Euler flag enumeration of Whitney stratified spaces

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Abstract

The flag vector contains all the face incidence data of a polytope, and in the poset setting, the chain enumerative data. It is a classical result due to Bayer and Klapper that for face lattices of polytopes, and more generally, Eulerian graded posets, the flag vector can be written as a cd-index, a non-commutative polynomial which removes all the linear redundancies among the flag vector entries. This result holds for regular CW complexes.

We relax the regularity condition to show the **cd**-index exists for Whitney stratified manifolds by extending the notion of a graded poset to that of a quasi-graded poset. This is a poset endowed with an order-preserving rank function and a weighted zeta function. This allows us to generalize the classical notion of Eulerianness, and obtain a **cd**-index in the quasi-graded poset arena. We also extend the semi-suspension operation to that of embedding a complex in the boundary of a higher dimensional ball and study the simplicial shelling components.

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1 Introduction

In this paper we extend the theory of face incidence enumeration of polytopes, and more generally, chain enumeration in graded Eulerian posets, to that of Whitney stratified spaces and quasi-graded posets.

The idea of enumeration using the Euler characteristic was suggested throughout Rota's work and influenced by Schanuel's categorical viewpoint [33, 40, 41, 42]. In order to carry out such a program that is topologically meaningful and which captures the broadest possible classes of examples, two key insights are required. First, the notion of grading in the face lattice of a polytope must be relaxed. Secondly, the usual zeta function in the incidence algebra must be extended to include the Euler characteristic as an important instance.

Recall an *Eulerian poset* is a graded partially ordered set (poset) P such that every nontrivial interval satisfies the Euler–Poincaré relation, that is, the number of elements of even rank equals the number of elements of odd rank. Equivalently, the Möbius function is given by $\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$, where ρ denotes the *rank function*. Another way to state this *Eulerian condition* is that the inverse of the zeta function $\zeta(x, y)$ (where $\zeta(x, y) = 1$ if $x \leq y$ and 0 otherwise), that is, the

Möbius function $\mu(x, y)$, is given by the function $(-1)^{\rho(y)-\rho(x)} \cdot \zeta(x, y)$. Families of Eulerian posets include (a) the face lattice of a convex polytope, (b) the face poset of a regular cell decomposition of a homology sphere, and (c) the elements of a finite Coxeter group ordered by the strong Bruhat order. In the case of a convex polytope the Eulerian condition expresses the fact that the link of each face has the Euler characteristic of a sphere.

The *f-vector* of a convex polytope enumerates, for each non-negative integer i, the number f_i of i-dimensional faces in the polytope. It satisfies the Euler–Poincaré relation. The problem of understanding the f-vectors of polytopes harks back to Steinitz [49], who completely described the 3-dimensional case. For polytopes of dimension greater than three the problem is still open. For *simplicial* polytopes, that is, each i-dimensional face is an i-dimensional simplex, the f-vectors satisfy linear relations known as the Dehn–Sommerville relations. Furthermore, the f-vectors of simplicial polytopes have been completely characterized by work of McMullen [37], Billera and Lee [7], and Stanley [44].

The *flag f-vector* of a graded poset counts the number of chains passing through a prescribed set of ranks. In the case of a polytope, it records all of the face incidence data, including that of the f-vector. Bayer and Billera proved that the flag f-vector of any Eulerian poset satisfies a collection of linear equalities now known as the *generalized Dehn–Sommerville relations* [2]. These linear equations may be interpreted as natural redundancies among the components of the flag f-vector. Bayer and Klapper removed these redundancies by showing that the space of flag f-vectors of Eulerian posets has a natural basis consisting of non-commutative polynomials in the two variables $\mathbf{c}$ and $\mathbf{d}$ [3]. The coefficients of this **cd**-*index* were later shown by Stanley to be non-negative in the case of spherically-shellable posets [46]. Other milestones for the **cd**-index include its inherent coalgebraic structure [19], its appearance in the proofs of inequalities for flag vectors [5, 15, 17, 32], its use in understanding the combinatorics of arrangements of subspaces and sub-tori [6, 23], and most recently, its connection to the Bruhat graph and Kazhdan–Lusztig theory [4, 22].

In this article we extend the **cd**-index and its properties to a more general situation, that of quasi-graded posets and Whitney stratified spaces. A quasi-grading on a poset P consists of a strictly order-preserving "rank" function $\rho : P \longrightarrow \mathbb{N}$ and a weighted zeta function $\bar{\zeta}$ in the incidence algebra $I(P)$ such that $\bar{\zeta}(x, x) = 1$ for all $x \in P$. See Section 2. A quasi-graded poset $(P, \rho, \bar{\zeta})$ will be said to be *Eulerian* if the function $(-1)^{\rho(y)-\rho(x)} \cdot \bar{\zeta}(x, y)$ is the inverse of $\bar{\zeta}(x, y)$ in the incidence algebra of P. This reduces to the classical definition of Eulerian if $(P, \rho, \bar{\zeta})$ is a ranked poset with the standard zeta function ζ.

We show that Eulerian $\bar{\zeta}$ functions exist on most posets (Proposition 3.2). Let P be a poset with a strictly order-preserving "rank" function ρ. Choose $\bar{\zeta}(x, y)$ arbitrarily whenever $\rho(y) - \rho(x)$ is odd. Then *there is a unique way to assign values to $\bar{\zeta}(x, y)$ whenever $\rho(y) - \rho(x)$ is even such that $(P, \rho, \bar{\zeta})$ is an Eulerian quasi-graded poset.* Theorem 4.2 states that the **cd**-index is defined for Eulerian quasi-graded posets. This result is equivalent to the flag $\bar{f}$-vector of an Eulerian quasi-graded poset satisfies the generalized Dehn–Sommerville relations (Theorem 4.3).

Additional properties of quasi-graded posets are discussed in Sections 3, 4 and 5. The Alexander duality formula has a natural generalization to Eulerian quasi-graded posets (Theorem 3.7 and Corollary 3.8). The **ab**-index and the **cd**-index of the Cartesian product $P \times B_1$ of a quasi-graded poset P and the Boolean algebra B_1 is given (Proposition 5.2). The **ab**-index of the Stanley product of two quasi-graded posets is the product of their **ab**-indexes (Lemma 5.3). The "zipping" operation (see [39]) can be defined for quasi-graded posets and the resulting **ab**-index is calculated (Proposition 5.8). Furthermore, the Eulerian property is preserved under the zipping operation (Theorem 5.9). Merging strata in a Whitney stratified manifold, the geometric analogue of poset zipping, is established later in Section 13.

Eulerian ranked posets arise geometrically as the face posets of regular cell decompositions of a sphere, whereas Eulerian quasi-graded posets arise geometrically from the more general case of Whitney stratifications. A *Whitney stratification* X of a compact topological space W is a decomposition of W into finitely many smooth manifolds which satisfy Whitney's "no-wiggle" conditions on how the strata fit together. See Section 6. These conditions guarantee (a) that X does not exhibit Cantor set-like behavior and (b) that the closure of each stratum is a union of strata. The faces of a convex polytope and the cells of a regular cell complex are examples of Whitney stratifications, but in general, a stratum in a stratified space need not be contractible. Moreover, the closure of a stratum of dimension d does not necessarily contain strata of dimension $d-1$, or for that matter, of any other dimension. Natural Whitney stratifications exist for real or complex algebraic sets, analytic sets, semi-analytic sets and for quotients of smooth manifolds by compact group actions.

The strata of a Whitney stratification (of a topological space W) form a poset, where the order relation $A < B$ is given by $A \subset \overline{B}$. Moreover, this set admits a natural quasi-grading which is defined by $\rho(A) = \dim(A) + 1$ and $\bar{\zeta}(A, B) = \chi(\mathrm{link}(A) \cap B)$ whenever $A < B$ are strata and χ is the Euler characteristic. See Definition 6.9. This is the setting for our Euler-characteristic enumeration.

Theorem 6.10 states that *the quasi-graded poset of strata of a Whitney stratified set is Eulerian* and therefore its **cd**-index is defined and its flag $\bar{f}$-vector satisfies the generalized Dehn–Sommerville relations. The background and results needed for this proof are developed in Sections 7, 8 and 9.

It is important to point out that, unlike the case of polytopes, the coefficients of the **cd**-index of Whitney stratified manifolds can be negative. See Examples 6.1, 6.14, 6.15 and 6.16. It is our hope that by applying topological techniques to stratified manifolds, we will yield a tractable interpretation of the coefficients of the **cd**-index. This may ultimately explain Stanley's non-negativity results for spherically shellable posets [46] and Karu's results for Gorenstein* posets [32].

One may also ask what linear inequalities hold among the entries of the weighted $\bar{f}$-vector of a Whitney stratified manifold, equivalently, what linear inequalities hold among the coefficients of the **cd**-index? See the concluding remarks for further details.

In his proof that the **cd**-index of a polytope is non-negative, Stanley introduced the notion of semisuspension. Given a polytopal complex that is homeomorphic to a ball, the semisuspension adds another facet whose boundary is the boundary of the ball. The resulting spherical CW complex has the same dimension, and the intervals in its face poset are Eulerian [46].

It is precisely the setting of Whitney stratified manifolds, and the larger class of Whitney stratified spaces, which is critical in order to study face enumeration of the semisuspension in higher dimensional spheres and more general topologically interesting examples. In Sections 10 and 11 the nth semisuspension and its **cd**-index is studied. In Theorem 11.4, by using the method of quasi-graded posets, we are able to give a short proof (that completely avoids the use of shellings) of a key result of Billera and Ehrenborg [5] that was needed for their proof that the n-dimensional simplex minimizes the **cd**-index among all n-dimensional polytopes.

In Section 12 we establish the Eulerian relation for the nth semisuspension (Theorem 12.1). This implies one cannot develop a local **cd**-index akin to the local h-vector Stanley devised in [45] to understand the effect of subdivisions on the h-vector.

In Section 14 the **cd**-index of the nth semisuspension of a non-pure shellable simplicial complex is determined. The **cd**-index of the shelling components are shown to satisfy a recursion involving a derivation which first appeared in [19]. By relaxing the notion of shelling, we furthermore show that the shelling components satisfy a Pascal type recursion. This yields new expressions for the shelling components and illustrates the power of leaving the realm of regular cell complexes for that of Whitney stratified spaces.

We end with open questions and comments in the concluding remarks.

2 Quasi-graded posets and their ab-index

Recall the *incidence algebra* of a poset is the set of all functions $f : I(P) \to \mathbb{C}$ where $I(P)$ denotes the set of intervals in the poset. The multiplication is given by $(f \cdot g)(x, y) = \sum_{x \leq z \leq y} f(x, z) \cdot g(z, y)$ and the identity is given by the delta function $\delta(x, y) = \delta_{x,y}$, where the second delta is the usual Kronecker delta function $\delta_{x,y} = 1$ if $x = y$ and zero otherwise. A poset is said to be *ranked* if every maximal chain in the poset has the same length. This common length is called the *rank* of the poset. A poset is said to be *graded* if it is ranked and has a minimal element $\widehat{0}$ and a maximal element $\widehat{1}$. For other poset terminology, we refer the reader to Stanley's treatise [48].

We introduce the notion of a quasi-graded poset. This extends the notion of a ranked poset.

Definition 2.1. *A quasi-graded poset $(P, \rho, \bar{\zeta})$ consists of*

(i) a finite poset P (not necessarily ranked),

(ii) a strictly order-preserving function ρ from P to $\mathbb{N}$, that is, $x < y$ implies $\rho(x) < \rho(y)$ and

(iii) a function $\bar{\zeta}$ in the incidence algebra $I(P)$ of the poset P, called the weighted zeta function*, such that $\bar{\zeta}(x, x) = 1$ for all elements x in the poset P.*

Observe that we do not require the poset to have a minimal element or a maximal element. Since $\bar{\zeta}(x, x) \neq 0$ for all $x \in P$, the function $\bar{\zeta}$ is invertible in the incidence algebra $I(P)$ and we denote its inverse by $\bar{\mu}$. See Lemma 3.4 for an expression for the inverse.

For $x \leq y$ in a quasi-graded poset $P = (P, \rho, \bar{\zeta})$, the *rank difference function* is given by $\rho(x, y) = \rho(y) - \rho(x)$. We say that a quasi-graded poset $(P, \rho, \bar{\zeta})$ with minimal element $\widehat{0}$ and maximal element $\widehat{1}$ has rank n if $\rho(\widehat{0}, \widehat{1}) = n$. The interval $[x, y]$ is itself a quasi-graded poset together with the rank function $\rho_{[x,y]}(w) = \rho(w) - \rho(x)$ and the weighted zeta function $\bar{\zeta}$.

Example 2.2. The classical example of a quasi-graded poset is (P, ρ, ζ), where P is a graded poset with rank function ρ and we take the weighted zeta function to be the usual zeta function ζ in the incidence algebra defined by $\zeta(x, y) = 1$ for all intervals $[x, y] \in P$. Here the inverse of the zeta function is the Möbius function, denoted by $\mu(x, y)$.

Let $(P, \rho, \bar{\zeta})$ be a quasi-graded poset with unique minimal element $\widehat{0}$ and unique maximal element $\widehat{1}$. The assumption of a quasi-graded poset having a $\widehat{0}$ and $\widehat{1}$ will be essential in order to define its **ab**-index and **cd**-index. For a chain $c = \{x_0 < x_1 < \cdots < x_k\}$ in the quasi-graded poset P, define $\bar{\zeta}(c)$ to be the product

$$\bar{\zeta}(c) = \bar{\zeta}(x_0, x_1) \cdot \bar{\zeta}(x_1, x_2) \cdots \bar{\zeta}(x_{k-1}, x_k). \tag{2.1}$$

Similarly, for the chain c define its *weight* to be

$$\mathrm{wt}(c) = (\mathbf{a} - \mathbf{b})^{\rho(x_0,x_1)-1} \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\rho(x_1,x_2)-1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\rho(x_{k-1},x_k)-1},$$

where $\mathbf{a}$ and $\mathbf{b}$ are non-commutative variables each of degree 1. The **ab**-*index of a quasi-graded poset* $(P, \rho, \bar{\zeta})$ is

$$\Psi(P, \rho, \bar{\zeta}) = \sum_{c} \bar{\zeta}(c) \cdot \mathrm{wt}(c), \tag{2.2}$$

where the sum is over all chains starting at the minimal element $\widehat{0}$ and ending at the maximal element $\widehat{1}$, that is, $c = \{\widehat{0} = x_0 < x_1 < \cdots < x_k = \widehat{1}\}$. When the rank function ρ and the weighted zeta function are clear from the context, we will write the shorter $\Psi(P)$. Observe that if a quasi-graded poset $(P, \rho, \bar{\zeta})$ has rank $n + 1$ then its **ab**-index is homogeneous of degree n.

The **ab**-index depends on the rank difference function $\rho(x, y)$ but not on the rank function itself. Hence we may uniformly shift the rank function without changing the **ab**-index. Later we will use the convention that $\rho(\widehat{0}) = 0$.

A different approach to the **ab**-index is via the flag f- and flag h-vectors. We extend this route by introducing the flag $\bar{f}$- and flag $\bar{h}$-vectors. Let $(P, \rho, \bar{\zeta})$ be a quasi-graded poset of rank $n + 1$ having a $\widehat{0}$ and $\widehat{1}$ such that $\rho(\widehat{0}) = 0$. For $S = \{s_1 < s_2 < \cdots < s_k\}$ a subset of $\{1, \ldots, n\}$, define the *flag $\bar{f}$-vector* by

$$\bar{f}_S = \sum_{c} \bar{\zeta}(c), \tag{2.3}$$

where the sum is over all chains $c = \{\widehat{0} = x_0 < x_1 < \cdots < x_{k+1} = \widehat{1}\}$ in P such that $\rho(x_i) = s_i$ for all $1 \leq i \leq k$. The *flag $\bar{h}$-vector* is defined by the relation (and by inclusion–exclusion, we also display its inverse relation)

$$\bar{h}_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot \bar{f}_T \quad \text{and} \quad \bar{f}_S = \sum_{T \subseteq S} \bar{h}_T. \tag{2.4}$$

For a subset $S \subseteq \{1, \ldots, n\}$ define the **ab**-monomial $u_S = u_1 u_2 \cdots u_n$ by $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. The **ab**-index of the quasi-graded poset $(P, \rho, \bar{\zeta})$ is then given by

$$\Psi(P, \rho, \bar{\zeta}) = \sum_S \bar{h}_S \cdot u_S,$$

where the sum ranges over all subsets S. Again, in the case when we take the weighted zeta function to be the usual zeta function ζ, the flag $\bar{f}$ and flag $\bar{h}$-vectors correspond to the usual flag f- and flag h-vectors.

