geodesic such that all of its iterates are nondegenerate, and assume that the negative bundle Γ_r is orientable for all r (cf. Lemma 6.5). Let $a = L(\gamma)$. Assume that $r \geq 2$. Let $\sigma_r, \overline{\sigma}_r, \tau_r, \overline{\tau}_r$ be the local (level) homology and cohomology classes described in equation (12.2.1). In the local level (co)homology group $H(\Lambda^{<ra} \cup \Sigma_r, \Lambda^{<ra})$, we have

$$\Delta_*(\sigma_r) = r\overline{\sigma}_r, \qquad \Delta_*(\overline{\sigma}_r) = 0,$$
$$\Delta^*(\tau_r) = r\overline{\tau}_r, \qquad \Delta^*(\overline{\tau}_r) = 0.$$

Using Theorem 12.3, if $\lambda_{j+k} = \lambda_{j+k}^{\min}$, then in $H_*(\Lambda^{<(j+k)a} \cup \Sigma_{j+k}, \Lambda^{<(j+k)a})$ we have

$$\{\sigma_j, \sigma_k\} = -\big(k + (-1)^{|\sigma_1|} j\big)\sigma_{j+k},$$
$$\{\sigma_j, \overline{\sigma}_k\} = (-1)^{|\sigma_1|} k\overline{\sigma}_{j+k};$$

while if $\lambda_{j+k} = \lambda_{j+k}^{\max}$, then in $H^*(\Lambda^{<(j+k)a} \cup \Sigma_{j+k}, \Lambda^{<(j+k)a})$ we have

$$\{\tau_j, \tau_k\} = \big(-k + (-1)^{|\tau_1|} j\big)\tau_{j+k},$$
$$\{\tau_j, \overline{\tau}_k\} = (-1)^{|\tau_1|} k\overline{\tau}_{j+k}.$$

17.4. Equivariant homology and cohomology
As in [CS], one may consider the $(T = S^1)$-equivariant homology $H_*^T(\Lambda)$ of the free loop space Λ. Let $ET \to BT$ be the classifying space and universal bundle for $T = S^1$; it is the limit of finite-dimensional approximations $S^{2n+1} \to \mathbb{C}P^n$. Let $\pi : \Lambda \times ET \to \Lambda_T = \Lambda \times_T ET$ be the Borel construction. There are Gysin (exact)

sequences (see [Sp, §5.7, p. 260]) with coefficients in $\mathbb{Z}$,

$$\longrightarrow H_{i+1}(\Lambda) \xrightarrow{\pi_*} H_{i+1}^T(\Lambda) \longrightarrow H_{i-1}^T(\Lambda) \xrightarrow{\pi^!} H_i(\Lambda) \longrightarrow ,$$

$$\longrightarrow H_T^i(\Lambda) \xrightarrow[\pi^*]{} H^i(\Lambda) \xrightarrow[\pi_!]{} H_T^{i-1}(\Lambda) \longrightarrow H_T^{i+1}(\Lambda) \longrightarrow .$$

The Chas-Sullivan "string bracket" (homology) product on equivariant homology is defined to be $(-1)^{i-n}$ times the composition

$$H_i^T(\Lambda)\times H_j^T(\Lambda) \xrightarrow{\pi^!\times\pi^!} H_{i+1}(\Lambda)\times H_{j+1}(\Lambda) \xrightarrow{*} H_{i+j+2-n}(\Lambda) \xrightarrow{\pi_*} H_{i+j+2-n}^T(\Lambda);$$

that is, $[\sigma, \delta] = (-1)^{|\sigma|}\pi_*\left(\pi^!(\sigma) * \pi^*(\delta)\right)$. The action of $T = S^1$ preserves the energy function, so *the (homology) string bracket extends to products on relative homology* $\check{H}_*^T(\Lambda^{\le a}, \Lambda^{\le a'})$ *and on level homology* $\check{H}_*^T(\Lambda^{\le a}, \Lambda^{<a})$ *which satisfy the same energy estimates as those in Proposition 5.3.*

Similarly, the (cohomology) product $\circledast$ gives rise to a product in equivariant cohomology as $(-1)^{i+n-1}$ times the composition

$$H_T^i(\Lambda) \times H_T^j(\Lambda) \xrightarrow{\pi^*\times\pi^*} H^i(\Lambda) \times H^j(\Lambda) \xrightarrow{\circledast} H^{i+j+n-1}(\Lambda) \xrightarrow{\pi_!} H_T^{i+j+n-2}(\Lambda),$$

or $\tau \circledcirc \omega = (-1)^{|\tau|}\pi_!\left(\pi^*(\tau) \circledast \pi^*(\omega)\right)$. It also gives products in relative equivariant cohomology $\check{H}_T^*(\Lambda^{\le a}, \Lambda^{\le b})$ with energy estimates as in Proposition 9.5.

17.5

The string bracket is discussed in [Ch] and [CS, p. 24] in the case when M is a surface of genus greater than 1. When $n = 2$, it gives a nontrivial map

$$H_0^T(\Lambda) \times H_0^T(\Lambda) \xrightarrow{[\cdot,\cdot]} H_0^T(\Lambda)$$

which turns out to be a product discovered by Goldman [Go] and Wolpert [Wo] (see the related [T]). In this case, the equivariant cohomology product $\circledcirc$ is also nontrivial in degree zero,

$$H_T^0(\Lambda, \Lambda_0) \times H_T^0(\Lambda, \Lambda_0) \xrightarrow{\circledcirc} H_T^0(\Lambda, \Lambda_0).$$

The group $H_T^0(\Lambda, \Lambda_0)$ can be identified with the set of maps from the set of free homotopy classes of loops in M to the coefficient group G, which take the homotopy class of trivial loops to the identity element in G.

Appendices

A. Čech homology and cohomology

A.1

Throughout this article, the symbols H_i and H^j denote the singular homology and cohomology, while $\check{H}_i$ and $\check{H}^j$ denote the Čech homology and cohomology as described, for example, in [ES, §9], [D, p. 339], [Br2, p. 315] (Čech homology), and [Sp, §6.7, Ex. 14, p. 327] (Čech cohomology).

The problem is that the space $\Lambda^{\leq a}$ and even its finite-dimensional approximation $\mathscr{M}_N^{\leq a}$ might be pathological if a is a critical value of the function F. However, for each regular value $a+\epsilon$, the space $\Lambda^{\leq a+\epsilon}$ has the homotopy type of a finite simplicial complex. Thus one might hope to describe the homology and cohomology of $\Lambda^{\leq a}$ using a limiting process. The Čech homology and cohomology are better behaved under limiting processes than the singular homology and cohomology. Unfortunately, the Čech homology does not always satisfy the exactness axiom for a homology theory (although the Čech cohomology does satisfy the exactness axiom; see the "solenoid" example in [ES]). We now review the relevant properties of these homology theories that are used in this article.

A.2

Let G be an Abelian group. If $A \subset X$ are topological spaces, then the composition of any two homomorphisms in the homology sequence for the pair $\check{H}_*(X, A; G)$ is always zero. If X and A are compact and if G is finite or if G is a field, then the homology sequence for $\check{H}_*(X, A; G)$ is exact.

If a topological space X has the homotopy type of a finite simplicial complex, then the natural transformations $H_j(X; G) \to \check{H}_j(X; G)$ and $\check{H}^j(X; G) \to H^j(X; G)$ are isomorphisms for all j.