Using Lemma 3.4 we have the next statement.

Lemma 2.3. *For a quasi-graded poset $(P, \rho, \bar{\zeta})$ of rank $n+1$ with minimal and maximal elements $\widehat{0}$ and $\widehat{1}$, the weighted Möbius function $\bar{\mu}(\widehat{0}, \widehat{1})$ is given by*

$$\bar{\mu}(\widehat{0}, \widehat{1}) = (-1)^{n+1} \cdot \bar{h}_{\{1,\ldots,n\}}.$$

We now give two recursions for the **ab**-index.

Proposition 2.4. *Let $(P, \rho, \bar{\zeta})$ be a quasi-graded poset where $\bar{\mu}$ is the inverse of $\bar{\zeta}$. The following two recursions hold for computing the* **ab**-*index of an interval $[x, z]$:*

$$\begin{aligned} \Psi([x, z], \rho, \bar{\zeta}) &= \bar{\zeta}(x, z) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,z)-1} \\ &\quad + \sum_{x<y<z} \Psi([x, y], \rho, \bar{\zeta}) \cdot \mathbf{b} \cdot \bar{\zeta}(y, z) \cdot (\mathbf{a} - \mathbf{b})^{\rho(y,z)-1}, \qquad (2.5) \\ \Psi([x, z], \rho, \bar{\zeta}) &= -\bar{\mu}(x, z) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,z)-1} \\ &\quad - \sum_{x<y<z} \Psi([x, y], \rho, \bar{\zeta}) \cdot \mathbf{a} \cdot \bar{\mu}(y, z) \cdot (\mathbf{a} - \mathbf{b})^{\rho(y,z)-1}. \qquad (2.6) \end{aligned}$$

Proof. Using the chain definition of the **ab**-index and conditioning on the largest element $y < z$ in the chain c, we obtain the recursion (2.5). Multiplying equation (2.5) by $\mathbf{a} - \mathbf{b}$ on the right and moving the term $\Psi([x, z], \rho, \bar{\zeta}) \cdot \mathbf{b}$ to the right-hand side, we obtain

$$\Psi([x, z], \rho, \bar{\zeta}) \cdot \mathbf{a} = \bar{\zeta}(x, z) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,z)} + \sum_{x<y\leq z} \Psi([x, y], \rho, \bar{\zeta}) \cdot \mathbf{b} \cdot \bar{\zeta}(y, z) \cdot (\mathbf{a} - \mathbf{b})^{\rho(y,z)}. \qquad (2.7)$$

Define three functions f, g and h in the incidence algebra of P by

$$f(x, y) = \begin{cases} \Psi([x, y], \rho, \bar{\zeta}) \cdot \mathbf{a} & \text{if } x < y, \\ 1 & \text{if } x = y, \end{cases} \qquad g(x, y) = \begin{cases} \Psi([x, y], \rho, \bar{\zeta}) \cdot \mathbf{b} & \text{if } x < y, \\ 1 & \text{if } x = y, \end{cases}$$

and

$$h(x, y) = \bar{\zeta}(x, y) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,y)}.$$

Equation (2.7) can then be written as $f = g \cdot h$ where the product is the convolution of the incidence algebra. Observe that h is invertible with its inverse is given by $h^{-1}(x, y) = \bar{\mu}(x, y) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,y)}$. By expanding the equivalent relation $g = f \cdot h^{-1}$, we obtain

$$\Psi([x, z], \rho, \bar{\zeta}) \cdot \mathbf{b} = \bar{\mu}(x, z) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,z)} + \sum_{x<y\leq z} \Psi([x, y], \rho, \bar{\zeta}) \cdot \mathbf{a} \cdot \bar{\mu}(y, z) \cdot (\mathbf{a} - \mathbf{b})^{\rho(y,z)}. \quad (2.8)$$

By moving the term $\Psi([x, z], \rho, \bar{\zeta}) \cdot \mathbf{a}$ to the left-hand side of equation (2.8) and canceling a factor of $\mathbf{b} - \mathbf{a}$ on the right, we obtain recursion (2.6). $\square$

Equation (2.6) is an alternative recursion for the **ab**-index which may be viewed as dual to (2.5). As a remark, the two recursions in Proposition 2.4 contain the boundary condition $\Psi([x, z], \rho, \bar{\zeta}) = \bar{\zeta}(x, z) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,z)-1} = -\bar{\zeta}(x, z) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,z)-1}$ for x covered by z.

We end this section with the essential result that the **ab**-index of a quasi-graded poset is a coalgebra homomorphism. Define a coproduct $\Delta : \mathbb{Z}\langle\mathbf{a}, \mathbf{b}\rangle \longrightarrow \mathbb{Z}\langle\mathbf{a}, \mathbf{b}\rangle \otimes \mathbb{Z}\langle\mathbf{a}, \mathbf{b}\rangle$ by $\Delta(1) = 0$, $\Delta(\mathbf{a}) = \Delta(\mathbf{b}) = 1 \otimes 1$ and for an **ab**-monomial $u = u_1 u_2 \cdots u_k$

$$\Delta(u) = \sum_{i=1}^{k} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_k.$$

The coproduct Δ extends to $\mathbb{Z}\langle\mathbf{a}, \mathbf{b}\rangle$ by linearity. It is straightforward to see that this coproduct is coassociative. The coproduct Δ first appeared in [19].

Theorem 2.5. *Let $(P, \rho, \bar{\zeta})$ be a quasi-graded poset. Then the following identity holds:*

$$\Delta(\Psi(P, \rho, \bar{\zeta})) = \sum_{\hat{0}<x<\hat{1}} \Psi([\hat{0}, x], \rho, \bar{\zeta}) \otimes \Psi([x, \hat{1}], \rho, \bar{\zeta}).$$

The proof is the same as in the case of a graded poset [19], and hence is omitted. One way to formulate this result is to say that the **ab**-index is a coalgebra homomorphism from the linear span of quasi-graded posets to the algebra $\mathbb{Z}\langle\mathbf{a}, \mathbf{b}\rangle$.

3 Eulerian posets and Alexander duality

We define a quasi-graded poset to be *Eulerian* if for all pairs of elements $x \leq z$ we have that

$$\sum_{x\leq y\leq z} (-1)^{\rho(x,y)} \cdot \bar{\zeta}(x, y) \cdot \bar{\zeta}(y, z) = \delta_{x,z}. \quad (3.1)$$

In other words, the function $\bar{\mu}(x, y) = (-1)^{\rho(x,y)} \cdot \bar{\zeta}(x, y)$ is the inverse of $\bar{\zeta}(x, y)$ in the incidence algebra. In the case $\bar{\zeta}(x, y) = \zeta(x, y)$, we refer to relation (3.1) as the *classical Eulerian relation.*

If x is covered by z, that is, there is no element y such that $x < y < z$, then the Eulerian condition states that either the rank difference $\rho(x,z)$ is odd or the weighted zeta function $\bar{\zeta}(x,z)$ equals zero. This statement will be generalized in Proposition 3.2 below. Similar to Lemma 3.1 in [20] and Exercise 3.174c in [48], we will first need the following lemma.

Lemma 3.1. *Let $(P, \rho, \bar{\zeta})$ be a quasi-graded poset with $\widehat{0}$ and $\widehat{1}$ of odd rank n such that every proper interval of P is Eulerian. Then $(P, \rho, \bar{\zeta})$ is an Eulerian quasi-graded poset.*

Proof. For any function f in the incidence algebra $I(P)$ satisfying $f(x,x) = 1$ for all x, we have

$$f(\widehat{0},\widehat{1}) + f^{-1}(\widehat{0},\widehat{1}) = -\sum_{\widehat{0}<y<\widehat{1}} f(\widehat{0},y) \cdot f^{-1}(y,\widehat{1}) = -\sum_{\widehat{0}<y<\widehat{1}} f^{-1}(\widehat{0},y) \cdot f(y,\widehat{1}).$$

Applying this relation to the case when $f = \bar{\zeta}$ gives

$$\begin{aligned}
\bar{\zeta}(\widehat{0},\widehat{1}) + \bar{\zeta}^{-1}(\widehat{0},\widehat{1}) &= -\sum_{\widehat{0}<y<\widehat{1}} \bar{\zeta}(\widehat{0},y) \cdot (-1)^{\rho(y,\widehat{1})} \cdot \bar{\zeta}(y,\widehat{1}) \\
&= \sum_{\widehat{0}<y<\widehat{1}} (-1)^{\rho(\widehat{0},y)} \cdot \bar{\zeta}(\widehat{0},y) \cdot \bar{\zeta}(y,\widehat{1}) \\
&= -\bar{\zeta}(\widehat{0},\widehat{1}) - \bar{\zeta}^{-1}(\widehat{0},\widehat{1}).
\end{aligned}$$

Here we are using that $\rho(\widehat{0},y) + \rho(y,\widehat{1}) = n$ which is odd. Hence $\bar{\zeta}^{-1}(\widehat{0},\widehat{1}) = (-1)^n \cdot \bar{\zeta}(\widehat{0},\widehat{1})$, that is, $\bar{\zeta}^{-1}(x,z) = (-1)^{\rho(x,z)} \cdot \bar{\zeta}(x,z)$ for all $x \leq z$ and thus the quasi-graded poset $(P, \rho, \bar{\zeta})$ is Eulerian. □

Proposition 3.2. *Let P be a poset with an order-preserving rank function ρ. Choose the values of $\bar{\zeta}(x,z)$ arbitrarily when the rank difference $\rho(x,z)$ is odd, and let*

$$\bar{\zeta}(x,z) = -1/2 \cdot \sum_{x<y<z} (-1)^{\rho(x,y)} \cdot \bar{\zeta}(x,y) \cdot \bar{\zeta}(y,z), \tag{3.2}$$

for $x < z$ when $\rho(x,z)$ is even. Then $(P, \rho, \bar{\zeta})$ is an Eulerian quasi-graded poset.

Proof. Lemma 3.1 guarantees that every odd interval is Eulerian and equation (3.2) guarantees that every even interval is Eulerian. □

Example 3.3. Let C be the n element chain $C = \{x_1 < x_2 < \cdots < x_n\}$ with the rank function $\rho(x_i) = i$. The number of intervals of odd rank in this poset is given by $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$. Hence we have $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$ degrees of freedom in choosing a weighted zeta function $\bar{\zeta}$ in order for the quasi-graded poset $(C, \rho, \bar{\zeta})$ to be Eulerian.

The next lemma is well-known. See for instance [18, Lemma 5.3]. Since the incidence algebra of a finite poset with n elements is a subalgebra of all $n \times n$ matrices, this lemma is an instance of the identity $(I + A)^{-1} = \sum_{k \geq 0} (-1)^k \cdot A^k$; see [48, Section 3.6].

Lemma 3.4. *Let P be a poset with minimal element $\widehat{0}$ and maximal element $\widehat{1}$. Let f be a function in the incidence algebra $I(P)$ such that $f(x,x) = 1$ for all x in P. Then the inverse function of f in the incidence algebra $I(P)$ can be computed by*

$$f^{-1}(\widehat{0},\widehat{1}) = \sum_{\widehat{0}=x_0<x_1<\cdots<x_k=\widehat{1}} (-1)^k \cdot f(x_0,x_1) \cdot f(x_1,x_2) \cdots f(x_{k-1},x_k).$$

Lemma 3.4 implies the next result.

Lemma 3.5. *Let P be a poset with $\widehat{0}$ and $\widehat{1}$. Let y be an element of P such that $\widehat{0} < y < \widehat{1}$ and let Q be the subposet $P - \{y\}$. Suppose f is a function in the incidence algebra of P satisfying $f(x,x) = 1$ for all $x \in P$. Then*

$$(f|_Q)^{-1}(\widehat{0},\widehat{1}) = f^{-1}(\widehat{0},\widehat{1}) - f^{-1}(\widehat{0},y) \cdot f^{-1}(y,\widehat{1}).$$

Iterating this lemma gives the following proposition.

Proposition 3.6. *Let P be a poset with $\widehat{0}$ and $\widehat{1}$, and let Q and R be two subposets of P such that $Q \cup R = P$ and $Q \cap R = \{\widehat{0},\widehat{1}\}$. Assume f is a function in the incidence algebra $I(P)$ satisfying $f(x,x) = 1$ for all $x \in P$. Then*

$$(f|_Q)^{-1}(\widehat{0},\widehat{1}) = \sum_{\substack{\widehat{0}=y_0<y_1<\cdots<y_k=\widehat{1} \\ y_i \in R}} (-1)^{k-1} \cdot f^{-1}(y_0,y_1) \cdot f^{-1}(y_1,y_2) \cdots f^{-1}(y_{k-1},y_k).$$

We now apply this result to Eulerian quasi-graded posets.

Theorem 3.7 (Alexander duality for quasi-graded posets)**.** *Let $(P,\rho,\bar{\zeta})$ be an Eulerian quasi-graded poset with $\widehat{0}$ and $\widehat{1}$ of rank $n+1$. Let Q and R be two subposets of P such that $Q \cup R = P$ and $Q \cap R = \{\widehat{0},\widehat{1}\}$. Then*

$$(\bar{\zeta}|_Q)^{-1}(\widehat{0},\widehat{1}) = (-1)^n \cdot (\bar{\zeta}|_R)^{-1}(\widehat{0},\widehat{1}).$$

Proof. Directly we have

$$\begin{aligned}
(\bar{\zeta}|_Q)^{-1}(\widehat{0},\widehat{1}) &= \sum_{\substack{\widehat{0}=y_0<y_1<\cdots<y_k=\widehat{1} \\ y_i \in R}} (-1)^{k-1} \cdot \bar{\mu}(y_0,y_1) \cdots \bar{\mu}(y_{k-1},y_k) \\
&= \sum_{\substack{\widehat{0}=y_0<y_1<\cdots<y_k=\widehat{1} \\ y_i \in R}} (-1)^{k-1} \cdot (-1)^{\rho(y_0,y_1)} \cdot \bar{\zeta}(y_0,y_1) \cdots (-1)^{\rho(y_{k-1},y_k)} \cdot \bar{\zeta}(y_{k-1},y_k) \\
&= (-1)^n \cdot \sum_{\substack{\widehat{0}=y_0<y_1<\cdots<y_k=\widehat{1} \\ y_i \in R}} (-1)^k \cdot \bar{\zeta}(y_0,y_1) \cdots \bar{\zeta}(y_{k-1},y_k) \\
&= (-1)^n \cdot (\bar{\zeta}|_R)^{-1}(\widehat{0},\widehat{1}).
\end{aligned}$$

□

Complementary pairs of posets can be constructed using rank selection.

Corollary 3.8. *Let $(P, \rho, \bar{\zeta})$ be an Eulerian quasi-graded poset of rank $n+1$ with $\widehat{0}$ and $\widehat{1}$. Then the symmetric relation*

$$\bar{h}_S = \bar{h}_{\overline{S}} \tag{3.3}$$

holds for all subsets $S \subseteq \{1, 2, \ldots, n\}$.

Proof. Let P_S be the S rank-selected quasi-graded subposet $\{x \in P : \rho(x) \in S\} \cup \{\widehat{0}, \widehat{1}\}$. We have the following string of equalities:

$$\bar{h}_S = (-1)^{|S|-1} \cdot (\bar{\zeta}|_{P_S})^{-1}(\widehat{0}, \widehat{1}) = (-1)^{n-|S|-1} \cdot (\bar{\zeta}|_{P_{\overline{S}}})^{-1}(\widehat{0}, \widehat{1}) = (-1)^{|\overline{S}|-1} \cdot (\bar{\zeta}|_{P_{\overline{S}}})^{-1}(\widehat{0}, \widehat{1}) = \bar{h}_{\overline{S}}. \quad \square$$

4 The cd-index and quasi-graded posets

Bayer and Billera determined all the linear relations which hold among the flag f-vector of (classical) Eulerian posets, known as the generalized Dehn–Sommerville relations [2]. Bayer and Klapper showed that the space of flag f-vectors of Eulerian posets have a natural basis expressed by the **cd**-index [3]. Since the degree of $\mathbf{c}$ is 1 and the degree of $\mathbf{d}$ is 2, the dimension of the span of flag f-vectors Eulerian posets of rank $n+1$ is given by the Fibonacci number F_n, where $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Stanley later gave a more elementary proof of the existence of the **cd**-index for Eulerian posets and showed the coefficients are non-negative for spherically-shellable posets [46].

Theorem 4.1 (Bayer–Klapper)**.** *For the face lattice of a polytope, more generally, any graded Eulerian poset P, its **ab**-index $\Psi(P)$ can be written uniquely as a polynomial in the non-commutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ of degree one and two, respectively.*

Generalizing the classical result of Bayer and Klapper for graded Eulerian posets, we have the analogue for quasi-graded posets.

Theorem 4.2. *For an Eulerian quasi-graded poset $(P, \rho, \bar{\zeta})$ its **ab**-index $\Psi(P, \rho, \bar{\zeta})$ can be written uniquely as a polynomial in the non-commutative variables $\mathbf{c} = \mathbf{a}+\mathbf{b}$ and $\mathbf{d} = \mathbf{ab}+\mathbf{ba}$. Furthermore, if the function $\bar{\zeta}$ is integer-valued then the **cd**-index only has integer coefficients.*

Proof. Adding equations (2.5) and (2.6) in Proposition 2.4 and recalling that $\bar{\mu}(x, y) = (-1)^{\rho(x,y)} \cdot \bar{\zeta}(x, y)$, we obtain an expression for $2 \cdot \Psi([x, z], \rho, \bar{\zeta})$ in terms of the **ab**-index of smaller intervals $[x, y]$ and the **ab**-polynomials $(1 - (-1)^k) \cdot (\mathbf{a} - \mathbf{b})^{k-1}$ and $(\mathbf{b} - (-1)^k \cdot \mathbf{a}) \cdot (\mathbf{a} - \mathbf{b})^{k-1}$. Using Stanley's observation that these two **ab**-polynomials can be written as **cd**-polynomials [46], and considering the two cases when k is odd and when k is even separately, the result follows.

If $\bar{\zeta}$ is integer-valued then it is clear that the **ab**-index only has integer coefficients. Finally, the fact that the **cd**-index only has integer coefficients follows from [22, Lemma 3.5]. $\square$

Theorem 4.2 gives a different proof of Corollary 3.8 since any **cd**-polynomial when expressed in the variables **a** and **b** is symmetric in **a** and **b**.

A different way to express the existence of the **cd**-index is as follows.

Theorem 4.3. *The flag $\bar{f}$-vector of an Eulerian quasi-graded poset of rank $n+1$ satisfies the generalized Dehn–Sommerville relations. More precisely, for a subset $S \subseteq \{1, \ldots, n\}$ and $i, k \in S \cup \{0, n+1\}$ with $i < k$ and $S \cap \{i+1, \ldots, k-1\} = \emptyset$, the following relation holds:*

$$\sum_{j=i}^{k} (-1)^j \cdot \bar{f}_{S \cup \{j\}} = 0. \tag{4.1}$$

The relations in Theorem 4.3 reduce to the classical Dehn–Sommerville relations [12, 43] for Eulerian simplicial posets. See [38] for results related to the classical Dehn–Sommerville relations.

The next result asserts that if the **cd**-index exists for every interval in a quasi-graded poset then the poset itself is Eulerian.

Proposition 4.4. *Let $(P, \rho, \bar{\zeta})$ be a quasi-graded poset such that the* **ab***-index of every interval can be expressed in terms of* **c** *and* **d***. Then the quasi-graded poset $(P, \rho, \bar{\zeta})$ is Eulerian.*

Proof. Let $[x, y]$ be an interval of rank $k+1$. For an **ab**-monomial m and $q(\mathbf{a}, \mathbf{b})$ a non-commutative polynomial in the variables **a** and **b**, let $[m]q(a, b)$ denote the coefficient of m in $q(\mathbf{a}, \mathbf{b})$. We have

$$\begin{aligned} \bar{\mu}(x, y) &= (-1)^{k+1} \cdot \bar{h}_{\{1,\ldots,k\}}([x, y]) \\ &= (-1)^{k+1} \cdot [\mathbf{b}^k]\Psi([x, y]) \\ &= (-1)^{k+1} \cdot [\mathbf{a}^k]\Psi([x, y]) \\ &= (-1)^{k+1} \cdot \bar{h}_{\emptyset}([x, y]) \\ &= (-1)^{k+1} \cdot \bar{\zeta}(x, y), \end{aligned}$$

where the third equality follows from expanding the **cd**-polynomial $\Psi([x, y])$ in terms of **a**'s and **b**'s. The resulting identity is the Eulerian relation (3.1). □

5 Poset operations

In this section we extend some standard poset operations to quasi-graded posets.

Given a quasi-graded poset $(P, \rho_P, \bar{\zeta}_P)$, its *dual* is the quasi-graded poset $(P^*, \rho_{P^*}, \bar{\zeta}_{P^*})$, where the partial order satisfies $y \leq_{P^*} x$ if $x \leq_P y$, the weighted zeta function is given by $\bar{\zeta}_{P^*}(x, y) = \bar{\zeta}_P(y, x)$, and the rank function is $\rho_{P^*}(x) = -\rho_P(x)$. When the poset P has a minimal and maximal element, we prefer to use the rank function $\rho_{P^*}(x) = \rho_P(\hat{1}) - \rho_P(x)$ in order that the minimal element in P^* has rank 0.

Let $*$ also denote the involution on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ that reverses each monomial. Then for a graded quasi-graded poset P we have that $\Psi(P^*, \rho_{P^*}, \bar{\zeta}_{P^*}) = \Psi(P, \rho_P, \bar{\zeta}_P)^*$.

The *Cartesian product* of two quasi-graded posets $(P, \rho_P, \bar{\zeta}_P)$ and $(Q, \rho_Q, \bar{\zeta}_Q)$ is the triple $(P \times Q, \rho, \bar{\zeta})$ where the rank function is given by the sum $\rho((x, y)) = \rho_P(x) + \rho_Q(y)$ and the weighted zeta function is the product $\bar{\zeta}((x, y), (z, w)) = \bar{\zeta}_P(x, z) \cdot \bar{\zeta}_Q(y, w)$.

It is straightforward to verify the following result.

Proposition 5.1. *If two quasi-graded posets $(P, \rho_P, \bar{\zeta}_P)$ and $(Q, \rho_Q, \bar{\zeta}_Q)$ are both Eulerian then so is their Cartesian product.*

One important case is the Cartesian product with the Boolean algebra B_1. The geometric motivation is that taking the Cartesian product of the face lattice of a polytope with B_1 corresponds to taking the pyramid of the polytope. To do this, define the derivation $G : \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle \rightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ by $G(\mathbf{a}) = \mathbf{ba}$ and $G(\mathbf{b}) = \mathbf{ab}$. Observe that G restricts to a derivation on $\mathbf{cd}$-polynomials by $G(\mathbf{c}) = \mathbf{d}$ and $G(\mathbf{d}) = \mathbf{cd}$. Also define the operator Pyr by $\mathrm{Pyr}(w) = w \cdot \mathbf{c} + G(w)$. These two operators first appeared in [19].

Proposition 5.2. *Let P be a quasi-graded poset with $\widehat{0}$ and $\widehat{1}$. Then the* $\mathbf{ab}$*-index of the Cartesian product of P with the Boolean algebra B_1 is given by*

$$\Psi(P \times B_1) = \mathbf{b} \cdot \Psi(P) + \Psi(P) \cdot \mathbf{a} + \sum_{\substack{x \in P \\ \widehat{0} < x < \widehat{1}}} \Psi([\widehat{0}, x]) \cdot \mathbf{ab} \cdot \Psi([x, \widehat{1}]), \tag{5.1}$$

$$\Psi(P \times B_1) = \mathrm{Pyr}(\Psi(P)). \tag{5.2}$$

Proof. The proof follows the same outline as in [19, Proposition 4.2]. Consider a chain $c = \{(\widehat{0}, \widehat{0}) = (x_0, y_0) < (x_1, y_1) < \cdots < (x_k, y_k) = (\widehat{1}, \widehat{1})\}$ in $P \times B_1$. Let i be the smallest index such that $y_i = \widehat{1}$ and let $x = x_i$. Assume that $\widehat{0} < x < \widehat{1}$. Notice that the element $(x, \widehat{0})$ may or may not be in the chain c. Let c' denote the chain $c - \{(x, \widehat{0})\}$ and let c'' denote the chain $c \cup \{(x, \widehat{0})\}$. The $\mathbf{ab}$-weights of the chains are respectively given by

$$\begin{aligned} w(c') &= w_{[\widehat{0},x]}(c_1) \cdot (\mathbf{a} - \mathbf{b}) \cdot \mathbf{b} \cdot w_{[x,\widehat{1}]}(c_2), \\ w(c'') &= w_{[\widehat{0},x]}(c_1) \cdot \mathbf{b} \cdot \mathbf{b} \cdot w_{[x,\widehat{1}]}(c_2), \end{aligned}$$

where c_1 and c_2 are the chains obtained by restricting the chain c to the interval $[(\widehat{0}, \widehat{0}), (x, \widehat{0})] \cong [\widehat{0}, x]$, respectively $[(x, \widehat{1}), (\widehat{1}, \widehat{1})] \cong [x, \widehat{1}]$. Let $(w, \widehat{0})$ be element preceding $(x, \widehat{0})$ in the chain c''. Since the weighted zeta function is multiplicative and $\bar{\zeta}_{B_1}(\widehat{0}, \widehat{1}) = 1$ we have $\bar{\zeta}((w, \widehat{0}), (x, \widehat{1})) = \bar{\zeta}((w, \widehat{0}), (x, \widehat{0}))$. Thus we conclude that the two chains c' and c'' have the same $\bar{\zeta}$-weight given by the product $\bar{\zeta}(c') = \bar{\zeta}(c'') = \bar{\zeta}(c_1) \cdot \bar{\zeta}(c_2)$. Summing over all chains c_1 and c_2 and over all elements x, we obtain the summation expression in (5.1). The other two terms on the right-hand side of (5.1) follow from the two cases $x = \widehat{0}$ and $x = \widehat{1}$.

The proof of the second identity follows the same outline as the proof of Theorem 5.2 in [19], using that the $\mathbf{ab}$-index is a coalgebra homomorphism. See Theorem 2.5. $\square$

The *Stanley product* of two quasi-graded posets $(P, \rho_P, \bar{\zeta}_P)$ and $(Q, \rho_Q, \bar{\zeta}_Q)$ with minimal and maximal elements is the triple $(R, \rho, \bar{\zeta})$. The poset R is described by $R = (P - \{\widehat{1}\}) \cup (Q - \{\widehat{0}\})$ with the partial order given by $x \leq_{P*Q} y$ if (i) $x \leq_P y$ and $x, y \in P - \{\widehat{1}\}$; (ii) $x \in P - \{\widehat{1}\}$ and $y \in Q - \{\widehat{0}\}$; or (iii) $x \leq_Q y$ and $x, y \in Q - \{\widehat{0}\}$. The rank function ρ is given by $\rho(x) = \rho_P(x)$ if $x \in P - \{\widehat{1}\}$ and $\rho(x) = \rho_Q(x) + \rho_P(\widehat{0}, \widehat{1}) - 1$ if $x \in Q - \{\widehat{0}\}$. Finally, the weighted zeta function is given by $\bar{\zeta}(x, y) = \bar{\zeta}_P(x, y)$ if $x, y \in P - \{\widehat{1}\}$; $\bar{\zeta}(x, y) = \bar{\zeta}_Q(x, y)$ if $x, y \in Q - \{\widehat{0}\}$; or $\bar{\zeta}(x, y) = \bar{\zeta}_P(x, \widehat{1}) \cdot \bar{\zeta}_Q(\widehat{0}, y)$ if $x \in P - \{\widehat{1}\}$ and $y \in Q - \{\widehat{0}\}$.

If $c = \{\widehat{0} = x_0 < x_1 < \cdots < x_p < y_1 < y_2 < \cdots < y_q = \widehat{1}\}$ is a chain in the product $P * Q$ then each $x_i \in P$ and each $y_j \in Q$. Let c_P and c_Q denote the two restricted chains $c_P = \{\widehat{0} = x_0 < x_1 < \cdots < x_p < \widehat{1}\}$, respectively $c_Q = \{\widehat{0} < y_1 < y_2 < \cdots < y_q = \widehat{1}\}$. Note that the zeta weight of the chains is multiplicative, that is, $\bar{\zeta}(c) = \bar{\zeta}_P(c_P) \cdot \bar{\zeta}_Q(c_Q)$. Thus summing over all chains yields the next result.

Lemma 5.3. *The* **ab***-index is multiplicative with respect to the Stanley product of quasi-graded posets, that is, for two quasi-graded posets P and Q:*

$$\Psi(P * Q) = \Psi(P) \cdot \Psi(Q).$$

Recall the *up set* $U(x)$ of an element x from a poset P is the set $U(x) = \{v \in P \ : \ x < v\}$.

Lemma 5.4. *Let $(P, \rho, \bar{\zeta})$ be a quasi-graded poset with two elements x and y such that $\rho(x) = \rho(y)$, $U(x) = U(y)$ and $\bar{\zeta}(x, v) = \bar{\zeta}(y, v)$ for all elements v in the joint up set. Let $(Q, \rho_Q, \bar{\zeta}_Q)$ be the quasi-graded poset defined by*

(i) $Q = P - \{x, y\} \cup \{w\}$,

(ii) Q inherits the order relation from P with the new relations $u < x$ or $u < y$ implies $u <_Q w$, and $x < v$ (hence $y < v$) implies $w <_Q v$.

(iii) Q inherits the rank function from P with the new value $\rho_Q(w) = \rho(x)$,

(iv) Q inherits the weighted zeta function from P with the new values $\bar{\zeta}_Q(u, w) = \bar{\zeta}(u, x) + \bar{\zeta}(u, y)$ and $\bar{\zeta}_Q(w, v) = \bar{\zeta}(x, v)$.

Then the **ab***-indexes of $(P, \rho, \bar{\zeta})$ and $(Q, \rho_Q, \bar{\zeta}_Q)$ are equal, that is, $\Psi(P, \rho, \bar{\zeta}) = \Psi(Q, \rho_Q, \bar{\zeta}_Q)$.*

The proof follows by straightforward chain enumeration.

The other operation we need for quasi-graded posets is zipping. This was originally developed by Reading for Eulerian posets [39]. For our purposes we are considering the dual situation with respect to Reading's original notion.

Definition 5.5. *A* zipper *in a quasi-graded poset $(P, \rho, \bar{\zeta})$ consists of three elements x, y and z such that*

(i) $\rho(x) = \rho(y) = \rho(z) + 1$,

(ii) *the element* z *is only covered by* x *and* y*,*

(iii) *for all* $v > z$ *with* $\rho(v) > \rho(x)$ *we have* $\bar{\zeta}(x, v) = \bar{\zeta}(y, v) = \bar{\zeta}(z, v)$*,*

(iv) $\bar{\zeta}(z, x) = \bar{\zeta}(z, y) = 1$*.*

As a consequence of condition *(iii)*, the elements x and y in a zipper have the same up set.

The next step is to zip the zipper.

Definition 5.6. *Let* $(P, \rho, \bar{\zeta})$ *be a quasi-graded poset with zipper* x*,* y *and* z*. The* zipped *quasi-graded poset* Q *consists of the elements* $Q = P - \{x, y, z\} \cup \{w\}$ *such that the poset* Q *inherits the partial order relation from* P *together with the following new relations: if* $u <_P x$ *or* $u <_P y$ *then* $u <_Q w$ *and if* $x <_P v$ *or* $y <_P v$ *then* $w <_Q v$*. The rank function of* Q *is also inherited from* P *with the new value* $\rho_Q(w) = \rho(x)$*. Finally, the weighted zeta function has the new values*

$$\bar{\zeta}_Q(w, v) = \bar{\zeta}(x, v) \quad \textit{and} \quad \bar{\zeta}_Q(u, w) = \bar{\zeta}(u, x) + \bar{\zeta}(u, y) - \bar{\zeta}(u, z).$$

Observe that we use the fact $\bar{\zeta}(u, v) = 0$ if $u \not\leq v$. Hence the last relation implies that $\bar{\zeta}_Q(u, w) = \bar{\zeta}(u, x)$ if $u < x$ and $u \not< y$, and vice-versa, with the roles of x and y exchanged. Similarly, $\bar{\zeta}_Q(u, w) = \bar{\zeta}(u, x) + \bar{\zeta}(u, y)$ if $u < x$, $u < y$ but $u \not< z$.