By [H, Theorem 3.33], if a topological space X is an increasing union of subspaces $X_1 \subset X_2 \subset \cdots$ and if every compact subset $K \subset X$ is contained in some X_n, then for all j the inclusions $X_n \to X$ induce isomorphisms

$$H_j(X; G) \cong \varinjlim H_j(X_n; G) \qquad \text{and} \qquad \check{H}_j(X; G) \cong \varinjlim \check{H}_j(X_n; G).$$

A.3

Let A be a closed subset of a paracompact Hausdorff space X. Let $U_1 \supset U_2 \supset \cdots$ be a sequence of subsets of X such that $\bigcap_{n=1}^{\infty} U_n = A$. Then the following table describes sufficient conditions that

$$\check{H}^q(A; G) \cong \varinjlim \check{H}^q(U_n; G) \quad \text{and} \quad \check{H}_q(A; G) \cong \varprojlim \check{H}_q(U_n; G):$$

	U_n open	U_n closed
cohomology	no restriction	X is compact
homology	X is a manifold	X is compact

These facts are classical, and the proofs may be found in the textbooks, for example, [ES, §§IX, X], [Sp, §6.6, Theorems 2, 6], [D, Chapter VIII, §§6.18, 13.4, 13.16], and [Br2]. (By [Sp, §6.8, Corollary 8], the Čech cohomology coincides with the Alexander-Spanier cohomology on the class of paracompact Hausdorff spaces.)

For the remainder of this appendix, continue with the notation M, Λ, F, Σ of §2.

LEMMA A.4

Let G be an Abelian group, and let $a \in \mathbb{R}$. Then the natural homomorphisms

$$H_*(\Lambda; G) \to \check{H}_*(\Lambda; G) \qquad \textit{and} \qquad H_*(\Lambda^{<a}; G) \to \check{H}_*(\Lambda^{<a}; G)$$

are isomorphisms. If $a \in \mathbb{R}$ is a regular value of F, or if a is a nondegenerate critical value of F in the sense of Bott, then the morphism $H_*(\Lambda^{\le a}; G) \to \check{H}_*(\Lambda^{\le a}; G)$ is an isomorphism. The same statements hold for Čech cohomology.

Proof

This follows from Proposition 3.3. The space $\Lambda^{<a}$ has the homotopy type of a finite-dimensional manifold, and if a is a regular value, then $\Lambda^{\le a}$ is homotopy equivalent to a finite-dimensional compact manifold with boundary. □

LEMMA A.5

If $a' < a \in \mathbb{R}$, then the inclusion $\Lambda^{\le a} \to \Lambda^{\le a+\epsilon}$ induces canonical isomorphisms

$$\check{H}_i(\Lambda^{\le a}; G) \cong \lim_{0 \longleftarrow \epsilon} H_i(\Lambda^{<a+\epsilon}; G), \tag{A.5.1}$$

$$\check{H}_i(\Lambda^{\le a}, \Lambda^{\le a'}; G) \cong \lim_{0 \longleftarrow \epsilon} H_i(\Lambda^{<a+\epsilon}, \Lambda^{<a'+\epsilon}; G), \tag{A.5.2}$$

with Čech homology on the left and singular homology on the right. If G is a field, if $\alpha \in H_i(\Lambda; G)$, and if $a = \mathsf{cr}(\alpha)$ is its critical level (§4), then there exists $\omega \in \check{H}_i(\Lambda^{\le a}; G)$ which maps to α.

Proof

By Proposition 3.3, the space $\Lambda^{\le a}$ is homotopy equivalent to the finite-dimensional space $\mathscr{M}_N^{\le a}$ which is contained in a manifold. Therefore

$$\check{H}_i(\Lambda^{\le a}) \cong H_i(\mathscr{M}_N^{\le a}) = \lim_{0 \longleftarrow \epsilon} H_i(\mathscr{M}_N^{<a+\epsilon}) \cong \lim_{0 \longleftarrow \epsilon} H_i(\Lambda^{\le a+\epsilon})$$

which proves (A.5.1). The relative case (A.5.2) is similar. Now, suppose that G is a field, and let $b_n \downarrow a = \mathsf{cr}(\alpha)$ be a convergent sequence of regular values of F. Then $\check{H}_i(\Lambda^{\le a}; G)$ is the limit of the sequence of finite-dimensional vector spaces

$$H_i(\Lambda^{\le b_1}) \leftarrow H_i(\Lambda^{\le b_2}) \leftarrow H_i(\Lambda^{\le b_3}) \leftarrow \cdots,$$

and for each $n \ge 1$, there is an element $\omega_n \in H_i(\Lambda^{\le b_n})$ that maps to α. Let $H_n = \mathrm{Image}(H_i(\Lambda^{\le b_n}; G) \to H_i(\Lambda^{\le b_1}; G))$. These form a decreasing chain of finite-dimensional vector spaces that therefore stabilize after some finite point, say,

$$H_N = \mathrm{Image}\big(H_i(\Lambda^{\le b_N}; G) \to H_i(\Lambda^{\le b_1}; G)\big) = \bigcap_{n=1}^{\infty} H_n = \check{H}_i(\Lambda^{\le a}; G).$$

It then suffices to take $\omega \in H_N$ to be the image of $\omega_N \in H_i(\Lambda^{\le b_N}; G)$. $\square$

PROPOSITION A.6

Fix $c \in \mathbb{R}$. Let $U \subset \Lambda$ be a neighborhood of $\Sigma^{=c}$. The inclusions

$$(\Lambda^{<c} \cup \Sigma^{=c}) \cap U \hookrightarrow \Lambda^{<c} \cup \Sigma^{=c} \hookrightarrow \Lambda^{\le c}$$

induce isomorphisms on Čech homology,

$$\begin{tikzcd}
\check{H}_i\big((\Lambda^{<c} \cup \Sigma^{=c}) \cap U, \Lambda^{<c} \cap U; G\big) \arrow[r] \arrow[d, "\beta"] & \check{H}_i(\Lambda^{\le c} \cap U, \Lambda^{<c} \cap U; G) \arrow[d, "\gamma"] & \\
\check{H}_i(\Lambda^{<c} \cup \Sigma^{=c}, \Lambda^{<c}; G) \arrow[r, "\alpha"] & \check{H}_i(\Lambda^{\le c}, \Lambda^{<c}; G) \arrow[r, "\tau"] & \lim\limits_{0 \leftarrow \epsilon} H_i(\Lambda^{<c+\epsilon}, \Lambda^{<c}; G)
\end{tikzcd}$$

Proof

It follows from excision that the relative homology group $\check{H}_i((\Lambda^{<c} \cup \Sigma^c) \cap U, \Lambda^{<c} \cap U)$ is independent of U. Taking $U = \Lambda$ gives the isomorphism β. The same argument applies to the isomorphism γ. The mapping τ is an isomorphism by §A.3. Finally, the mapping α is an isomorphism because the inclusion $(\Lambda^{<c} \cup \Sigma^c, \Lambda^{<c}) \to (\Lambda^{\le c}, \Lambda^{<c})$ is a homotopy equivalence. A homotopy inverse is given by the time t flow $\psi_t : \Lambda^{\le c} \to \Lambda^{\le c}$ of the vector field $-\mathrm{grad}(F)$, for any choice of $t > 0$ (cf. [Kl, §1], [C, §I.3]). $\square$

B. Thom isomorphisms

B.1

The constructions in this article necessitate the use of various relative versions of the Thom isomorphism for finite- and infinite-dimensional spaces in singular and Čech homology and cohomology. In this section, we review these standard facts.