Example 5.7. As a remark, we do not need (the dual of) Reading's original condition that $x \wedge y = z$ in the definition of a zipper since we do not require the weighted zeta function of a zipped quasi-graded poset $\bar{\zeta}_Q$ to always be equal to 1. As a concrete example, consider the the face lattice of the 2-gon with vertices v_1 and v_2 and edges e_1 and e_2. This poset is the rank 3 butterfly poset and it is Eulerian with the weighted zeta function $\bar{\zeta}$ equal to the zeta function ζ. The triple e_1, e_2 and v_1 is a zipper. After zipping we obtain the length 3 chain $Q = \{\widehat{0} \prec v_2 \prec w \prec \widehat{1}\}$. The weighted zeta function for Q is given by $\bar{\zeta}_Q(v_2, w) = 2$ and 1 everywhere else. It is easily checked that the resulting poset is Eulerian.

Proposition 5.8. *Let* $(P, \rho, \bar{\zeta})$ *be a quasi-graded poset with minimal and maximal elements. Assume that* P *has a zipper* x*,* y *and* z*. Then the* **ab***-indexes of the zipped poset* $(Q, \rho_Q, \bar{\zeta}_Q)$ *and of the interval* $([\widehat{0}, w], \rho_Q, \bar{\zeta}_Q)$ *are given by*

$$\Psi(Q) = \Psi(P) - \Psi([\widehat{0}, z]) \cdot \mathbf{d} \cdot \Psi([z, \widehat{1}]), \tag{5.3}$$

$$\Psi([\widehat{0}, w]) = \Psi([\widehat{0}, x]) + \Psi([\widehat{0}, y]) - \Psi([\widehat{0}, z]) \cdot \mathbf{c}. \tag{5.4}$$

Proof. We begin by summing the weights of chains from P which go through the elements x, y and z. The chains that contain z (and possibly x, y or neither) are enumerated by

$$\Psi([\widehat{0}, z]) \cdot (2 \cdot \mathbf{bb} + \mathbf{b}(\mathbf{a} - \mathbf{b})) \cdot \Psi([z, \widehat{1}]). \tag{5.5}$$

Here we use that $\bar{\zeta}(x,v) = \bar{\zeta}(y,v) = \bar{\zeta}(z,v)$. The chains in P that contain x but not z are enumerated by

$$\bar{\zeta}(\widehat{0},x)\cdot(\mathbf{a}-\mathbf{b})^{\rho(x)-1}\cdot\mathbf{b}\cdot\Psi([x,\widehat{1}]) + \sum_{\substack{\widehat{0}<u<x \\ u\neq z}} \Psi([\widehat{0},u])\cdot\mathbf{b}\cdot\bar{\zeta}(u,x)\cdot(\mathbf{a}-\mathbf{b})^{\rho(u,x)-1}\cdot\mathbf{b}\cdot\Psi([x,\widehat{1}]). \tag{5.6}$$

Similarly, the chains in P through y which do not contain z are enumerated by

$$\bar{\zeta}(\widehat{0},y)\cdot(\mathbf{a}-\mathbf{b})^{\rho(y)-1}\cdot\mathbf{b}\cdot\Psi([y,\widehat{1}]) + \sum_{\substack{\widehat{0}<u<y \\ u\neq z}} \Psi([\widehat{0},u])\cdot\mathbf{b}\cdot\bar{\zeta}(u,y)\cdot(\mathbf{a}-\mathbf{b})^{\rho(u,y)-1}\cdot\mathbf{b}\cdot\Psi([y,\widehat{1}]). \tag{5.7}$$

Finally, we enumerate the chains in the poset Q through the new element w:

$$\bar{\zeta}_Q(\widehat{0},w)\cdot(\mathbf{a}-\mathbf{b})^{\rho_Q(w)-1}\cdot\mathbf{b}\cdot\Psi([w,\widehat{1}]) + \sum_{\widehat{0}<u<w} \Psi([\widehat{0},u])\cdot\mathbf{b}\cdot\bar{\zeta}_Q(u,w)\cdot(\mathbf{a}-\mathbf{b})^{\rho_Q(u,w)-1}\cdot\mathbf{b}\cdot\Psi([w,\widehat{1}]). \tag{5.8}$$

Adding (5.6) and (5.7) and subtracting (5.8) gives

$$\begin{aligned}
&\bar{\zeta}(\widehat{0},z)\cdot(\mathbf{a}-\mathbf{b})^{\rho(z)}\cdot\mathbf{b}\cdot\Psi([x,\widehat{1}]) + \sum_{\widehat{0}<u<z} \Psi([\widehat{0},u])\cdot\mathbf{b}\cdot\bar{\zeta}(u,z)\cdot(\mathbf{a}-\mathbf{b})^{\rho(u,z)}\cdot\mathbf{b}\cdot\Psi([x,\widehat{1}]) \\
&= \Psi([\widehat{0},z])\cdot(\mathbf{a}-\mathbf{b})\cdot\mathbf{b}\cdot\Psi([x,\widehat{1}]).
\end{aligned} \tag{5.9}$$

The chains that jump from u to x where $u \not< y$ cancel with the corresponding chains jumping from u to w. A similar cancellation occurs with the roles of x and y exchanged. Hence the difference $\Psi(P) - \Psi(Q)$ is given by the sum of (5.5) and (5.9).

For the interval $[\widehat{0},w]$ in the poset Q we have

$$\begin{aligned}
\Psi([\widehat{0},w]) &= \bar{\zeta}_Q(\widehat{0},w)\cdot(\mathbf{a}-\mathbf{b})^{\rho(w)-1} + \sum_{\widehat{0}<u<w} \Psi([\widehat{0},u])\cdot\mathbf{b}\cdot\bar{\zeta}_Q(u,w)\cdot(\mathbf{a}-\mathbf{b})^{\rho(u,w)-1} \\
&= \bar{\zeta}(\widehat{0},x)\cdot(\mathbf{a}-\mathbf{b})^{\rho(x)-1} + \sum_{\substack{\widehat{0}<u<x \\ u\neq z}} \Psi([\widehat{0},u])\cdot\mathbf{b}\cdot\bar{\zeta}(u,x)\cdot(\mathbf{a}-\mathbf{b})^{\rho(u,x)-1} \\
&\quad + \bar{\zeta}(\widehat{0},y)\cdot(\mathbf{a}-\mathbf{b})^{\rho(y)-1} + \sum_{\substack{\widehat{0}<u<y \\ u\neq z}} \Psi([\widehat{0},u])\cdot\mathbf{b}\cdot\bar{\zeta}(u,y)\cdot(\mathbf{a}-\mathbf{b})^{\rho(u,y)-1} \\
&\quad - \bar{\zeta}(\widehat{0},z)\cdot(\mathbf{a}-\mathbf{b})^{\rho(z)} - \sum_{\widehat{0}<u<z} \Psi([\widehat{0},u])\cdot\mathbf{b}\cdot\bar{\zeta}(u,z)\cdot(\mathbf{a}-\mathbf{b})^{\rho(u,z)} \\
&= \Psi([\widehat{0},x]) - \Psi([\widehat{0},z])\cdot\mathbf{b} + \Psi([\widehat{0},y]) - \Psi([\widehat{0},z])\cdot\mathbf{b} - \Psi([\widehat{0},z])\cdot(\mathbf{a}-\mathbf{b}),
\end{aligned}$$

where in the second and third equality the computations are taking place in the original poset P. This is the desired expression. □

Theorem 5.9. *For quasi-graded posets the zipping operation preserves the Eulerian property.*

Proof. Consider an interval $[u,v]$ in the zipped poset Q, where P is the original quasi-graded poset with zipper x, y and z. First we claim that this interval has a **cd**-index. There are four cases to consider. (i) If w does not belong to the interval $[u,v]$ then $[u,v]$ is the same as in the original poset P and hence is Eulerian. (ii) If $u=w$ then $[w,v]$ is isomorphic to $[x,v]$ and is Eulerian. (iii) If $v=w$ then the interval $[u,w]$ has a **cd**-index by equation (5.4). (iv) Finally, if $u<w<v$ then the interval $[u,v]$ has a **cd**-index by equation (5.3). In each case the interval has a **cd**-index and by Proposition 4.4 we conclude that the poset Q is Eulerian. □

6 Whitney stratified sets

We begin with a modest example.

Example 6.1. Consider the non-regular CW complex Ω consisting of one vertex v, one edge e and one 2-dimensional cell c such that the boundary of c is the union $v\cup e$, that is, boundary of the complex Ω is a one-gon. Its face poset is the four element chain $\mathscr{F}(\Omega)=\{\widehat{0}<v<e<c\}$. This is not an Eulerian poset. The classical definition of the **ab**-index, in other words, using $\bar{\zeta}(x,y)=1$ for all $x\leq y$, yields that the **ab**-index of Ω is $\mathbf{a}^2$. Note that $\mathbf{a}^2$ cannot be written in terms of $\mathbf{c}$ and $\mathbf{d}$.

Observe that the edge e is attached to the vertex v twice. Hence it is natural to change the value of $\bar{\zeta}(v,e)$ to be 2 and to keep the remaining values of $\bar{\zeta}(x,y)$ to be 1. The face poset $\mathscr{F}(\Omega)$ is now Eulerian, its **ab**-index is given by $\bar{\zeta}(\Omega)=\mathbf{a}^2+\mathbf{b}^2$ and hence its **cd**-index is $\bar{\zeta}(\Omega)=\mathbf{c}^2-\mathbf{d}$.

The motivation for the value 2 in Example 6.1 is best expressed in terms of the Euler characteristic of the link. The link of the vertex v in the edge e is two points whose Euler characteristic is 2. In order to view this example in the right topological setting, we review the notion of a Whitney stratification. For more details, see [14], [26], [27, Part I §1.2], and [35].

A subset S of a topological space M is *locally closed* if S is a relatively open subset of its closure $\overline{S}$. Equivalently, for any point $x\in S$ there exists a neighborhood $U_x\subseteq S$ such that the closure $\overline{U_x}\subseteq S$ is closed in M. Another way to phrase this is a subset $S\subset M$ is locally closed if and only if it is the intersection of an open subset and a closed subset of M.

Definition 6.2. *Let W be a closed subset of a smooth manifold M which has been decomposed into a finite union of locally closed subsets called strata:*

$$W=\bigcup_{X\in\mathcal{P}}X.$$

Furthermore suppose this decomposition satisfies the condition of the frontier*:*

$$X\cap\overline{Y}\neq\emptyset\Longleftrightarrow X\subseteq\overline{Y}.$$

This implies the closure of each stratum is a union of strata, and it provides the index set $\mathcal{P}$ with the partial ordering:

$$X\subseteq\overline{Y}\Longleftrightarrow X\leq_{\mathcal{P}}Y.$$

This decomposition of W is a Whitney stratification *if*

1. *Each $X \in \mathcal{P}$ is a (locally closed, not necessarily connected) smooth submanifold of M.*

2. *If $X <_{\mathcal{P}} Y$ then Whitney's conditions (A) and (B) hold: Suppose $y_i \in Y$ is a sequence of points converging to some $x \in X$ and that $x_i \in X$ converges to x. Also assume that (with respect to some local coordinate system on the manifold M) the secant lines $\ell_i = \overline{x_i y_i}$ converge to some limiting line ℓ and the tangent planes $T_{y_i}Y$ converge to some limiting plane τ. Then the following inclusions hold:*

$$\text{(A) } T_x X \subseteq \tau \qquad \text{and} \qquad \text{(B) } \ell \subseteq \tau.$$

Remark 6.3. For convenience we will henceforth also assume that W is pure dimensional, meaning that if $\dim(W) = n$ then the union of the n-dimensional strata of W forms a dense subset of W. Strata of dimension less than n are referred to as *singular strata*.

Local structure of stratified sets. Lemmas 6.4 and 6.6 below are "standard" but the proofs (which we omit) involve the full strength of Thom's *first isotopy lemma*; see [14, 26, 27, 35, 36, 50].

Let $A, A' \subset M$ be smooth submanifolds. They are said to be *transverse in M at a point* $x \in A \cap A'$ if $T_xA + T_xA' = T_xM$. The submanifolds A, A' are said to be transverse (in M) if they are transverse at every point of their intersection.

Two Whitney stratified subsets $W, W' \subset M$ are said to be *transverse at a point* $x \in W \cap W'$ if the stratum A of W that contains x is transverse at the point x to the stratum A' of W' that contains x. The Whitney stratified sets W, W' are said to be *transverse* if they are transverse at every point, that is, if every stratum A of W is transverse to every stratum A' of W'. In this case, the intersection $W \cap W'$ is Whitney stratified with strata of the form $A \cap A'$ where A is a stratum of W and A' is a stratum of W'.

Transversality is an *open* condition: if $A \leq_{\mathcal{P}} B$ and if $A' \leq_{\mathcal{P}} B'$ are strata of W, W' respectively and if A is transverse to A' at a point $x \in A \cap A'$ then there exists a neighborhood $U \subset M$ of the point x such that $W \cap U$ is transverse to $W' \cap U$, in other words, the following four conditions hold: (i) A is transverse to A' at every point $y \in A \cap A' \cap U$, (ii) A is transverse to B' at every point $y \in A \cap B' \cap U$, (iii) B is transverse to A' at every point $y \in B \cap A' \cap U$ and (iv) B is transverse to B' at every point $y \in B \cap B' \cap U$. Transversality is also a *dense* condition: two compact Whitney stratified subsets $W, W' \subset \mathbb{R}^n$ of Euclidean space may be made transverse by moving one of them by a translation, say, $W' \to W' + a$, where the vector $a \in \mathbb{R}^n$ may be chosen to be arbitrarily small.

Let W be a Whitney stratified closed subset of a smooth manifold M. Let X be a stratum of W and let $x \in X$. Let $N_x \subset M$ be a *normal slice* to X at x, that is, a smooth submanifold of M that is transverse to X, with $N_x \cap X = \{x\}$ (so that $\dim(N_x) + \dim(X) = \dim(M)$). Let $B_\epsilon(x)$ be a closed ball (with respect to some local coordinate system for M) of radius ϵ centered at x. If ϵ is chosen sufficiently small then

1. the boundary $\partial B_\epsilon(x)$ is transverse to N_x,

2. the intersection $\partial B_\epsilon(x) \cap N_x$ is transverse to every stratum of W.

Consequently the intersection

$$\mathrm{link}_W(X,x) := N_x \cap \partial B_\epsilon(x) \cap W$$

is Whitney stratified by its intersection with the strata of W. The link is well-defined in the following sense.

Lemma 6.4. *Assume the stratum X is connected. Let $x' \in X$, let $N'_{x'}$ be a choice of normal slice to X at the point x' and let $B'_{\epsilon'}(x')$ be a choice of closed ball centered at x'. Then for ϵ, ϵ' sufficiently small the intersections*

$$\mathrm{link}_W(X,x) := N_x \cap \partial B_\epsilon(x) \cap W \quad \textit{and} \quad \mathrm{link}_W(X,x') := N'_{x'} \cap \partial B'_{\epsilon'}(x') \cap W$$

are homeomorphic by a homeomorphism ψ that is smooth on each stratum such that $\psi(x) = x'$.

We may therefore refer to "the" link of the stratum X by choosing a point $x \in X$ and writing $\mathrm{link}_W(X) = \mathrm{link}_W(X,x)$. If $X \subset \overline{Y}$ are strata of W and if $x \in X$ then we write

$$\mathrm{link}_Y(X) = N_x \cap \partial B_\epsilon(x) \cap Y$$

for the intersection of the link of X with the stratum Y, and similarly for $\mathrm{link}_{\overline{Y}}(X)$. These intersections are also independent of the same choices as described above.

The link is preserved under transverse intersection in the following sense. Let $P \subset M$ be a smooth submanifold that is transverse to every stratum of W. Let X be a stratum of W and let $x \in X \cap P$. Then there is a stratum-preserving homeomorphism that is smooth on each stratum,

$$\mathrm{link}_{P\cap W}(P \cap X, x) \cong \mathrm{link}_W(X,x). \tag{6.1}$$

This follows from Lemma 6.4 by choosing the normal slice N_x to be contained in P. With this choice, $\mathrm{link}_{P\cap W}(P \cap X, x) = \mathrm{link}_W(X,x)$.

Remark 6.5. The "top" stratum, or largest stratum of a Whitney stratified set W need not be connected. Normally, one requires that the singular strata of a Whitney stratified set W should be connected. However it is possible to allow disconnected singular strata provided

(i) that the condition of the frontier continues to hold, and

(ii) that the links L_1, L_2 of any two connected components A_1, A_2 of a singular stratum A are isomorphic, meaning that there is a stratum preserving homeomorphism $h_{12} : L_1 \longrightarrow L_2$ that is smooth on each stratum.

(Here, the stratification of L_i is the natural one given by the intersection of L_i with the strata of W, for $i = 1, 2$.)

Lemma 6.6. *Let $W \subset M$ be a Whitney stratified closed subset of a smooth manifold M. Let X be a stratum of W and let $x \in X$. Then there exists a basis $\mathcal{B}_x$ for the neighborhoods of x in W such that every neighborhood $U \in \mathcal{B}_x$ has a local product structure, that is, there exists a stratum-preserving homeomorphism*

$$U \cong \mathbb{R}^{\dim(X)} \times \mathrm{cone}(\mathrm{link}_W(x)) \tag{6.2}$$

that is smooth on each stratum which takes the basepoint $\{x\}$ to $\{0\} \times \{$cone point$\}$.

It follows that a Whitney stratification of a closed set $W \subset M$ is *locally trivial*: if $x, y \in X$ and if $\mathcal{B}_x, \mathcal{B}_y$ denote the corresponding bases of neighborhoods of x and y (respectively) then for any $U_x \in \mathcal{B}_x$ and any $U_y \in \mathcal{B}_y$ there is a stratum-preserving homeomorphism

$$h : U_x \longrightarrow U_y \tag{6.3}$$

that is smooth on each stratum such that $h(x) = y$.

Remark 6.7. Complex algebraic, complex analytic, real algebraic, real analytic, semi-algebraic, semi-analytic, and sub-analytic sets all admit Whitney stratifications [13, 28, 29, 30, 34, 36, 50, 51]. The existence of Whitney stratifications, together with the local triviality theorems, constitute one of the great triumphs of stratification theory because they provide a deep understanding of the local structure of the singularities of algebraic and analytic sets. In particular they say that these sets do not exhibit fractal or Cantor set-like behavior.

Remark 6.8. An example of an algebraic set W with a decomposition into smooth manifolds that is *not* locally trivial is provided by *Whitney's cusp*. See [35, Example 2.6] and [52]. It is the variety $x^3 + y^2 = x^2z^2$. It can be decomposed into two smooth manifolds as follows. The "small stratum" is the line $x = y = 0$, and all the rest is the "large stratum". This decomposition is not locally trivial at the origin, although it satisfies Whitney's condition (A) everywhere. Thus, condition (A) does not suffice to guarantee local triviality of the stratification. Whitney guessed (correctly) that requiring condition (B) would restore local triviality. Indeed the above decomposition fails condition (B) at the origin. To restore condition (B) it is necessary to refine the stratification by declaring the origin to be a third stratum. The resulting decomposition is locally trivial.

We next state the key definition for developing face incidence enumeration for Whitney stratified spaces.

Definition 6.9. *Let W be a Whitney stratified closed subset of a smooth manifold M. Define the face poset $\mathcal{F} = \mathcal{F}(W)$ of W to be the quasi-graded poset consisting of the poset of strata $\mathcal{P}$ adjoined with a minimal element $\widehat{0}$. The rank function is given by*

$$\rho(X) = \begin{cases} \dim(X) + 1 & \textit{if } X > \widehat{0}, \\ 0 & \textit{if } X = \widehat{0}, \end{cases}$$

and the weighted zeta function is

$$\bar{\zeta}(X, Y) = \begin{cases} \chi(\mathrm{link}_Y(X)) & \textit{if } X > \widehat{0}, \\ \chi(Y) & \textit{if } X = \widehat{0}. \end{cases}$$

Theorem 6.10. *Let W be Whitney stratified closed subset of a smooth manifold M. Then the face poset of W is an Eulerian quasi-graded poset.*

The proof of Theorem 6.10 occupies Sections 8 through 9.

We now give a few examples of Whitney stratifications beginning with the classical polygon.

Example 6.11. Consider a two dimensional cell c with its boundary subdivided into n vertices $v_1, \ldots, v_n$ and n edges $e_1, \ldots, e_n$. There are three ways to view this as a Whitney stratification.

(1) Declare each of the $2n+1$ cells to be individual strata. This is the classical view of an n-gon. Here the weighted zeta function is the classical zeta function, that is, always equal to 1 (assuming $n \geq 2$).

(2) Declare each of the n edges to be one stratum $e = \cup_{i=1}^{n} e_i$, that is, we have the $n+2$ strata $v_1, \ldots, v_n, e, c$. Here the non-one values of the weighted zeta function are given by $\bar{\zeta}(\widehat{0}, e) = n$ and $\bar{\zeta}(v_i, e) = 2$.

(3) Lastly, we can have the three strata $v = \cup_{i=1}^{n} v_i$, $e = \cup_{i=1}^{n} e_i$ and c. Now non-one values of the weighted zeta function are given by $\bar{\zeta}(\widehat{0}, v) = \bar{\zeta}(\widehat{0}, e) = n$ and $\bar{\zeta}(v, e) = 2$.

In contrast, we cannot have $v, e_1, \ldots, e_n, c$ as a stratification, since the link of a point p in e_i depends on the point p in v chosen.

By Lemma 5.4 the **cd**-index of each of the three Whitney stratifications in Example 6.11 are the same, that is, $\mathbf{c}^2 + (n-2) \cdot \mathbf{d}$. Hence we have the immediate corollary.

Corollary 6.12. *The* **cd**-*index of an n-gon is given by* $\mathbf{c}^2 + (n-2) \cdot \mathbf{d}$ *for* $n \geq 1$.

The last stratification in the previous example can be extended to any simple polytope.

Example 6.13. Let P be an n-dimensional simple polytope, that is, every interval $[x, y]$ in the face lattice, where $\widehat{0} < x \leq y$, is isomorphic to a Boolean algebra. We obtain a different stratification of the ball by joining all the facets together to one strata. By Lemma 5.4 we note that the **cd**-index does not change, since the information is carried in the weighted zeta function. We continue by joining all the subfacets together to one strata. Again the **cd**-index remains unchanged. In the end we obtain a stratification where the union of all the i-dimensional faces forms the ith strata. The face poset of this stratification is the $(n+2)$-element chain $C = \{\widehat{0} = x_0 < x_1 < \cdots < x_{n+1} = \widehat{0}\}$, with the rank function $\rho(x_i) = i$ and weighted zeta function $\bar{\zeta}(\widehat{0}, x_i) = f_{i-1}(P)$ and $\bar{\zeta}(x_i, x_j) = \binom{n+1-i}{n+1-j}$. Again, by Lemma 5.4 we have $\Psi(C, \rho, \bar{\zeta}) = \Psi(P)$.

A similar stratification can be obtained for any regular polytope.

Example 6.14. Consider the stratification of an n-dimensional manifold with boundary, denoted $(M, \partial M)$, into its boundary ∂M and its interior M°. The face poset is $\{\widehat{0} < \partial M < M^\circ\}$ with the elements having ranks 0, n and $n+1$, respectively. The weighted zeta function is given by $\bar{\zeta}(\widehat{0}, \partial M) = \chi(\partial M)$, $\bar{\zeta}(\widehat{0}, M^\circ) = \chi(M)$ and $\bar{\zeta}(\partial M, M^\circ) = 1$. If n is even then ∂M is an odd-dimensional manifold without boundary and hence its Euler characteristic is 0. In this case the **ab**-index is $\Psi(M) = \chi(M) \cdot (\mathbf{a} - \mathbf{b})^n$. If n is odd then we have the relation $\chi(\partial M) = 2 \cdot \chi(M)$ and hence the **ab**-index is given by $\Psi(M) = \chi(M) \cdot (\mathbf{a} - \mathbf{b})^n + 2 \cdot \chi(M) \cdot (\mathbf{a} - \mathbf{b})^{n-1} \cdot \mathbf{b}$. Passing to the **cd**-index we conclude

$$\Psi(M) = \begin{cases} \chi(M) \cdot (\mathbf{c}^2 - 2\mathbf{d})^{n/2} & \text{if } n \text{ is even,} \\ \chi(M) \cdot (\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} \cdot \mathbf{c} & \text{if } n \text{ is odd.} \end{cases}$$

The next example is a higher dimensional analogue of the one-gon in Example 6.1.

Example 6.15. Consider the subdivision Ω_n of the n-dimensional ball $\mathbb{B}^n$ consisting of a point p, an $(n-1)$-dimensional cell c and the interior b of the ball. If $n \geq 2$, the face poset is $\{\widehat{0} < p < c < b\}$ with the elements having ranks 0, 1, n and $n+1$, respectively. In the case $n = 1$, the two elements p and c are incomparable. The weighted zeta function is given by $\bar{\zeta}(\widehat{0}, p) = \bar{\zeta}(\widehat{0}, c) = \bar{\zeta}(\widehat{0}, b) = 1$, $\bar{\zeta}(p, c) = 1 + (-1)^n$, and $\bar{\zeta}(p, b) = \bar{\zeta}(c, b) = 1$. Thus the **ab**-index is

$$\Psi(\Omega_n) = (\mathbf{a}-\mathbf{b})^n + \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{n-1} + (\mathbf{a}-\mathbf{b})^{n-1} \cdot \mathbf{b} + (1 + (-1)^n) \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot \mathbf{b}. \tag{6.4}$$

When n is even the expression (6.4) simplifies to

$$\begin{aligned}\Psi(\Omega_n) &= \mathbf{a} \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot \mathbf{a} + \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot \mathbf{b} \\ &= \frac{1}{2} \cdot \left[(\mathbf{a}-\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot (\mathbf{a}-\mathbf{b}) + (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot (\mathbf{a}+\mathbf{b})\right] \\ &= \frac{1}{2} \cdot \left[(\mathbf{c}^2 - 2\mathbf{d})^{n/2} + \mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^{(n-2)/2} \cdot \mathbf{c}\right]. \end{aligned} \tag{6.5}$$

When n is odd the expression (6.4) simplifies to

$$\begin{aligned}\Psi(\Omega_n) &= \mathbf{a} \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot \mathbf{a} - \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot \mathbf{b} \\ &= \frac{1}{2} \cdot \left[(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot (\mathbf{a}-\mathbf{b}) + (\mathbf{a}-\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot (\mathbf{a}+\mathbf{b})\right] \\ &= \frac{1}{2} \cdot \left[\mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} + (\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} \cdot \mathbf{c}\right]. \end{aligned} \tag{6.6}$$

As a remark, these **cd**-polynomials played an important role in proving that the **cd**-index of a polytope is coefficient-wise minimized on the simplex, namely, $\Psi(\Omega_n) = (-1)^{n-1} \cdot \alpha_n$, where α_n are defined in [5]. See Theorem 11.4 for a generalization of one of the main identities in [5].

The weighted zeta function of a stratification can take negative values, as our last example illustrates.

Example 6.16. Let M be a solid 3-dimensional torus such that its boundary is the 2-dimensional torus $\mathbb{T}^2$. Consider the stratification of $\mathbb{T}^2$ into n points $V = \{v_1, v_2, \ldots, v_n\}$ and the n-punctured torus $\mathbb{T}^2 - V$. Its weighted zeta function is negative on the interval $[\widehat{0}, \mathbb{T}^2 - V]$, that is, $\bar{\zeta}(\widehat{0}, \mathbb{T}^2 - V) = \chi(\mathbb{T}^2 - V) = -n$. The **cd**-index of this stratification of the solid torus is $n \cdot \mathbf{dc} - n \cdot \mathbf{cd}$.

7 Properties of the Euler characteristic

In this section we state two results involving the Euler characteristic of the boundary of a manifold and an inclusion-exclusion expression for the Euler characteristic. Both will support results leading to the proof of Theorem 6.10 in Section 9.

Lemma 7.1. *Let $(A, \partial A)$ be a connected n-dimensional manifold (possibly non-orientable) with (possibly empty) boundary. Then the Euler characteristic of the boundary is given by*

$$\chi(\partial A) = \left(1 - (-1)^{\dim(A)}\right) \cdot \chi(A). \tag{7.1}$$

Proof. (Sketch) This follows from Poincaré duality $H^i(A, \partial A; \mathbb{Z}_2) \cong H_{n-i}(A; \mathbb{Z}_2)$ and the long exact homology sequence for the pair $(A, \partial A)$, together with the fact that the Euler characteristic does not depend on the coefficient field of the homology groups. □

Lemma 7.2. *Let $X = \bigcup_{i=1}^r U_i$ be an open cover of a topological space X. For any $R \subseteq \{1, 2, \ldots, r\}$ let $U_R = \bigcap_{i \in R} U_i$. Then we have the following inclusion-exclusion formula for the Euler characteristic:*

$$\chi(X) = \sum_{\phi \neq R \subseteq [r]} (-1)^{|R|+1} \cdot \chi(U_R). \tag{7.2}$$

Proof. (Sketch) This follows from induction and the Mayer–Vietoris theorem or equivalently, from the Mayer–Vietoris spectral sequence. □

8 Tube systems and control data

After Whitney introduced his conditions (A) and (B), René Thom, in a remarkable paper [50], described a daring technique for proving that Whitney stratifications were locally trivial. Thom constructed the required homeomorphism (6.2–6.3) as the time 1 continuous flow of a certain discontinuous (but "controlled") vector field. Although his outline was sound, there were many serious gaps and difficulties with the exposition that were finally resolved in J. Mather's wonderful notes [35], which are still the best source for this material. The first step involves the construction of "control data", that is, a system of compatible tubular neighborhoods of the strata. Further references for tube systems include Section 2.5 in [14], Section 2.2 in [26] and Part I, Section 1.5 in [27].

Let A be a smooth submanifold of a smooth manifold M. Let $\nu(A)$ denote the normal bundle of A in M. Choose a smooth inner product $\langle \cdot, \cdot \rangle$ on $\nu(A)$ and let $\nu(A)_{<\epsilon} = \{v \in \nu(A) : \langle v, v \rangle < \epsilon\}$. This quantity may be thought of as the set of vectors whose distance squared from A is less than ϵ. A *tubular neighborhood* of A is such a choice of inner product on the normal bundle $\nu(A)$ together with a smooth embedding $\phi : \nu(A)_{<\epsilon} \longrightarrow M$ of the ϵ neighborhood of the zero section such that the restriction of ϕ to the zero section is the identity map. It follows that the image $T_A = T_A(\epsilon) \subset M$ is an open neighborhood of A in M. The projection $\pi : \nu(A) \longrightarrow A$ determines a smooth projection $\pi_A : T_A \longrightarrow A$ and the inner product determines a smooth "tubular function" $\rho_A : T_A \longrightarrow [0, \epsilon)$ such that $(\pi_A, \rho_A) : T_A \longrightarrow A \times (0, \epsilon)$ is a smooth proper submersion (and hence is a smooth fiber bundle).

Let $W \subset M$ be a closed subset with a fixed Whitney stratification. A *system of control data* on W consists of a choice of tubular neighborhood T_A for each stratum A of W with the following properties:

1. $T_A \cap T_B = \phi$ unless $A < B$ or $B < A$,
2. if $A < B$ then $\pi_A \pi_B(x) = \pi_A(x)$ for all $x \in T_A \cap T_B$

3. if $A < B$ then $\rho_A \pi_B(x) = \rho_A(x)$ for all $x \in T_A \cap T_B$
4. The mapping $(\pi_A, \rho_A) : W \cap T_A \longrightarrow A \times (0, \epsilon)$ is a stratified submersion, that is, a locally trivial fiber bundle (with fiber $\mathrm{link}_W(A)$) whose restriction to each stratum is a submersion.

Every Whitney stratified set admits a system of control data, and we henceforth assume that control data for W has been chosen.

Let A be a stratum of W and let $\Sigma(A) = \overline{A} - A$ be the closed union of strata in the closure of A. We may assume that ρ_A extends to the $\overline{T}_A - \Sigma(A)$ and takes the value ϵ on the boundary ∂T_A which is the image of the ϵ-sphere bundle in the normal bundle of A. If ϵ is chosen sufficiently small then the manifold $\partial T_A \subset M$ is transverse to all the strata $B > A$ (and it does not intersect any strata $C < A$) so it is Whitney stratified by its intersection with the strata of W.