Recall that a neighborhood N of a closed subset X of a topological space Y is a *tubular neighborhood* if there exists a finite-dimensional (normal) real vector bundle $\pi : E \to X$, and a homeomorphism $\psi : E \to N \subset Y$ which takes the zero section to X by the identity mapping. In this case, excising $Y - N$ gives an isomorphism

$$H(E, E - X; G) \cong H(N, N - X; G) \cong H(Y, Y - X; G),$$

where H denotes either singular homology or cohomology (with coefficients in an Abelian group G) and where $E - X$ is the complement of the zero section. Let us take the coefficient group to be $G = \mathbb{Z}$ if the normal bundle E is orientable (in which case, we fix an orientation), and $G = \mathbb{Z}/(2)$ otherwise. The Thom class

$$\mu_E \in H^n(E, E - X; G)$$

is the unique cohomology class that evaluates to 1 on the chosen homology generator of each fiber $\pi^{-1}(x)$. The cup product with this class gives the Thom isomorphism in cohomology,

$$H^i(X; G) \cong H^i(E; G) \to H^{i+n}(E, E - X; G) \cong H^{i+n}(Y, Y - X; G), \quad \text{(B.1.1)}$$

and the cap product with this class gives the Thom isomorphism in homology,

$$H_i(X; G) \cong H_i(E; G) \leftarrow H_{i+n}(E, E - X; G) \cong H_{i+n}(Y, Y - X; G) \quad \text{(B.1.2)}$$

(see [Sp, Chapter 5, §7, p. 259]). The same results hold for Čech homology and cohomology. We need to establish relative versions of these isomorphisms.

PROPOSITION B.2

Let $A \subset X$ be closed subsets of a topological space Y. Assume that X has a tubular neighborhood N in Y corresponding to a homeomorphism $\phi : E \to N$ of a normal bundle $E \to X$ of fiber dimension n. If E is orientable, then choose an orientation and set $G = \mathbb{Z}$ (otherwise, set $G = \mathbb{Z}/(2)$). Then the Thom isomorphism induces an isomorphism

$$H^i(X, X - A; G) \cong H^{i+n}(Y, Y - A; G), \quad \text{(B.2.1)}$$

in singular cohomology, and an isomorphism

$$H_i(X, X - A; G) \cong H_{i+n}(Y, Y - A; G), \quad \text{(B.2.2)}$$

in singular homology. Taking $A = X$ gives Gysin homomorphisms

$$H^i(X; G) \cong H^{i+n}(Y, Y - X; G) \to H^{i+n}(Y; G), \quad \text{(B.2.3)}$$

$$H_{i+n}(Y; G) \to H_{i+n}(Y, Y - X; G) \cong H_i(X; G), \quad \text{(B.2.4)}$$

denoted $h_!$ and $h^!$, respectively, where $h : X \to Y$ denotes the inclusion. If $U \subset Y$ is open, then taking $A = X - X \cap U$ gives Gysin homomorphisms

$$H^i(X, X \cap U; G) \cong H^{i+n}(Y, Y - A; G) \to H^{i+n}(Y, U; G), \tag{B.2.5}$$

$$H_{i+n}(Y, U; G) \to H_{i+n}(Y, Y - A; G) \cong H_i(X, X \cap U; G). \tag{B.2.6}$$

Proof

We describe the argument for (B.2.1); the argument for (B.2.2) is the same, with the arrows reversed. We suppress the coefficient group G in order to simplify the notation in the following argument. Let $E^A = \pi^{-1}(X-A) \subset E$, and let $E^0 = E - X$. The sets E^0, E^A are open in X so they form an excisive pair* giving the excision isomorphism

$$H^i(E^0 \cup E^A, E^0) \xrightarrow{\cong} H^i(E^A, E^0 \cap E^A).$$

The cup product with the Thom class $\mu_E \in H^n(E, E^0; \mathbb{Z})$ gives a mapping

$$H^i(X, X - A) \cong H^i(E, E^A) \to H^{i+n}(E, E^0 \cup E^A)$$

which we claim is an isomorphism. This follows from the five lemma and the exact sequence of the triple

$$E^0 \subset (E^0 \cup E^A) \subset E.$$

In fact, the diagram in Figure 11 commutes.

$$\begin{tikzcd}
H^{i+n}(E, E^0 \cup E^A) \arrow[r] & H^{i+n}(E, E^0) \arrow[r] & \begin{array}{c} H^{i+n}(E^0 \cup E^A, E^0) \\ = H^{i+n}(E^A, E^0 \cap E^A) \end{array} \\
H^i(E, E^A) \arrow[r] \arrow[u] & H^i(E) \arrow[r] \arrow[u, "\cong"'] & H^i(E^A) \arrow[u, "\cong"'] \\
H^i(X, X - A) \arrow[r] \arrow[u, equal] & H^i(X) \arrow[r] \arrow[u, equal] & H^i(X - A) \arrow[u, equal]
\end{tikzcd}$$

Figure 11

*A pair $A, B \subset X$ is excisive if $A \cup B = A^o \cup B^o$, where A^o denotes the relative interior of A in $A \cup B$ (cf. [Sp, p. 188]).

So the left-hand vertical mapping is an isomorphism. Since A is closed in Y, we may excise $Y - N$ from $Y - A$ to obtain an isomorphism

$$H^{i+n}(Y, Y-A) \cong H^{i+n}(N, N-A) \cong H^{i+n}(E, E^0 \cup E^A). \qquad \square$$

B.3

We also need in §7 the following standard facts concerning the Thom isomorphism. Suppose that $E_1 \to A$ and $E_2 \to A$ are oriented vector bundles of ranks d_1 and d_2. If both are oriented, let $G = \mathbb{Z}$ (and choose orientations of each); otherwise, let $G = \mathbb{Z}/(2)$ be the coefficient group for homology. Let $E = E_1 \oplus E_2$. The diagram of projections

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E_1 \arrow[d, "\pi_1"] \\ E_2 \arrow[r, "\pi_2"] & A \end{tikzcd}$$

gives identifications $E \cong \pi_1^*(E_2) \cong \pi_2^*(E_1)$ of the total space E as a vector bundle $\pi_1^*(E_2)$ over E_1 (resp., as a vector bundle $\pi_2^*(E_1)$ over E_2). The Thom class $\mu_2 \in H^{d_2}(E_2, E_2 - A)$ pulls up to a class

$$\pi_1^*(\mu_2) \in H^{d_2}(E, E - E_1)$$

and similarly with the indices reversed. Then the relative cup product

$$\begin{aligned} H^{d_1}(E, E - E_1) \times H^{d_2}(E, E - E_2) \to\ & H^{d_1+d_2}\big(E, (E - E_1) \cup (E - E_2)\big) \\ =\ & H^{d_1+d_2}(E, E - A) \end{aligned}$$

takes $(\pi_1^*(\mu_2), \pi_2^*(\mu_1))$ to the Thom class

$$\mu_E = \pi_1^*(\mu_2) \smile \pi_2^*(\mu_1).$$

Consequently, the Thom isomorphism for E is the composition of the Thom isomorphisms

$$H^i(A) \xrightarrow[\smile\mu_1]{} H^{i+d_1}(E_1, E_1 - A) \xrightarrow[\smile\mu_2]{} H^{i+d_1+d_2}(E, E - A).$$

COROLLARY B.4

In the situation of Proposition B.2, suppose that $A \subset X \subset Y$ are closed sets, suppose that A has a tubular neighborhood in X with oriented normal bundle Υ of rank m, and suppose that X has a tubular neighborhood in Y with oriented normal bundle E of rank n. Then $\mu_{E\oplus\Upsilon} = \pi_\Upsilon^*(\mu_E) \smile \pi_E^*(\mu_\Upsilon)$ is a Thom class in $H^{n+m}(E \oplus \Upsilon, E \oplus \Upsilon - A)$ and the composition of Thom isomorphisms across the bottom, in the diagram

$$\begin{tikzcd} H^i(X) \arrow[rr, "\cong"] \arrow[d] & & H^{i+n}(Y, Y-X) \arrow[d, "\psi"] \\ H^i(A) \arrow[r, "(\mathrm{B.1.1})", "\smile\mu_\Upsilon"'] & H^{i+m}(X, X-A) \arrow[r, "(\mathrm{B.2.1})", "\smile\mu_E"'] & H^{i+n+m}(Y, Y-A) \end{tikzcd}$$

is the Thom isomorphism $\smile (\mu_{E\oplus\Upsilon})$. The diagram gives rise to a Gysin homomorphism $\psi : H^r(Y, Y - X) \to H^{r+m}(Y, Y - A)$ which may be interpreted as the cup product with the Thom class

$$\mu_\Upsilon \in H^m(\Upsilon, \Upsilon - A) \cong H^m(X, X - A) \cong H^m\big(E, E - \pi_E^{-1}(A)\big)$$

in the following sequence of homomorphisms

$$\begin{tikzcd} H^r(Y, Y-X) \arrow[d, "\cong"] & & H^{r+m}(Y, Y-A) \\ H^r(E, E-X) \arrow[r, "\cup\mu_\Upsilon"'] & H^{r+m}\big(E, (E-\pi_E^{-1}(A)) \cup (E-X)\big) = H^{r+m}(E, E-A) \arrow[ur, "\cong"] & \end{tikzcd}$$

C. Proof of Proposition 9.4

As in §9.3, let Ω be the space of loops that are based at the point $x_0 \in M$, and let $i : \Omega \to \Lambda$ denote the inclusion. The relative Gysin homomorphism $i_! : H^a(\Omega, x_0) \to H^{a+n}(\Lambda, \Lambda_0)$ is defined to be the composition (for sufficiently small $\epsilon > 0$),

$$H^a(\Omega, \Omega_{<\epsilon}) = H^a(\Omega, \Omega - \Omega^{\geq\epsilon}) \cong H^{a+n}(\Lambda, \Lambda - \Omega^{\geq\epsilon}) \to H^{a+n}(\Lambda, \Lambda_{<\epsilon}), \tag{C.1.1}$$

where the middle mapping is the Thom isomorphism (B.2.1). Equation 9.4.1 is obvious. The proof of equation (9.4.2) involves a commutative diagram (the notation is explained below),

$$
\begin{tikzcd}[column sep=small]
H^a(\Omega,\Omega_{<\epsilon})\times H^b(\Lambda,\Lambda_{<\epsilon}) \arrow[r, "\tau\times I"] \arrow[d] & H^{a+n}(\Lambda,\Lambda-\Omega^{\geq\epsilon})\times H^b(\Lambda,\Lambda_{<\epsilon}) \arrow[r] \arrow[d] & H^{a+n}(\Lambda,\Lambda_{<\epsilon})\times H^b(\Lambda,\Lambda_{<\epsilon}) \arrow[d] \\
H^{a+b}\big((\Omega,\Omega_{<\epsilon})\times(\Lambda,\Lambda_{<\epsilon})\big) \arrow[r, "\tau"] \arrow[d] & H^{a+b+n}\big((\Lambda,\Lambda-\Omega^{\geq\epsilon})\times(\Lambda,\Lambda_{<\epsilon})\big) \arrow[r] \arrow[dd] & H^{a+b+n}\big((\Lambda,\Lambda_{<\epsilon})\times(\Lambda,\Lambda_{<\epsilon})\big) \arrow[d] \\
H^{a+b}(\mathscr{F}_\Omega,\mathscr{F}_\Omega^{\bullet,<\epsilon}\cup\mathscr{F}_\Omega^{<\epsilon,\bullet}) \arrow[d, equal] & & H^{a+b+n}(\mathscr{F},\mathscr{F}^{\bullet,<\epsilon}\cup\mathscr{F}^{<\epsilon,\bullet}) \arrow[d, equal] \\
H^{a+b}(\mathscr{F}_\Omega,\mathscr{F}_\Omega-\mathscr{F}_\Omega^{\geq\epsilon,\geq\epsilon}) \arrow[r, "\tau"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\boxed{A}"] & H^{a+b+n}(\mathscr{F},\mathscr{F}-\mathscr{F}_\Omega^{\geq\epsilon,\geq\epsilon}) \arrow[r] \arrow[d, "\tau"'] & H^{a+b+n}(\mathscr{F},\mathscr{F}-\mathscr{F}^{\geq\epsilon,\geq\epsilon}) \arrow[d, "\tau"] \\
H^{a+b+n}(\Omega,\Omega-\mathscr{F}_\Omega^{\geq\epsilon,\geq\epsilon}) \arrow[r, "\tau"] \arrow[d, "J^*"'] & H^{a+b+2n}(\Lambda,\Lambda-\mathscr{F}_\Omega^{\geq\epsilon,\geq\epsilon}) \arrow[r] \arrow[d, "J^*"'] & H^{a+b+2n}(\Lambda,\Lambda-\mathscr{F}^{\geq\epsilon,\geq\epsilon}) \arrow[d, "J^*"] \\
H^{a+b+n}\big((\Omega,\Omega_{<\epsilon})\times(I,\partial I)\big) \arrow[r, "\tau"] \arrow[d] & H^{a+b+2n}\big((\Lambda,\Lambda-\Omega^{\geq\epsilon})\times(I,\partial I)\big) \arrow[r] \arrow[d] & H^{a+b+2n}\big((\Lambda,\Lambda_{<\epsilon})\times(I,\partial I)\big) \arrow[d] \\
H^{a+b+n-1}(\Omega,\Omega_{<\epsilon}) \arrow[r, "\tau"] & H^{a+b+n-1}(\Lambda,\Lambda-\Omega^{\geq\epsilon}) \arrow[r] & H^{a+b+2n-1}(\Lambda,\Lambda_{<\epsilon})
\end{tikzcd}
$$

In this diagram, we use the standard notation for pairs,

$$(A,B)\times(C,D)=\big(A\times C,(A\times D)\cup(B\times C)\big).$$

The symbol I denotes the unit interval $[0,1]$, and $\partial I=\{0,1\}$ is its boundary. The symbol $\mathscr{F}_\Omega^{\bullet,<\epsilon}$ denotes the figure-eight space of loops based at x_0 for which the first loop is arbitrary and the second has energy less than ϵ and similarly for $\mathscr{F}_\Omega^{\geq\epsilon,\geq\epsilon}$. The mappings denoted τ are Thom isomorphisms.

The diagram commutes because it consists of restriction maps and Thom isomorphisms; the square labeled $\boxed{A}$ commutes by Corollary B.4. It is confusing but straightforward to check that the mapping $J:\Lambda\times I\to\Lambda$ takes both $\Lambda\times\partial I$ and $(\Lambda-\Omega^{\geq\epsilon})\times I$ into the set $\Lambda-\mathscr{F}_\Omega^{\geq\epsilon,\geq\epsilon}$, so the middle vertical J^* mapping is defined.

Let us start with an element (x,y) in the upper left-hand corner. We claim that its image down the left-hand column is $x\circledast i^*(y)$. In fact, one could add an additional column to the left of the diagram, which begins

$$
\begin{aligned}
H^a(\Omega,\Omega_{<\epsilon})\times H^b(\Omega,\Omega_{<\epsilon}) &\to H^{a+b}\big((\Omega,\Omega_{<\epsilon})\times(\Omega,\Omega_{<\epsilon})\big)\\
&\to H^{a+b}(\mathscr{F}_\Omega,\mathscr{F}_\Omega^{\bullet,<\epsilon}\cup\mathscr{F}_\Omega^{<\epsilon,\bullet})
\end{aligned}
$$

and which coincides with the left column of the diagram from the third entry on downwards. Mapping (x,y) to the new column gives $(x,i^*(y))$. Following this element down the new column gives $x\circledast i^*(y)$.

Therefore, following (x,y) down the first column and across the bottom gives the element $i_!(x\circledast i^*(y))$. Following (x,y) across the top gives $(i_!(x),y)$ and continuing down the right-hand column gives $i_!(x) \circledast y$. This completes the proof of equation (9.4.2).