For each stratum A define $A^0 = A - \bigcup_{B<A} \overline{T}_B$. This is an open subset of the stratum A. The family of lines can be used to describe a homeomorphism $A^0 \longrightarrow A$. Moreover, A^0 is naturally the interior of a manifold with boundary (corners), $A^1 := A - \bigcup_{B<A} T_B \subset A$ whose boundary is a union

$$\partial A^1 = \bigcup_{B<A} \left(\partial T_B - \bigcup_{C \neq B} T_C \right)$$

of pieces of the sphere bundles of strata $B < A$. Let $\overline{X}$ be the closure of a single stratum of W. Then the system of control data $(T_A, \pi_A, \rho_A, \Phi^A)$ when intersected with $\overline{X}$ gives a system of control data for $\overline{X}$.

If A is a stratum of W and if $x \in A$ then $\pi_A^{-1}(x) = N_x$ is a "normal slice" to A at x as described in Section 6. Thus the link of the stratum A, at the point $x \in A$ is (homeomorphic to) the set

$$\mathrm{link}(A) = \pi_A^{-1}(x) \cap \partial T_A \cap W.$$

For any stratum $B > A$ the *link of the stratum A in B* is the (not necessarily compact, not necessarily connected) manifold

$$\mathrm{link}_B(A) = \pi_A^{-1}(x) \cap \partial T_A \cap B.$$

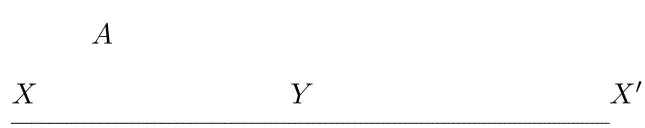

Figure 1: Strata: $X < Y < A$

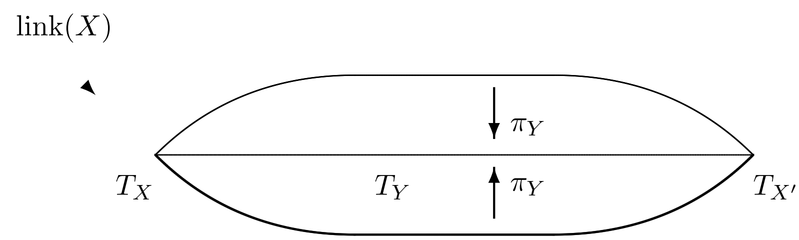

Figure 2: Tubular neighborhoods

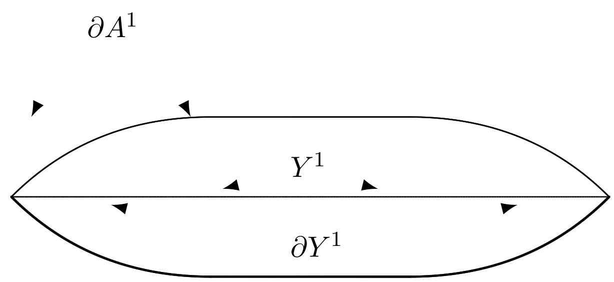

Figure 3: Manifolds with boundary

Proposition 8.1. *Let $B_1 < B_2 < \cdots < B_k$ be a chain of strata and let $\pi_j : T_{B_j} \longrightarrow B_j$ be the projection. Define*

$$\partial T(B_1, B_2, \ldots, B_k) := \partial T_{B_1} \cap \partial T_{B_2} \cap \cdots \cap \partial T_{B_k}. \tag{8.1}$$

If $\epsilon > 0$ is sufficiently small then for any stratum A of W the Euler characteristic of the intersection $\partial T(B_1, B_2, \ldots, B_k) \cap A$ factors:

$$\chi\left(\partial T(B_1, B_2, \ldots, B_k) \cap A\right) = \chi(B_1) \cdot \chi(\mathrm{link}_{B_2}(B_1)) \cdot \chi(\mathrm{link}_{B_3}(B_2)) \cdots \chi(\mathrm{link}_A(B_k)). \tag{8.2}$$

Proof. To simplify the notation, we write π_1, ρ_1 rather than π_{B_1}, ρ_{B_1}, etc. We will prove by induction on t that if ϵ is sufficiently small then

(a) the collection of submanifolds $\partial T_{B_i} \subset M$ is mutually transverse,

(b) $\partial T(B_1, B_2, \ldots, B_k)$ is transverse to every stratum A of W, and

(c) the intersection $\partial T(B_1, B_2, \ldots, B_k) \cap W$ may be described as an iterated stratified fibration:

$$
\begin{tikzcd}
 & \operatorname{link}_W(B_k) \arrow[dl] \\
\partial T(B_1, B_2, \ldots, B_k) \cap W \arrow[d, "\pi_t"] & \operatorname{link}(B_{k-1}) \cap B_k \arrow[dl] \\
\partial T(B_1, B_2, \ldots, B_{k-1}) \cap B_k \arrow[d, "\pi_{t-1}"] & \operatorname{link}(B_{k-2}) \cap B_{k-1} \arrow[dl] \\
\partial T(B_1, B_2, \ldots, B_{k-2}) \cap B_{k-1} \arrow[d] & \\
\cdots \arrow[d] & \operatorname{link}(B_1) \cap B_2 \arrow[dl] \\
\partial T(B_1) \cap B_2 \arrow[d, "\pi_1"] & \\
B_1. &
\end{tikzcd}
$$

Consequently, for any stratum $A > B_k$ the intersection $\partial T(B_1, B_2, \ldots, B_k) \cap A$ also fibers in this way, from which equation (8.2) follows immediately.

For $k = 1$, part (a) is vacuous and parts (b) and (c) are simply a restatement of property (4) in the definition of control data.

For the inductive step, suppose we have proven (a), (b), (c) for a chain of strata $B_2 < B_3 < \cdots < B_k$. In particular,

$$\partial T(B_2, B_3, \ldots, B_k) \cap W \longrightarrow \cdots \longrightarrow B_2 \tag{8.3}$$

is an iterated fibration. Consider the effect of adding a stratum B_1 to the bottom of this chain. By property (4), the mapping

$$(\pi_1, \rho_1) : T(B_1) \cap W \longrightarrow B_1 \times (0, \epsilon)$$

is a stratified fibration: it is a submersion on each stratum. Composing this with the tower (8.3) gives an iterated stratified fibration

$$\partial T(B_2, B_3, \ldots, B_k) \cap T(B_1) \cap W \longrightarrow \cdots \longrightarrow B_2 \cap T(B_1) \longrightarrow B_1 \times (0, \epsilon).$$

The projection to $(0, \epsilon)$ is a submersion on each stratum, which is equivalent to the statement that for all $s \in (0, \epsilon)$ the submanifold $\rho_1^{-1}(s) \subset M$ is transverse to every stratum in this tower. Shrinking ϵ if necessary, the submanifold $\rho_1^{-1}(\epsilon) = \partial T(B_1)$ is therefore transverse to every stratum that appears in this tower. Thus, restricting to $\rho_1^{-1}(\epsilon)$ gives the desired iterated stratified fibration

$$\partial T(B_2, \ldots, B_k) \cap \partial T(B_1) \longrightarrow \cdots \longrightarrow B_2 \cap \partial T(B_1) \longrightarrow B_1. \qquad \square$$

9 The main formula

In this section we will give the proof of our main result from Section 6, that the face poset of a Whitney stratified closed subset of a smooth manifold is indeed an Eulerian quasi-graded poset.

Recall that we have homotopy equivalences $A^0 \subset A^1 \subset A$ for a stratum A.

Proposition 9.1. *Let A be a stratum of W. Then the boundary ∂A^1 is a topological manifold and*

$$\chi(\partial A^1) = \sum_{B<A} (-1)^{\dim(B)} \cdot \chi(B) \cdot \chi(\mathrm{link}_A(B)). \tag{9.1}$$

Consequently, the following identity holds:

$$\left(1 + (-1)^{\dim(A)+1}\right) \cdot \chi(A) + \sum_{B<A} (-1)^{\dim(B)+1} \cdot \chi(B) \cdot \chi(\mathrm{link}_A(B)) = 0. \tag{9.2}$$

Proof. Let us cover the boundary ∂A^1 with ϵ-neighborhoods of the closed sets $S_B = \partial T_B - \bigcup_{C \neq B} T_C$ as B ranges over the strata $B < A$. By equation (7.2) of Lemma 7.2, the Euler characteristic $\chi(\partial A^1)$ is an alternating sum of the Euler characteristics of the multi-intersections of these sets S_B. As observed in Section 8 such an intersection is empty unless the corresponding strata form a chain. Let $\mathcal{P}$ denote the partially ordered set of strata $B < A$. If $\mathbf{R} \subseteq \mathcal{P}$ let $\partial T(\mathbf{R}) = \bigcap_{B \in \mathbf{R}} S_B$. By Lemma 7.2 we have

$$\chi(\partial A^1) = \sum_{\phi \neq \mathbf{R} \subseteq \mathcal{P}} (-1)^{|\mathbf{R}|+1} \cdot \chi(\partial T(\mathbf{R}) \cap A).$$

As observed in Proposition 8.1, the term $\chi(\partial T(\mathbf{R}) \cap A)$ is zero unless $\mathbf{R} = (R_1 < R_2 < \cdots < R_k)$ is a chain of strata, with $R_k < A$, in which case the Euler characteristic is given by equation (8.2). We will need to focus on the top element $R_{|\mathbf{R}|}$ of each chain. For notational convenience let us define

$$\chi(\mathbf{R}) = \chi(R_1) \cdot \chi(\mathrm{link}_{R_2}(R_1)) \cdot \chi(\mathrm{link}_{R_3}(R_2)) \cdots \chi(\mathrm{link}_{R_k}(R_{k-1})).$$

Then equation (8.2) gives $\chi(\partial T(\mathbf{R}) \cap A) = \chi(\mathbf{R}) \cdot \chi(\mathrm{link}_A(R_k))$ and we have

$$\chi(\partial A^1) = \sum_{\mathbf{R}<A} (-1)^{|\mathbf{R}|+1} \cdot \chi(\mathbf{R}) \cdot \chi(\mathrm{link}_A(R_{|\mathbf{R}|})). \tag{9.3}$$

If $|\mathbf{R}| > 1$ then it is possible to remove the top element and obtain a smaller chain $\mathbf{T} = (T_1 < T_2 < \cdots < T_{k-1}) = (R_1 < R_2 < \cdots < R_{k-1})$ such that $\mathbf{R} = (\mathbf{T} < R_{|\mathbf{R}|})$. Let $B = R_k = R_{|\mathbf{R}|}$ denote this top element and note that $R_{k-1} = T_{|\mathbf{T}|}$. Then the corresponding term in equation (9.3) is therefore

$$(-1)^{|\mathbf{R}|+1} \cdot \chi(\mathbf{T}) \cdot \chi(\mathrm{link}_B(T_{|\mathbf{T}|})) \cdot \chi(\mathrm{link}_A(B)).$$

If $|\mathbf{R}| = 1$ then the chain $\mathbf{R}$ consists of a single stratum $B \in \mathcal{P}$ and the contribution from this term is just $\chi(B) \cdot \chi(\mathrm{link}_A(B))$.

With this notation we are able to group together the terms involving $\chi(\mathrm{link}_A(B))$ as $B \in \mathcal{P}$ varies, that is, we group terms according to the last factor in equation (8.2) to obtain

$$\chi(\partial A^1) = \sum_{B \in \mathcal{P}} \chi(\mathrm{link}_A(B)) \cdot \left(\chi(B) + \sum_{\mathbf{T}<B} (-1)^{|\mathbf{T}|} \cdot \chi(\mathbf{T}) \cdot \chi(\mathrm{link}_B(T_{|\mathbf{T}|})) \right).$$

By equation (9.3) the inner sum is just $-\chi(\partial B^1)$, so the quantity inside the parentheses is $\chi(B) - \chi(\partial B^1) = (-1)^{\dim(B)} \cdot \chi(B)$ by equation (7.1). Consequently

$$\chi(\partial A^1) = \sum_{B \in \mathcal{P}} \chi(\mathrm{link}_A(B)) \cdot (-1)^{\dim B} \cdot \chi(B),$$

which is equation (9.1). Equation (9.2) follows from equation (9.1) and Lemma 7.1. $\square$

We are now ready to give the proof of the main result from Section 6.

Proof of Theorem 6.10. Let $X < Z$ be elements of the face poset $\mathcal{F}(P)$, where we allow the possibility that $X = \widehat{0}$. Let $W' := \mathrm{link}_W(X) \cap \overline{Z}$ be the intersection of the link of X with the closure of Z. This is again a Whitney stratified subset of M and its poset $\mathcal{P}'$ of strata is equal to the interval $[X, Z] \subseteq \mathcal{P}$ because the strata of W' are all of the form $Y' = Y \cap \mathrm{link}_W(X)$ where $X \leq Y \leq Z$. The minimal element $\widehat{0}'$ of $\mathcal{P}'$ corresponds to the empty stratum, that is, $\widehat{0}' = X \cap \mathrm{link}_W(X) = \phi$. Moreover, we claim that the weighted zeta function for W', denoted by $\bar{\zeta}'$, coincides with that of W restricted to the interval $[X, Z]$, that is,

$$\bar{\zeta}(X, Y) = \bar{\zeta}'(X', Y').$$

In fact, by equation (6.1), taking $P = \mathrm{link}_M(X)$, there is a stratum-preserving homeomorphism

$$\mathrm{link}_W(Y) \cong \mathrm{link}_{W \cap P}(Y \cap P)$$

and hence, intersecting with $\overline{Z}$ gives a homeomorphism

$$\mathrm{link}_W(Y) \cap \overline{Z} \cong \mathrm{link}(Y') \cap \overline{Z}'$$

where $Y' = Y \cap P = Y \cap \mathrm{link}(X)$ and $Z' = Z \cap P = Z \cap \mathrm{link}(X)$ are the corresponding strata of W'.

The rank function ρ' of $\mathcal{P}'$ is

$$\rho(Y') = \dim(Y') + 1 = \dim(Y) + 1 - \dim(X) - 1 = \rho(X, Y).$$

Therefore we may apply Equation (9.2) to the stratified space W' and thereby obtain

$$\sum_{X \leq Y \leq Z} (-1)^{\rho(X,Y)} \cdot \bar{\zeta}(X, Y) \cdot \bar{\zeta}(Y, Z) = \delta_{X,Z}.$$

$\square$

10 The semisuspension

Let Γ be a polytopal complex, that is, a regular cell complex whose cells are polytopes. Assume the dimension of Γ is k. Let $n > k$ be an integer. We define the *nth semisuspension* of Γ, denoted $\mathrm{Semi}(\Gamma, n)$, to be the family of CW complexes obtained by embedding Γ in the boundary of an n-dimensional ball $\mathbb{B}^n$. Thus we are adding the two strata $\partial\mathbb{B}^n - \Gamma$ and the interior of $\mathbb{B}^n$ to the complex Γ to obtain the stratification. Note that one really has a family of embeddings. For example, one can embed a circle into the boundary of a 4-dimensional ball so that the result is any given knot. Nevertheless, we will show the face poset of $\mathrm{Semi}(\Gamma, n)$ is well-defined. Furthermore, in the case Γ is homeomorphic to a k-dimensional ball, the semisuspension $\mathrm{Semi}(\Gamma, n)$ is unique up to homeomorphism.