Now, let us prove equation (9.4.3): $\langle a \circledast b, Z\rangle = \langle a, [X]\rangle \cdot \langle b, [Y]\rangle$. Let (a, b) originate in the upper right corner of the following diagram. Then going down the right-hand column and across the bottom gives $\langle a \circledast b, Z\rangle$:

$$\begin{tikzcd}
H^i(X) \times H^j(Y) \arrow[d] & H^i(\Omega, x_0) \times H^j(\Omega, x_0) \arrow[l] \arrow[d] \\
H^{i+j}(X \times Y) \arrow[d, "="] & H^{i+j}(\mathscr{F}_\Omega, \Omega \times x_0 \cup x_0 \times \Omega) \arrow[l] \arrow[d, "="] \\
H^{i+j}(X \bullet Y) \arrow[dd, "\cup J^*(\mu_{\mathscr{F}})"'] & H^{i+j}(\mathscr{F}_\Omega, \mathscr{F}_\Omega - \mathscr{F}_\Omega^{>0,>0}) \arrow[l] \arrow[d, "\cup\mu_{\mathscr{F}}"', "\text{(B.2.5)}"] \\
 & H^{i+j+n}(\Omega, \Omega - \mathscr{F}_\Omega^{>0,>0}) \arrow[d, "J^*"] \\
H^{i+j+n}(Z \times I, Z \times \partial I) \arrow[d, "\cong"] & H^{i+j+n}\big(\Omega \times I, \partial(\Omega \times I)\big) \arrow[l] \arrow[d, "\cong"] \\
H^{i+j+n-1}(Z) & H^{i+j+n-1}(\Omega, x_0) \arrow[l]
\end{tikzcd}$$

We claim that going across the top and down the left side gives $\langle a, [X]\rangle \cdot \langle b, [Y]\rangle$. To see this, it suffices to know that the Gysin map $H^{i+j}(X \bullet Y) \to H^{i+j+n-1}(Z \times I, Z \times \partial I)$ is given by the cup product with $J^*(\mu_{\mathscr{F}})$. In fact, the following diagram is a fiber (or "Cartesian") square so the Thom class of $X \bullet Y$ in $Z \times I$ is $J^*(\mu_{\mathscr{F}})$:

$$\begin{tikzcd}
Z \times [0,1] \arrow[r, "J"] & \Omega \arrow[r] & \Lambda \\
(X \bullet Y) \times \left\{\frac{1}{2}\right\} \arrow[u] \arrow[r, "J"] & \mathscr{F}_\Omega \arrow[u] \arrow[r] & \mathscr{F} \arrow[u, "\phi_{1/2}"']
\end{tikzcd}$$

This completes the proof of equation (9.4.3). $\square$

D. Critical submanifolds

D.1

In this article, we need to use a particular form (Theorem D.2) of the fundamental lemma of Morse theory. We provide a proof, since the exact statement does not appear in the literature, but related statements may be found in [Mo1], [Bo2], [Mi], [Bo1], [L], [Kl, Corollary 2.4.11, §3.2], and [R]. (By Lemma A.4, the homology groups that appear in the following theorem may be taken to be either Čech or singular.)

THEOREM D.2

Let X be a Riemannian Hilbert manifold, and let $f : X \to \mathbb{R}$ be a smooth function that satisfies condition C of Palais and Smale. Let Σ be a finite-dimensional connected nondegenerate critical submanifold in the strong sense that the eigenvalues of $d^2 f$ on the normal bundle of Σ are bounded away from zero. Let $\lambda < \infty$ be the index of Σ, and let $d < \infty$ be the dimension of Σ. Let $c = f(\Sigma)$ be the critical value. Suppose that there is a smooth connected manifold V with

$$\dim(V) = \dim(\Sigma) + \lambda$$

and smooth embeddings

$$\Sigma \xrightarrow[\sigma]{} V \xrightarrow[\rho]{} X$$

so that $\rho \circ \sigma : \Sigma \to \Sigma$ is the identity and $f \circ \rho(x) < c$ whenever $x \in V - \Sigma$. Then ρ induces an isomorphism

$$H_i(V, V - \Sigma; G) \xrightarrow{\cong} H_i(X^{<c} \cup \Sigma, X^{<c}; G)$$

for any coefficient group G and for all $i \geq 0$. In fact, ρ induces a local diffeomorphism of pairs $(V, V - \Sigma) \cong (\Sigma^-, \Sigma^- - \Sigma)$, where Σ^- is defined below.

Composing with the Thom isomorphism (B.1.2) gives a further isomorphism

$$H_i(V, V - \Sigma; G) \cong H_{i-\lambda}(\Sigma; G),$$

where $G = \mathbb{Z}$ if the normal bundle of Σ in V is orientable, and $G = \mathbb{Z}/(2)$ otherwise.

D.3. Proof

This essentially follows from [C, Theorem 7.3, p. 72] or [Kl, Corollary 2.4.8, Proposition 2.4.9]. The tangent bundle $TX|\Sigma$ decomposes into an orthogonal sum of vector bundles $\Gamma^+ \oplus \Gamma^0 \oplus \Gamma^-$ spanned by the positive, null, and negative eigenvectors (respectively) of the self-adjoint operator associated to $d^2 f$. The inclusion $\Sigma \to X$ induces an isomorphism $T\Sigma \cong \Gamma^0$ so we may identify the normal bundle of Σ in X with $\Gamma^+ \oplus \Gamma^-$.

For ϵ sufficiently small, the restriction of the exponential map

$$\exp : (\Gamma^+ \oplus \Gamma^-)_\epsilon \to X$$

is a homeomorphism onto some neighborhood $U \subset X$. Let $\Sigma^- = \exp(\Gamma^-_\epsilon) \subset X$. This submanifold is often described as “the unstable manifold that hangs down from Σ,” for if ϵ is sufficiently small and if $0 \neq a \in \Gamma^-_\epsilon$, then $f(\exp(a)) < c$. Its tangent bundle, when restricted to Σ, is

$$T\Sigma^-|\Sigma = \Gamma^0 \oplus \Gamma^-.$$

The projection $\Gamma^+ \oplus \Gamma^- \to \Gamma^-$ induces a projection $\pi : U \to \Sigma^-$ which is homotopic to the identity by the homotopy

$$\pi_t\big(\exp(a \oplus b)\big) = \exp(ta \oplus b),$$

where $t \in [0, 1]$, $a \in \Gamma^+_\epsilon$, $b \in \Gamma^-_\epsilon$, and where π_1 is the identity and $\pi_0 = \pi$. Moreover, the kernel of the differential $d\pi(x) : T_xX \to T_x\Sigma^-$ at any point $x \in \Sigma$ is precisely the positive eigenspace, $\Gamma^+_x \subset T_xX$. Let us identify the manifold V with its image $\rho(V) \subset X$ so that $TV|\Sigma \subset \Gamma^0 \oplus \Gamma^-$. It follows that the restriction of π,

$$\pi : V \cap U \to \Sigma^-, \tag{D.3.1}$$

has a nonvanishing differential at every point $x \in \Sigma \subset V$, and consequently the mapping (D.3.1) is a diffeomorphism in some neighborhood of Σ. It follows that π induces an isomorphism

$$\pi_* : H_i(V, V - \Sigma) \to H_i(\Sigma^-, \Sigma^- - \Sigma) \cong H_i(D\Gamma^-, \partial D\Gamma^-), \tag{D.3.2}$$

where $D\Gamma^-$ denotes a sufficiently small disk bundle in Γ^- and $\partial D\Gamma^-$ is its boundary. On the other hand, by Morse theory (the above-mentioned [C, Theorem 7.3] or [Kl, Proposition 2.4.9]), the space $U^{\le c+\delta}$ has the homotopy type of the adjunction space $U^{\le c-\delta} \cup_{\partial D\Gamma^-} D\Gamma^-$. This gives the standard isomorphism of Morse theory,

$$H_i(D\Gamma^-, \partial D\Gamma^-) \cong H_i(U^{\le c+\delta}, U^{\le c-\delta}). \tag{D.3.3}$$