The *face poset* $\mathscr{F}(\mathrm{Semi}(\Gamma, n))$ of the nth semisuspension of Γ consists of the face poset of the complex Γ with two extra elements $*$ and $\widehat{1}$ of rank n and $n+1$, respectively, where the element $*$ corresponds to $\mathbb{S}^{n-1} - \Gamma$ and the maximal element corresponds to the interior of the ball $\mathbb{B}^n$. The order relations are $x \leq *$ for $x \in \Gamma$ and $* \prec \widehat{1}$. The weighted zeta function $\bar{\zeta}$ is given by

$$\bar{\zeta}(x, y) = \begin{cases} \chi(\mathbb{S}^{n-1-\rho(x)} - \mathrm{link}_\Gamma(x)) & \text{if } x < * \text{ and } y = *, \\ 1 & \text{otherwise.} \end{cases} \tag{10.1}$$

Observe that when $n = k+1$ the element $*$ would have the same rank as any of the k-dimensional facets x and hence $\bar{\zeta}(x, *) = 0$. By Alexander duality we know that for a closed subset A of $\mathbb{S}^m$ the Euler characteristic of the complement $\chi(\mathbb{S}^m - A)$ is given by $\chi(A)$ if m is odd and $2 - \chi(A)$ if m is even. Using the reduced Euler characteristic we can write this relation as $\chi(\mathbb{S}^m - A) = 1 - (-1)^m \cdot \widetilde{\chi}(A)$. Thus the weighted zeta function can be rewritten as

$$\bar{\zeta}(x, y) = \begin{cases} 1 + (-1)^{\rho(x,*)} \cdot \widetilde{\chi}(\mathrm{link}_\Gamma(x)) & \text{if } x < y \text{ and } y = *, \\ 1 & \text{otherwise.} \end{cases} \tag{10.2}$$

Also note that for a facet x of Γ, the link $\mathrm{link}_\Gamma(x)$ is the complex consisting of the empty set, and hence its reduced Euler characteristic is -1.

Note that equation (10.2) does not depend on the particular embedding of the complex Γ into the n-dimensional sphere. Summarizing this discussion and using Theorem 6.10, we have the following result.

Proposition 10.1. *Let Γ be a k-dimensional polytopal complex and let $n > k$ be an integer. Then the face poset of the nth semisuspension $\mathscr{F}(\mathrm{Semi}(\Gamma, n))$ having the weighted zeta function $\bar{\zeta}$ in* (10.1) *does not depend on the embedding of Γ into the boundary of $\mathbb{B}^n$. Furthermore, the face poset $\mathcal{F}(\mathrm{Semi}(\Gamma, n))$ is an Eulerian quasi-graded poset.*

We now can relate the **cd**-index of the $(n+1)$st semisuspension to that of the nth semisuspension.

Proposition 10.2. *Let Γ be a polytopal complex of dimension less than n. Then the following identity holds:*

$$\Psi(\mathrm{Semi}(\Gamma, n+1)) = \Psi(\mathrm{Semi}(\Gamma, n)) \cdot \mathbf{c} - \Psi([\widehat{0}, *]) \cdot \mathbf{d},$$

where the interval $[\widehat{0}, *]$ occurs in the face poset of the semisuspension $\mathrm{Semi}(\Gamma, n)$*.*

Proof. We begin by expanding the **cd**-index of the semisuspension $\mathrm{Semi}(\Gamma, n+1)$ using the chain definition of the **ab**-index. Note there are four types of chains, depending on whether the $*$-element is in the chain or not, and whether the chain contains a non-empty element of the complex Γ or not. In order to distinguish between the two face posets $\mathcal{F}(\mathrm{Semi}(\Gamma, n))$ and $\mathcal{F}(\mathrm{Semi}(\Gamma, n+1))$, we mark the weighted zeta function, the rank function and the element $*$ of the second poset with primes. We have

$$\begin{aligned}\Psi(\mathrm{Semi}(\Gamma, n+1)) = (\mathbf{a}-\mathbf{b})^{n+1} + \sum_{x \in \Gamma - \{\widehat{0}\}} \Psi([\widehat{0}, x]) \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\rho'(x,\widehat{1})-1} \\ + \bar{\zeta}'(\widehat{0}, *') \cdot (\mathbf{a}-\mathbf{b})^{n} \cdot \mathbf{b} + \sum_{x \in \Gamma - \{\widehat{0}\}} \Psi([\widehat{0}, x]) \cdot \mathbf{b} \cdot \bar{\zeta}'(x, *') \cdot (\mathbf{a}-\mathbf{b})^{\rho'(x,*')-1} \cdot \mathbf{b}.\end{aligned}$$

Note that $\bar{\zeta}'(x, *') = 1 + (-1)^{\rho'(x,*')} \cdot \widetilde{\chi}(\mathrm{link}_\Gamma(x)) = 2 - \big(1 + (-1)^{\rho(x,*)} \cdot \widetilde{\chi}(\mathrm{link}_\Gamma(x))\big) = 2 - \bar{\zeta}(x, *)$. Hence the above expression translates to

$$\begin{aligned}\Psi(\mathrm{Semi}(\Gamma, n+1)) &= (\mathbf{a}-\mathbf{b})^{n+1} + \sum_{x \in \Gamma - \{\widehat{0}\}} \Psi([\widehat{0}, x]) \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\rho(x,\widehat{1})-1} \cdot (\mathbf{a}-\mathbf{b}) \\ &\quad + (2 - \bar{\zeta}(\widehat{0}, *)) \cdot (\mathbf{a}-\mathbf{b})^{n} \cdot \mathbf{b} \\ &\quad + \sum_{x \in \Gamma - \{\widehat{0}\}} \Psi([\widehat{0}, x]) \cdot \mathbf{b} \cdot (2 - \bar{\zeta}(x, *)) \cdot (\mathbf{a}-\mathbf{b})^{\rho(x,*)-1} \cdot (\mathbf{a}-\mathbf{b}) \cdot \mathbf{b} \\ &= (\mathbf{a}-\mathbf{b})^{n} \cdot \mathbf{c} + \sum_{x \in \Gamma - \{\widehat{0}\}} \Psi([\widehat{0}, x]) \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\rho(x,\widehat{1})-1} \cdot \mathbf{c} \\ &\quad - \bar{\zeta}(\widehat{0}, *) \cdot (\mathbf{a}-\mathbf{b})^{n-1} \cdot (\mathbf{a}-\mathbf{b}) \cdot \mathbf{b} \\ &\quad - \sum_{x \in \Gamma - \{\widehat{0}\}} \Psi([\widehat{0}, x]) \cdot \mathbf{b} \cdot \bar{\zeta}(x, *) \cdot (\mathbf{a}-\mathbf{b})^{\rho(x,*)-1} \cdot (\mathbf{a}-\mathbf{b}) \cdot \mathbf{b} \\ &= \Psi(\mathrm{Semi}(\Gamma, n)) \cdot \mathbf{c} - \bar{\zeta}(\widehat{0}, *) \cdot (\mathbf{a}-\mathbf{b})^{n-1} \cdot \mathbf{d} \\ &\quad - \sum_{x \in \Gamma - \{\widehat{0}\}} \Psi([\widehat{0}, x]) \cdot \mathbf{b} \cdot \bar{\zeta}(x, *) \cdot (\mathbf{a}-\mathbf{b})^{\rho(x,*)-1} \cdot \mathbf{d} \\ &= \Psi(\mathrm{Semi}(\Gamma, n)) \cdot \mathbf{c} - \Psi([\widehat{0}, *]) \cdot \mathbf{d},\end{aligned}$$

where we used $-(\mathbf{a}-\mathbf{b}) \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} - \mathbf{d}$ in the third step. □

11 Inclusion-exclusion for the semisuspension and its cd-index

This section begins with an inclusion-exclusion relation for the nth semisuspension of polytopal complexes.

Theorem 11.1. *Let Γ and Δ be two polytopal complexes such that their union $\Gamma \cup \Delta$ is a polytopal complex of dimension less than n. Then the following inclusion-exclusion relation holds:*

$$\Psi(\mathrm{Semi}(\Gamma, n)) + \Psi(\mathrm{Semi}(\Delta, n)) = \Psi(\mathrm{Semi}(\Gamma \cap \Delta, n)) + \Psi(\mathrm{Semi}(\Gamma \cup \Delta, n)). \tag{11.1}$$

Proof. Consider a chain $c = \{\widehat{0} < x_1 < \cdots < x_i = x < \widehat{1}\}$ in the face poset of $\text{Semi}(\Gamma \cup \Delta, n))$ whose largest element x belongs to $\Gamma \cup \Delta$. If the element x is in the intersection $\Gamma \cap \Delta$ then this chain is enumerated twice on both sides of the identity. If the element x lies in Γ but not in $\Gamma \cap \Delta$ then this chain is enumerated once on both sides. Symmetrically, if the element x lies in Δ but not in the intersection then the chain is again enumerated once on both sides.

It remains to consider chains $c = \{\widehat{0} < x_1 < \cdots < x_i = x < * < \widehat{1}\}$ that contain the element $*$. Again let x be the largest element in the chain contained in the union $\Gamma \cup \Delta$. If x does not belong to the intersection $\Gamma \cap \Delta$ then the chain c is enumerated in one term from each side of (11.1). The case that remains is when x belongs to the intersection $\Gamma \cap \Delta$. Note that the link of x in Γ, Δ, $\Gamma \cap \Delta$ and $\Gamma \cup \Delta$ are all polytopal complexes and the reduced Euler characteristic of polytopal complexes behaves as a valuation, that is, $\widetilde{\chi}(\text{link}_{\Gamma}(x)) + \widetilde{\chi}(\text{link}_{\Delta}(x)) = \widetilde{\chi}(\text{link}_{\Gamma\cap\Delta}(x)) + \widetilde{\chi}(\text{link}_{\Gamma\cup\Delta}(x))$. Hence we have

$$\begin{aligned}
\bar{\zeta}_{\Gamma}(x,*) + \bar{\zeta}_{\Delta}(x,*) &= 1 + (-1)^{\rho(x,*)} \cdot \widetilde{\chi}(\text{link}_{\Gamma}(x)) + 1 + (-1)^{\rho(x,*)} \cdot \widetilde{\chi}(\text{link}_{\Delta}(x)) \\
&= 1 + (-1)^{\rho(x,*)} \cdot \widetilde{\chi}(\text{link}_{\Gamma\cap\Delta}(x)) + 1 + (-1)^{\rho(x,*)} \cdot \widetilde{\chi}(\text{link}_{\Gamma\cup\Delta}(x)) \\
&= \bar{\zeta}_{\Gamma\cap\Delta}(x,*) + \bar{\zeta}_{\Gamma\cup\Delta}(x,*),
\end{aligned}$$

using equation (10.2). Hence the zeta weight of the chain c satisfies $\bar{\zeta}_{\Gamma}(c)+\bar{\zeta}_{\Delta}(c) = \bar{\zeta}_{\Gamma\cap\Delta}(c)+\bar{\zeta}_{\Gamma\cup\Delta}(c)$ and thus contributes the same amount to both sides of (11.1). $\square$

Corollary 11.2. *Let $\Gamma_1, \ldots, \Gamma_r$ be r polytopal complexes such that their union has dimension less than n. Then the following inclusion-exclusion relation holds:*

$$\Psi\left(\text{Semi}\left(\bigcup_{i=1}^{r} \Gamma_i, n\right)\right) = \sum_{\emptyset \neq I \subseteq \{1,\ldots,r\}} (-1)^{|I|-1} \cdot \Psi\left(\text{Semi}\left(\bigcap_{i\in I} \Gamma_i, n\right)\right).$$

For a polytopal complex Γ with facets $F_1, \ldots, F_m$, define the *nerve complex* $\mathcal{N}(\Gamma)$ to be the simplicial complex with vertex set $\{1, \ldots, m\}$ and the subset I is a face if the intersection $\bigcap_{i\in I} F_i$ is non-empty. We need the following version of the nerve theorem. A weaker version is due to Borsuk [11, Theorem 1]. For other versions and references, see Björner's overview article [8, Section 4]. The version we state here follows from Björner, Korte and Lovász [9, Theorem 4.5].

Theorem 11.3 (Nerve theorem for polytopal complexes)**.** *For a polytopal complex Γ, the complex Γ and the nerve complex $\mathcal{N}(\Gamma)$ are homotopy equivalent.*

We can now generalize Proposition 4.3 in [5]. Recall that Ω_n denotes the stratification of an n-dimensional closed ball into a point, an $(n-1)$-dimensional cell and an n-dimensional cell. See Example 6.15.

Theorem 11.4. *Let Γ be a polytopal complex of dimension less than n. Assume that Γ has facets $F_1, \ldots, F_r$. Then the* **cd***-index of the semisuspension* $\text{Semi}(\Gamma, n)$ *is given by*

$$\Psi(\text{Semi}(\Gamma, n)) = -\sum_{F} \widetilde{\chi}(\text{link}_{\Gamma}(F)) \cdot \Psi(F) \cdot \Psi(\Omega_{n-\dim(F)}),$$

where the sum is over all possible intersections F of the facets $F_1, \ldots, F_r$.

Proof. By the inclusion-exclusion Corollary 11.2, we have that

$$\begin{aligned}\Psi(\mathrm{Semi}(\Gamma, n)) &= \sum_{\emptyset \neq I \subseteq \{1,\ldots,r\}} (-1)^{|I|-1} \cdot \Psi\left(\mathrm{Semi}\left(\bigcap_{i\in I} F_i, n\right)\right) \\ &= \sum_{F} \left(\sum_{\substack{\emptyset \neq I \subseteq \{1,\ldots,r\} \\ \bigcap_{i\in I} F_i = F}} (-1)^{|I|-1} \right) \cdot \Psi(\mathrm{Semi}(F, n)), \qquad (11.2)\end{aligned}$$

where the outer sum is over all possible intersections F of the facets $F_1, \ldots, F_r$. We express the inner sum using the nerve complex of the link. For F a face of Γ let $J(F)$ be the non-empty index set $J(F) = \{i \in I \,:\, F \subseteq F_i\}$. The inner sum of (11.2) is given by

$$\begin{aligned}\sum_{\substack{\emptyset \neq I \subseteq J(F) \\ \bigcap_{i\in I} F_i = F}} (-1)^{|I|-1} &= \sum_{\emptyset \neq I \subseteq J(F)} (-1)^{|I|-1} - \sum_{\substack{I \subseteq J(F) \\ \bigcap_{i\in I} F_i \supsetneq F}} (-1)^{|I|-1} \\ &= 1 - \chi(\mathcal{N}(\mathrm{link}_\Gamma(F))) \\ &= -\widetilde{\chi}(\mathcal{N}(\mathrm{link}_\Gamma(F))) \\ &= -\widetilde{\chi}(\mathrm{link}_\Gamma(F)), \qquad (11.3)\end{aligned}$$

where in the last step we used that the nerve complex of a polytopal complex is homotopy equivalent to the original complex and hence they have the same (reduced) Euler characteristic. Finally, observe that the face poset of $\mathrm{Semi}(F, n)$ is given by the Stanley product

$$\mathcal{F}(\mathrm{Semi}(F, n)) = \mathcal{F}(F) * \mathcal{F}(\Omega_{n-\dim(F)}). \qquad (11.4)$$

By combining equations (11.2), (11.3) and (11.4), the result follows. $\square$

Theorem 11.4 generalizes Proposition 4.3 in [5] which considered the case when $F_1, \ldots, F_r$ is the initial line shelling segment of an n-dimensional polytope. Their proof is based on shelling, whereas the proof we give here for Theorem 11.4 is an application of inclusion-exclusion.

12 The Eulerian relation for the semisuspension

Let Γ be a regular subdivision of an n-dimensional ball $\mathbb{B}^n$ such that the interior of the ball is one of the faces. Let Λ be a regular subdivision of Γ such that the interior of the ball is yet again a face of Λ. For a face F of Γ we define $\Lambda|_F$ to be the subdivision of F induced by Λ. There are two extremal cases. When F is the empty set, let $\Lambda|_F$ be the empty subdivision of the empty face. In this case the semisuspension $\mathrm{Semi}(\Lambda|_F, n)$ is the $(n-1)$-dimensional sphere and the interior of the n-dimensional ball. The second extremal case is when $F = \widehat{1}$, and we let $\Lambda|_F$ and $\mathrm{Semi}(\Lambda|_F, n)$ denote the subdivision Λ of the n-dimensional sphere.

Theorem 12.1. *Let Γ be a regular subdivision of the n-ball $\mathbb{B}^n$ and let Λ be a regular subdivision of Γ such that both subdivisions have the interior of the ball as a face. Then the alternating sum of*

cd-*indexes of semisuspensions is equal to zero, that is,*

$$\sum_{F \in \Gamma} (-1)^{\rho(F,\widehat{1})} \cdot \Psi(\mathrm{Semi}(\Lambda|_F, n)) = 0.$$

Proof. The chains enumerated by the term $\Psi(\mathrm{Semi}(\Lambda|_F, n))$ fall into four cases: (i) the empty chain $\{\widehat{0} < \widehat{1}\}$, (ii) the chain $\{\widehat{0} < * < \widehat{1}\}$, (iii) chains containing non-trivial elements from Λ but not containing $*$, and finally, (iv) chains containing $*$ and non-trivial elements from Λ.