All these isomorphisms fit together in a commutative diagram:

$$\begin{tikzcd}
H_i(U^{\le c}, U^{<c}) \arrow[r, "\pi_*"] & H_i(\Sigma^-, \Sigma^- - \Sigma) \arrow[r, "i_*"] & H_i(U^{\le c}, U^{<c}) \arrow[r, "(A.6)"] \arrow[d, "(A.6)"] & H_i(U^{<c} \cup \Sigma, U^{<c}) \\
H_i(V, V - \Sigma) \arrow[u, "\rho_*"] \arrow[ur, "(D.3.2)"] \arrow[r, "(D.3.2)"'] & H_i(D\Gamma^-, \partial D\Gamma^-) \arrow[u] \arrow[ur] \arrow[r, "(D.3.3)"'] & H_i(U^{\le c+\delta}, U^{\le c-\delta}) &
\end{tikzcd}$$

Each of the arrows labeled by an equation number is an isomorphism, so i_* is an isomorphism. But $i_*\pi_*$ is the identity, so π_* is also an isomorphism and hence also ρ_*. $\square$

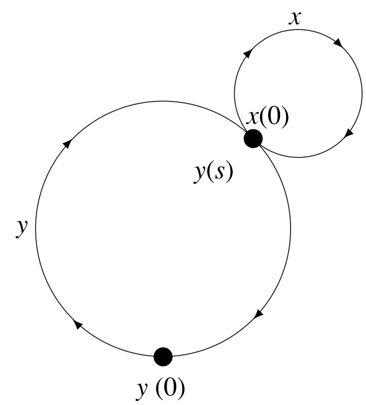

Figure 12. Joining two loops at time s

E. Proof of Theorem 17.2

E.1

The proof of Theorem 17.2 involves a second construction of the cohomology bracket, along the same lines as the definition of the $\circledast$-product. As in §2, let Λ be the free loop space of mappings $x : \mathbb{R}/\mathbb{Z} \to M$ (or $x : [0,1] \to M$). For the purposes of this appendix only, let $\widehat{\Lambda}$ be the free loop space of H^1-mappings $\mathbb{R}/2\mathbb{Z} \to M$ (or $[0,2] \to M$). If $x, y \in \Lambda$, if $s \in \mathbb{R}/\mathbb{Z}$ (or $s \in [0,1]$), and if $x(0) = y(s)$, define $x \cdot_s y \in \widehat{\Lambda}$ (see Figure 12) by

$$x \cdot_s y(t) = \begin{cases} y(t) & \text{if } 0 \le t \le s, \\ x(t-s) & \text{if } s \le t \le 1+s, \\ y(t) = y(t-1) & \text{if } 1+s \le t \le 2. \end{cases}$$

Define $\{\Lambda, \Lambda\}$ to be the set of triples $(x, y, s) \in \Lambda \times \Lambda \times \mathbb{R}/2\mathbb{Z}$ such that

$$\begin{cases} x(0) = y(s) & \text{if } 0 \le s \le 1, \\ y(0) = x(s) & \text{if } 1 \le s \le 2. \end{cases}$$

Define $\Phi_1 : \{\Lambda, \Lambda\} \to \widehat{\Lambda}$ by

$$\Phi_1(x, y, s) = \begin{cases} x \cdot_s y & \text{if } 0 \le s \le 1, \\ y \cdot_{(s-1)} x & \text{if } 1 \le s \le 2. \end{cases}$$

We have embeddings

$$\boxed{\Lambda \times \Lambda \times \mathbb{R}/(2\mathbb{Z}) \xleftarrow{h} \{\Lambda, \Lambda\} \xrightarrow{\Phi} \widehat{\Lambda} \times \mathbb{R}/2\mathbb{Z}} \tag{E.1.1}$$

where $\Phi(x, y, s) = (\Phi_1(x, y, s), s)$. The images $\Phi(\{\Lambda, \Lambda\})$ and $h(\{\Lambda, \Lambda\})$ have normal bundles and tubular neighborhoods, and in fact they are given by the pullback of the diagonal Δ under the mappings

$$\begin{tikzcd}
\Lambda \times \Lambda \times \mathbb{R}/2\mathbb{Z} \arrow[r, "\alpha"] & M \times M & \widehat{\Lambda} \times \mathbb{R}/2\mathbb{Z} \arrow[l, "\beta"'] \\
h(\{\Lambda, \Lambda\}) \arrow[u] \arrow[r] & \Delta \arrow[u] & \Phi(\{\Lambda, \Lambda\}) \arrow[u] \arrow[l]
\end{tikzcd}$$

where

$$\alpha(x, y, s) = \begin{cases} (x(0), y(s)) & \text{if } 0 \le s \le 1, \\ (x(s), y(0)) & \text{if } 1 \le s \le 2, \end{cases}$$

and $\beta(w, s) = (w(s), w(s+1))$. (Each half of $\{\Lambda, \Lambda\}$ has a smooth tubular neighborhood and normal bundle in $\Lambda \times \Lambda \times \mathbb{R}/2\mathbb{Z}$, but there is a "kink" where the two halves are joined so we obtain only a topological tubular neighborhood and normal bundle of $h(\{\Lambda, \Lambda\})$.) In particular,

$$\Phi(\{\Lambda, \Lambda\}) = \left\{(w, s) \in \widehat{\Lambda} \times \mathbb{R}/2\mathbb{Z} : \; w(s) = w(s \pm 1)\right\}. \tag{E.1.2}$$

E.2

We claim that the bracket $\{x, y\} \in H_{\deg(x)+\deg(y)-n+1}(\Lambda)$ is obtained by passing from left to right in (E.1.1); that is, it is the image of $x \times y \times [\mathbb{R}/2\mathbb{Z}]$ under the composition

$$\begin{tikzcd}
H_i(\Lambda) \times H_j(\Lambda) \times H_1(\mathbb{R}/2\mathbb{Z}) \arrow[r, "\epsilon\times"] & H_{i+j+1}(\Lambda \times \Lambda \times \mathbb{R}/2\mathbb{Z}) \arrow[d, "h^!"', "(\mathrm{B.2.4})"] \\
H_{i+j-n+1}(\widehat{\Lambda}) & H_{i+j+1-n}(\{\Lambda, \Lambda\}) \arrow[l, "\Phi_1"']
\end{tikzcd}$$

where $\deg(x) = i$, $\deg(y) = j$, $\epsilon = (-1)^{n(n-j-1)}$, and $[\mathbb{R}/2\mathbb{Z}] \in H_1(\mathbb{R}/2\mathbb{Z})$ denotes the orientation class. First, we show this agrees with the definition of $\{x, y\}$ in [CS].

The projection $\pi : \{\Lambda, \Lambda\} \to \mathbb{R}/2\mathbb{Z}$ is locally trivial, and $\pi^{-1}(0) \cong \pi^{-1}(1) \cong \mathscr{F}$ is the figure-eight space. Let $\{\Lambda, \Lambda\}_{[0,1]} = \pi^{-1}([0, 1])$, and let $\partial\{\Lambda, \Lambda\}_{[0,1]} = \pi^{-1}(\{0\} \cup \{1\})$. Then the bracket product in [CS] is a sum of two terms,

$$\{x, y\} = x \heartsuit y - (-1)^{(i-n+1)(j-n+1)} y \heartsuit x$$

(but [CS] use a $*$ rather than a $\heartsuit$) which may be identified as the two images of

$$h^!(\epsilon x \times y \times [\mathbb{R}/2\mathbb{Z}]) \in H_{i+j-n+1}(\{\Lambda, \Lambda\})$$

in

$$H_{i+j-n+1}\left(\Phi(\{\Lambda, \Lambda\}_{[0,1]}), \partial\Phi(\{\Lambda, \Lambda\}_{[0,1]})\right)$$

and

$$H_{i+j-n+1}\left(\Phi(\{\Lambda, \Lambda\}_{[1,2]}), \partial\Phi(\{\Lambda, \Lambda\}_{[1,2]})\right),$$

respectively. The projection to $\widehat{\Lambda}$ adds these together (with the appropriate sign).

The proof that the construction of §E.2 agrees with (17.1.1) is essentially the same as the proof of [CS, Corollary 5.3]. Using this fact, the proof of (1), (2), and (3) in §17.1 is then the same as in [CS, §4].

E.3

In this section, we define the reparametrization function

$$\widehat{J} : \widehat{\Lambda} \times \mathbb{R}/2\mathbb{Z} \times [0, 2] \to \widehat{\Lambda} \times \mathbb{R}/2\mathbb{Z}$$

which is analogous to the function J of §9.1. First, some notation. For $r \in [0, 2]$, let $\widehat{\theta}_{1\to r} : [0, 2] \to [0, 2]$ be the piecewise linear function taking $0 \mapsto 0$, $1 \mapsto r$, and $2 \mapsto 2$. It is just the function θ of §9.1, but the domain and range have been stretched to $[0, 2]$. For any real number s, define the translation $\chi_s : \mathbb{R}/2\mathbb{Z} \to RZ$ by $\chi_s(t) = t + s$. Define

$$\widehat{J}(w, s, r) = (w \circ \chi_s \circ \theta_{1\to r} \circ \chi_{-s}, s), \tag{E.3.1}$$

and set $\widehat{J}_r(w, s) = \widehat{J}(w, s, r)$. In analogy with our notation for $\mathscr{F}$ in §9.1, let $\{\Lambda, \Lambda\}^{>0,>0}$ be the set of $(x, y, s) \in \{\Lambda, \Lambda\}$ such that $F(x) > 0$ and $F(y) > 0$. Let $\widehat{\Lambda}_0 = \widehat{\Lambda}^{=0}$ denote the constant loops in $\widehat{\Lambda}$. We claim that $\widehat{J}$ takes both of the sets

$$\widehat{\Lambda}_0 \times \mathbb{R}/2\mathbb{Z} \times [0, 2] \qquad \text{and} \qquad \widehat{\Lambda} \times \mathbb{R}/2\mathbb{Z} \times \partial[0, 2] \tag{E.3.2}$$

into the set $\widehat{\Lambda} \times \mathbb{R}/2\mathbb{Z} - \Phi(\{\Lambda, \Lambda\}^{>0,>0})$.

This is obvious for the first of these sets, while the verification for the second set involves four cases: $s \leq 1$ or $s \geq 1$ vs. $r = 0$ or $r = 2$. In each case, the function $\chi_s \circ \widehat{\theta}_{1\to r} \circ \chi_{-s}$ is constant, with value s, on the interval $[s, s+1]$ (mod 2). Therefore $J(w, s, r) = (\gamma, s)$, where $\gamma = x \cdot_s y$ $(s \leq 1)$ or $\gamma = x \cdot_{(s-1)} y$ $(s \geq 1)$, and either x or y is a constant loop.

E.4
The geometric construction of the cohomology bracket is the following composition,

$$\begin{tikzcd}
H^i(\Lambda,\Lambda_0)\times H^j(\Lambda,\Lambda_0)\times H^0(\mathbb{R}/2\mathbb{Z}) \arrow[r, "\epsilon\times"] & H^{i+j}\big((\Lambda,\Lambda_0)\times(\Lambda,\Lambda_0)\times\mathbb{R}/2\mathbb{Z}\big) \arrow[d, "h^*"'] \\
& H^{i+j}\left(\{\Lambda,\Lambda\},\{\Lambda,\Lambda\}-\{\Lambda,\Lambda\}^{>0,>0}\right) \arrow[d, "\Phi_!"', "(\mathrm{B}.2.1)"] \\
& H^{i+j+n}\left(\widehat{\Lambda}\times\mathbb{R}/2\mathbb{Z},\widehat{\Lambda}\times\mathbb{R}/2\mathbb{Z}-\Phi(\{\Lambda,\Lambda\}^{>0,>0})\right) \arrow[d, "\widehat{J}^*"'] \\
H^{i+j+n-2}(\widehat{\Lambda},\widehat{\Lambda}_0) & H^{i+j+n}\big((\widehat{\Lambda},\widehat{\Lambda}_0)\times\mathbb{R}/2\mathbb{Z}\times([0,2],\partial[0,2])\big) \arrow[l, "\pi_!"']
\end{tikzcd}$$

where π denotes the projection to $\widehat{\Lambda}$. So the bracket is obtained by passing from left to right in the following diagram:

$$\boxed{\Lambda\times\Lambda\times\mathbb{R}/2\mathbb{Z} \xleftarrow[h]{} \{\Lambda,\Lambda\} \xrightarrow{\Phi} \widehat{\Lambda}\times\mathbb{R}/2\mathbb{Z} \xleftarrow{\widehat{J}} \widehat{\Lambda}\times\mathbb{R}/2\mathbb{Z}\times[0,2] \longrightarrow \widehat{\Lambda}}$$

E.5
For $r\in[0,2]$, set $\widehat{J}_r(w,s)=\widehat{J}(w,s,r)$. Let $\mathscr{T}:\{\Lambda,\Lambda\}\to\{\Lambda,\Lambda\}$ by $\mathscr{T}(x,y,s)=(y,x,s+1)$, and (by abuse of notation) set $\mathscr{T}:\widehat{\Lambda}\times\mathbb{R}/2\mathbb{Z}\to\widehat{\Lambda}\times\mathbb{R}/2\mathbb{Z}$ by $\mathscr{T}(z,s)=(z\circ\chi_1,s+1)$. (So $\mathscr{T}$ moves the basepoint halfway around the loop.) Then the following diagram commutes:

$$\begin{tikzcd}
\{\Lambda,\Lambda\} \arrow[r, "\Phi"] \arrow[d, "\mathscr{T}"] & \widehat{\Lambda}\times\mathbb{R}/2\mathbb{Z} \arrow[r, "\widehat{J}_r"] \arrow[d, "\mathscr{T}"] & \widehat{\Lambda}\times\mathbb{R}/2\mathbb{Z} \arrow[d, "\mathscr{T}"] \\
\{\Lambda,\Lambda\} \arrow[r, "\Phi"] & \widehat{\Lambda}\times\mathbb{R}/2\mathbb{Z} \arrow[r, "\widehat{J}_r"] & \widehat{\Lambda}\times\mathbb{R}/2\mathbb{Z}
\end{tikzcd}$$

As in §E.2, the bracket is a sum of two terms,

$$\{x,y\}=x\clubsuit y-(-1)^{(|x|+1)(|y|+1)}y\clubsuit x,$$

which are interchanged by the involution $\mathscr{T}$. The proof that the construction in §E.4 agrees with formula (17.1.2) is essentially the same as the proof of [CS, Corollary 5.3]. The proof of (A), (B), and (C) in Theorem 17.2 is then similar to the argument in [CS, §4]. □

F. Associativity of $\circledast$

F.1
In this section, we complete the proof of Proposition 9.2. The following statements refer to Figure 13; we omit the parallel diagram that is obtained by taking the cohomology of each of the spaces and pairs that appear in this diagram. Each mapping denoted τ denotes an inclusion with normal bundle. The corresponding homomorphism in the cohomology diagram is the Thom isomorphism. In the cohomology diagram, the squares involving arrows denoted τ commute because the relevant normal bundles pull back.

Starting with $x \times y \times z$ in the upper left corner, the product $x \circledast (y \circledast z)$ is obtained by going across the top row then down the right side of the diagram, while the product $(x \circledast y) \circledast z$ is obtained by going down the left side of the diagram and then along the bottom row. Here, the symbol $\mathscr{F}_{1/3}$ denotes $\phi_{1/3(F)}$ and the space $\mathscr{C}$ denotes the space of (three-leaf) clovers, that is, the preimage of the (small) diagonal under the mapping

$$(\mathbf{ev}_0, \mathbf{ev}_{1/3}, \mathbf{ev}_{2/3}) : \Lambda \to M \times M \times M.$$

It has a normal bundle in Λ which is isomorphic to $TM \oplus TM$. The symbol $\mathscr{C}^{>0,>0,\geq 0}$ denotes those loops consisting of three composable loops $\alpha \cdot \beta \cdot \gamma$, with α, β glued at time $1/3$ and with β, γ glued at time $2/3$, such that two (or more) of these "leaves" have positive energy (i.e., one or fewer of these loops is constant). Hence $\mathscr{C} - \mathscr{C}^{>0,>0,\geq 0}$ consists of clovers such that two or more of the leaves are constant. The square marked $\boxed{1}$ is Cartesian: the lower right corner is the intersection of the upper right and lower left corners. The symbol Δ denotes a diagonal mapping and id denotes an identity mapping.

F.2
Using the obvious extension of the notation for $\theta_{1/2 \to s} : I \to I$ (with $\theta(0) = 0$ and $\theta(1) = 1$), the mappings J_i and $\widehat{J}_i$ are (re)defined by

$$J_1(s, \gamma) = \gamma \circ \theta_{\substack{1/3 \to (2/3)s, \\ 2/3 \to 2/3}}, \qquad J_2(\gamma, t) = \gamma \circ \theta_{\substack{1/3 \to 1/3, \\ 2/3 \to 1/3 + (2/3)t}}$$

$$\widehat{J}_1(s, \gamma) = \gamma \circ \theta_{1/3 \to s}, \qquad \widehat{J}_2(\gamma, t) = \gamma \circ \theta_{2/3 \to t},$$

so that

$$J_1 \circ (\mathsf{id} \times \widehat{J}_2)(s, \gamma, t) = \gamma \circ \theta_{2/3 \to t} \circ \theta_{\substack{1/3 \to (2/3)s \\ 2/3 \to 2/3}} \qquad = \gamma \circ \theta_{\substack{1/3 \to st \\ 2/3 \to t}} \tag{F.2.1}$$

$$J_2 \circ (\widehat{J}_1 \times \mathsf{id})(\gamma, s, t) = \gamma \circ \theta_{1/3 \to s} \circ \theta_{\substack{1/3 \to 1/3 \\ 2/3 \to 1/3 + (2/3)t}} \qquad = \gamma \circ \theta_{\substack{1/3 \to s \\ 2/3 \to s + (1-s)t}} \tag{F.2.2}$$

$$
\begin{tikzcd}
(\Lambda,\Lambda_0)\times(\Lambda,\Lambda_0)\times(\Lambda,\Lambda_0) & (\Lambda,\Lambda_0)\times(\mathscr{F}_{2/3},\mathscr{F}_{2/3}-\mathscr{F}_{2/3}^{>0,>0}) \arrow[l, "\mathrm{id}\times\Delta"'] \arrow[r, "\mathrm{id}\times\tau"'] & (\Lambda,\Lambda_0)\times(\Lambda,\Lambda-\mathscr{F}_{2/3}^{>0,>0}) & (\Lambda,\Lambda_0)\times(\Lambda,\Lambda_0)\times(I_2,\partial I_2) \arrow[l, "\mathrm{id}\times J_2"'] \\
(\mathscr{F}_{1/3},\mathscr{F}_{1/3}-\mathscr{F}_{1/3}^{>0,>0})\times(\Lambda,\Lambda_0) \arrow[u, "\Delta\times\mathrm{id}"] \arrow[d, "\tau\times\mathrm{id}"] \arrow[ur, phantom, "\boxed{1}"] & (\mathcal{C},\mathcal{C}-\mathcal{C}^{>0,>0,\geq 0}) \arrow[l] \arrow[u] \arrow[r, "\tau"'] \arrow[d, "\tau"] & (\mathscr{F}_{1/3},\mathscr{F}_{1/3}-\mathcal{C}^{>0,>0,\geq 0}) \arrow[u] \arrow[d, "\tau"] & (\mathscr{F}_{1/3},\mathscr{F}_{1/3}-\mathscr{F}_{1/3}^{>0,>0})\times(I_2,\partial I_2) \arrow[l, "J_2"'] \arrow[u, "\Delta\times\mathrm{id}"] \arrow[d, "\tau\times\mathrm{id}"] \\
(\Lambda,\Lambda-\mathscr{F}_{1/3}^{>0,>0})\times(\Lambda,\Lambda_0) & (\mathscr{F}_{2/3},\mathscr{F}_{2/3}-\mathcal{C}^{>0,>0,\geq 0}) \arrow[l] \arrow[r, "\tau"'] & (\Lambda,\Lambda-\mathcal{C}^{>0,>0,\geq 0}) & (\Lambda,\Lambda-\mathscr{F}_{1/3}^{>0,>0})\times(I_2,\partial I_2) \arrow[l, "J_2"'] \arrow[dl, phantom, "\boxed{2}"] \\
(I_1,\partial I_1)\times(\Lambda,\Lambda_0)\times(\Lambda,\Lambda_0) \arrow[u, "J_1\times\mathrm{id}"] & (I_1,\partial I_1)\times(\mathscr{F}_{2/3},\mathscr{F}_{2/3}-\mathscr{F}_{2/3}^{>0,>0}) \arrow[l] \arrow[u, "J_1"] \arrow[r, "\tau"'] & (I_1,\partial I_1)\times(\Lambda,\Lambda-\mathscr{F}_{2/3}^{>0,>0}) \arrow[u, "J_1"] & (I_1,\partial I_1)\times(\Lambda,\Lambda_0)\times(I_2,\partial I_2) \arrow[l, "\mathrm{id}\times\hat{J}_2"'] \arrow[u, "\hat{J}_1\times\mathrm{id}"]
\end{tikzcd}
$$

Figure 13. Associativity of ⊛-product

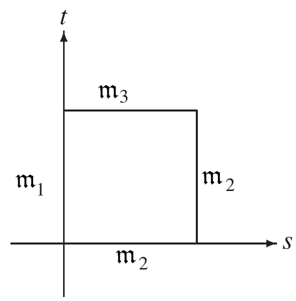

Figure 14. Boundary behavior

We need to prove that the corresponding cohomology diagram commutes. The only part that is not obvious is the square designated $\boxed{2}$ in the diagram. This square commutes up to (relative) homotopy for the following reason. Let $\mathfrak{m}$ denote the set of continuous nondecreasing mappings $\theta : [0, 1] \to [0, 1]$ such that $\theta(0) = 0, \theta(1) = 1$, and θ is linear on $[0, 1/3]$, on $[1/3, 2/3]$, and on $[2/3, 1]$. For $i = 1, 2, 3$, let $\mathfrak{m}_i$ denote the collection of all $\theta \in \mathfrak{m}$ such that θ is constant on $[(i-1)/3, i/3]$. The functions $\theta_{\text{etc.}}$ appearing on the right-hand side of (F.2.1) and (F.2.2) may be interpreted as continuous mappings $(s, t) \in I_1 \times I_2 \to \mathfrak{m}$ with the following boundary behavior: $\{0\} \times I_2 \to \mathfrak{m}_1$, $\{1\} \times I_2 \to \mathfrak{m}_2$; $I_1 \times \{0\} \to \mathfrak{m}_2$; and $I_1 \times \{1\} \to \mathfrak{m}_3$. This boundary behavior is indicated in Figure 14.

But the collection of such maps $I_1 \times I_2 \to \mathfrak{m}$ is convex, so the mappings (F.2.1) and (F.2.2) are homotopic. This completes the proof that the $\circledast$-product is associative. □

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Goresky
School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540, USA; goresky@math.ias.edu

Hingston
Department of Mathematics, College of New Jersey, Ewing, New Jersey O8628, USA; hingston@tcnj.edu