The alternating sum of the weights of the chains of type (i) is given by

$$\sum_{\widehat{0} \leq F \leq \widehat{1}} (-1)^{\rho(F,\widehat{1})} \cdot (\mathbf{a} - \mathbf{b})^n = 0, \tag{12.1}$$

since the face poset of any regular subdivision satisfies the classical Eulerian relation. Similarly, the alternating sum of the weights of the chains of type (ii) is given by

$$(-1)^{n+1} \cdot \left(1 + (-1)^{n-1}\right) \cdot (\mathbf{a} - \mathbf{b})^{n-1} \cdot \mathbf{b} + \sum_{\widehat{0} < F < \widehat{1}} (-1)^{\rho(F,\widehat{1})} \cdot (\mathbf{a} - \mathbf{b})^{n-1} \cdot \mathbf{b} = 0. \tag{12.2}$$

Observe that the first term corresponds to $F = \widehat{0}$ (see Example 6.14) and that there is no contribution from $F = \widehat{1}$.

Now consider chains of type (iii). Note that there is no contribution from the term $F = \widehat{0}$. From the remaining terms we have the contribution

$$\begin{aligned} &\sum_{\widehat{0} < F \leq \widehat{1}} (-1)^{\rho(F,\widehat{1})} \cdot \sum_{\substack{\widehat{0} < x < \widehat{1} \\ \sigma(x) \leq F}} \Psi([\widehat{0}, x]) \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,\widehat{1})-1} \\ &= \sum_{\widehat{0} < x < \widehat{1}} \left(\sum_{\sigma(x) \leq F \leq \widehat{1}} (-1)^{\rho(F,\widehat{1})} \right) \cdot \Psi([\widehat{0}, x]) \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,\widehat{1})-1} = 0, \end{aligned} \tag{12.3}$$

since the inner sum is equal to zero. Here we let $\sigma(x)$ denote the smallest dimensional face in Γ containing the face $x \in \Lambda$.

Finally, consider the chains of type (iv). Here the only contribution is from the terms $\widehat{0} < F < \widehat{1}$:

$$\begin{aligned} &\sum_{\widehat{0} < F < \widehat{1}} (-1)^{\rho(F,\widehat{1})} \cdot \sum_{\substack{\widehat{0} < x < \widehat{1} \\ \sigma(x) \leq F}} \Psi([\widehat{0}, x]) \cdot \mathbf{b} \cdot \bar{\zeta}(x, *) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,*)-1} \cdot \mathbf{b} \\ &= \sum_{\widehat{0} < x < \widehat{1}} \left(\sum_{\sigma(x) \leq F < \widehat{1}} (-1)^{\rho(F,\widehat{1})} \cdot \bar{\zeta}(x, *) \right) \cdot \Psi([\widehat{0}, x]) \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,*)-1} \cdot \mathbf{b} = 0. \end{aligned} \tag{12.4}$$

Again we claim that the inner sum is equal to zero. Observe that if we have the strict inequality $\sigma(x) < F$ then the face x is on the boundary of F and $\mathrm{link}_F(x)$ is contractible, that is, the reduced

Euler characteristic is 0 and the weighted zeta function is given by $\bar{\zeta}(x,*) = 1$. On the other hand if the equality $\sigma(x) = F$ holds then the face x is on the interior $\sigma(x)$ and $\text{link}_{\sigma(x)}(x)$ is sphere of dimension $\rho(\sigma(x)) - \rho(x) - 1$. Its reduced Euler characteristic is $(-1)^{\rho(\sigma(x))-\rho(x)-1}$. Hence we have $\bar{\zeta}(x,*) = 1 + (-1)^{\rho(x,*)} \cdot (-1)^{\rho(\sigma(x))-\rho(x)-1} = 1 + (-1)^{\rho(\sigma(x),\widehat{1})}$ and the inner sum is given by

$$(-1)^{\rho(\sigma(x),\widehat{1})} \cdot \left(1 + (-1)^{\rho(\sigma(x),\widehat{1})}\right) + \sum_{\sigma(x)<F<\widehat{1}} (-1)^{\rho(F,\widehat{1})} = \sum_{\sigma(x)\leq F\leq\widehat{1}} (-1)^{\rho(F,\widehat{1})} = 0.$$

The Eulerian relation now follows by summing the four identities (12.1), (12.2), (12.3) and (12.4). □

13 Merging strata

We now consider the operation of merging three strata and its effect on the **cd**-index. This is the geometric analogue of zipping elements in quasi-graded posets.

Theorem 13.1. *Let W be a Whitney stratification of a manifold M such that M^0 is one of the strata. Assume that x, y and z are three strata such that $\dim(x) = \dim(y) = \dim(z) + 1$ and when replacing the three strata x, y and z with a single stratum $w = x \cup y \cup z$ one obtains a Whitney stratification W'. Then the **cd**-index changes according to*

$$\Psi(W') = \Psi(W) - \Psi([\widehat{0}, z]) \cdot \mathbf{d} \cdot \Psi([x, \widehat{1}]).$$

Proof. Note that $W' = W - \{x, y, z\} \cup \{w\}$. We begin to show that x, y and z form a zipper in the face poset of W. Condition (i) in Definition 5.5 follows from the dimension condition in the statement of the theorem.

Since W' is a Whitney stratification, we know that the link of a point $p \in w$ is independent of the choice of p if the point p belongs to x, y or z. Hence the strata z is only covered by the strata x and y, verifying condition (ii). By similar reasoning condition (iii) follows. Pick a point p in z. Locally the neighborhood of p in z is $\mathbb{R}^{\dim(z)}$ and the neighborhood of p in w is $\mathbb{R}^{\dim(z)+1}$. Thus the neighborhood of p in x (or y) is a half space. Hence $\text{link}_x(z)$ is a point and we conclude that $\bar{\zeta}(x, z) = 1$ (and $\bar{\zeta}(y, z) = 1$).

Next we verify that the zipped poset is indeed the face poset of the stratification W'. All we need to verify is that their weighted zeta functions agree, since they already have the same poset structure and rank function. Observe that $\text{link}_v(x)$ is the same as $\text{link}_v(w)$ since x is contained in w. Hence we have $\bar{\zeta}_{W'}(w, v) = \bar{\zeta}(x, v)$.

Next we must show $\bar{\zeta}_{W'}(u, w) = \bar{\zeta}(u, x) + \bar{\zeta}(u, y) - \bar{\zeta}(u, z)$ for all $u \in P$. Here we have several cases to verify. If the strata u is comparable to x (or y) only, two terms are equal to zero and the identity holds. If the strata u is less than z (and hence x and y) then the identity follows by the principle of inclusion-exclusion for the Euler characteristic. See Lemma 7.2. Finally, if the strata u is less than x and y, but not z, then $\bar{\zeta}(u,z) = 0$ and the identity follows by the additivity of the Euler characteristic.

Hence the face poset of W' is the result of zipping the face poset of W and the identity follows from Proposition 5.8. $\square$

14 Shelling components for non-pure simplicial complexes

We now turn our attention to computing the **cd**-index of the nth semisuspension of a (non-pure) shellable simplicial complex. The first step is to define the **cd**-index of the simplicial shelling components.

For $i \leq k$ let $\Delta_{k,i}$ be the simplicial complex consisting of $i+1$ facets of the k-dimensional simplex. Define the quasi-graded poset $P_{n,k,i}$ for $0 \leq i \leq k \leq n$ to be the face poset of the semisuspension

$$P_{n,k,i} = \mathcal{F}(\mathrm{Semi}(\Delta_{k,i}, n)).$$

Define the **cd**-index of the simplicial shelling component, $\check{\Phi}(n,k,i)$, to be the difference

$$\check{\Phi}(n,k,i) = \Psi(P_{n,k,i}) - \Psi(P_{n,k,i-1}) \quad \text{for } 1 \leq i \leq k \leq n,$$

and $\check{\Phi}(n,k,0) = \Psi(P_{n,k,0})$ for $0 \leq k \leq n$. See Table 1 for the degree 2 and 3 cases. Observe that

$$\Psi(P_{n,k,i}) = \sum_{j=0}^{i} \check{\Phi}(n,k,j). \tag{14.1}$$

The polynomials $\check{\Phi}(n,n,i)$ (the case $k = n$) were introduced by Stanley [46].

To obtain a recursion for the shelling components, we need the following identity.

Proposition 14.1. *For $0 \leq i \leq k \leq n$ the following identity holds:*

$$\Psi(P_{n,k,i} \times B_1) + \Psi(P_{n+1,k+1,0}) = \Psi(P_{n+1,k+1,i+1}) + \Psi(P_{n,k,i} * B_2). \tag{14.2}$$

Proof. View the only facet of $\Delta_{k+1,0}$ to be one of the facets not among the facets of $\Delta_{k+1,i}$. Then we have the union $\Delta_{k+1,i} \cup \Delta_{k+1,0} = \Delta_{k+1,i+1}$ and the intersection $\Delta_{k+1,i} \cap \Delta_{k+1,0} = \Delta_{k,i}$. Hence by the inclusion-exclusion relation in Theorem 11.1, we have

$$\Psi(\mathrm{Semi}(\Delta_{k+1,i}, n+1)) + \Psi(\mathrm{Semi}(\Delta_{k+1,0}, n+1)) = \Psi(\mathrm{Semi}(\Delta_{k+1,i+1}, n+1)) + \Psi(\mathrm{Semi}(\Delta_{k,i}, n+1)).$$

Or equivalently,

$$\Psi(P_{n+1,k+1,i}) + \Psi(P_{n+1,k+1,0}) = \Psi(P_{n+1,k+1,i+1}) + \Psi(P_{n+1,k,i}). \tag{14.3}$$

Applying Proposition 10.2 to the last term of (14.3) gives

$$\Psi(P_{n+1,k,i}) = \Psi(P_{n,k,i}) \cdot \mathbf{c} - \Psi([\widehat{0},*]) \cdot \mathbf{d}, \tag{14.4}$$

where the interval $[\widehat{0},*]$ is in the quasi-graded poset $P_{n,k,i}$.

Now observe that the quasi-graded poset $P_{n,k,i} \times B_1$ is the face poset of the cone of $\mathrm{Semi}(\Delta_{k,i}, n)$. This cone has two facets, namely, $(\widehat{1},\widehat{0})$ and $(*,\widehat{1})$, and they share a subfacet $(*,\widehat{0})$. Replacing these three strata with a new strata $*$, we obtain by Theorem 13.1 that

$$\Psi(P_{n,k,i} \times B_1) - \Psi([\widehat{0},(*,\widehat{0})]) \cdot \mathbf{d} = \Psi(\mathrm{Semi}(\Delta_{k+1,i}, n+1)) = \Psi(P_{n+1,k+1,i}). \tag{14.5}$$

Observe that the interval $[\widehat{0},(*,\widehat{0})]$ in $P_{n,k,i} \times B_1$ is identical to the interval $[\widehat{0},*]$ in $P_{n,k,i}$.

Finally, using $\Psi(P_{n,k,i} \times B_1) = \Psi(P_{n,k,i}) \cdot \mathbf{c}$ by Lemma 5.3, adding equations (14.3), (14.4) and (14.5), and canceling terms yields equation (14.2). □

The case $n = k$ in Proposition 14.1 appears in Theorem 8.1 in [19]. Refer to Section 5 for the definition of the derivation G.

Theorem 14.2. *The* $\mathbf{cd}$*-index of the simplicial shelling components satisfy the recursion*

$$G(\check{\Phi}(n,k,i)) = \check{\Phi}(n+1,k+1,i+1)$$

with the boundary conditions

$$\check{\Phi}(n,k,0) = \Psi(B_k) \cdot \Psi(\Omega_{n-k+1}),$$

for $k \geq 1$ *and*

$$\check{\Phi}(n,0,0) = \begin{cases} (\mathbf{c}^2 - 2\mathbf{d})^{n/2} & \text{if } n \text{ is even,} \\ (\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} \cdot \mathbf{c} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Substituting the sum (14.1) into Proposition 14.1 yields

$$\sum_{j=0}^{i} \mathrm{Pyr}(\check{\Phi}(n,k,j)) + \check{\Phi}(n+1,k+1,0) = \sum_{j=0}^{i+1} \check{\Phi}(n+1,k+1,j) + \sum_{j=0}^{i} \check{\Phi}(n,k,j) \cdot \mathbf{c}.$$

Using that $\mathrm{Pyr}(w) = w \cdot \mathbf{c} + G(w)$ and canceling terms, the identity simplifies to

$$\sum_{j=0}^{i} G(\check{\Phi}(n,k,j)) = \sum_{j=1}^{i+1} \check{\Phi}(n+1,k+1,j). \tag{14.6}$$

The recursion follows immediately from (14.6). The boundary condition follows from $P_{n,k,0} = B_k * \mathcal{F}(\Omega_{n-k+1})$ when $k \geq 1$ and Example 6.14 when $k = 0$. □

We recall the notion of shellability of non-pure complexes due to Björner and Wachs [10]. Let Δ be a non-pure simplicial complex, that is, a simplicial complex with its maximal faces not necessarily all of the same dimension. We say Δ is *non-pure shellable* if either $\dim(\Delta) = 0$ (and hence any ordering of the facets is a shelling order) or $\dim(\Delta) \geq 1$ and there is an ordering $F_1, \ldots, F_m$ of the facets, called a *shelling order*, satisfying $\left(\bigcup_{j=1}^{r-1} \overline{F_j}\right) \cap \overline{F_r}$ is pure of dimension $\dim(F_r) - 1$ for $r = 2, \ldots, m$. We say that a facet F_r has shelling type (k, i) if $\dim(F_r) + 1 = k$ and that the intersection $\left(\bigcup_{j=1}^{r-1} \overline{F_j}\right) \cap \overline{F_r}$ has i facets (of dimension $k-2$). Note that the first facet F_1 has type $(\dim(F_1)+1, 0)$. Furthermore, the only facet of shelling type $(k, 0)$ is the first facet. The h-triangle entry $h_{k,i}$ is the number of facets of shelling type (k, i). It is a well-known result that the h-triangle does not depend on the particular shelling order and is equivalent to the f-triangle [10].

Theorem 14.3. *Let Δ be a non-pure shellable simplicial complex of dimension at most n. Then the* **cd***-index of the semisuspension of Δ is given by*

$$\Psi(\mathrm{Semi}(\Delta, n)) = \sum_{k=0}^{n} \sum_{i=0}^{k} h_{k,i} \cdot \check{\Phi}(n, k, i).$$

Proof. The proof is by induction on the number of facets in the complex Δ. If Δ has one facet, say of dimension k, then $\mathscr{F}(\mathrm{Semi}(\Delta, n)) = B_{k+1} * \Omega_{n-k} = P_{n,k,0}$. Hence $\Psi(\mathrm{Semi}(\Delta, n)) = \check{\Phi}(n, k, 0)$ and the only non-zero entry in the h-triangle is $h_{k,0} = 1$.

Now assume that the result is true for the complex Δ and we adjoin another facet F of shelling type (k, i). Note that $i \geq 1$. By the inclusion-exclusion Theorem 11.1, we have

$$\Psi(\mathrm{Semi}(\Delta \cup F, n)) - \Psi(\mathrm{Semi}(\Delta, n)) = \Psi(\mathrm{Semi}(F, n)) - \Psi(\mathrm{Semi}(\Delta \cap F, n)). \tag{14.7}$$

Again by inclusion-exclusion, the right-hand side of (14.7) is given by

$$\Psi(\mathrm{Semi}(\Delta_{k,i}, n)) - \Psi(\mathrm{Semi}(\Delta_{k,i-1}, n)) = \Psi(P_{n,k,i}) - \Psi(P_{n,k,i-1}) = \check{\Phi}(n, k, i),$$

completing the induction step. $\square$

In the proof of Theorem 14.3 we did not use the fact that F_1 through F_m are facets of the simplicial complex Δ, only that F_r is a facet of $\overline{F_1} \cup \cdots \cup \overline{F_{r-1}} \cup \overline{F_r}$. Using this observation, we prove the following Pascal type relation.

Proposition 14.4. *For $0 \leq i \leq k < n$ we have the identity*

$$\check{\Phi}(n, k+1, i) = \check{\Phi}(n, k, i) + \check{\Phi}(n, k+1, i+1).$$

Proof. We begin to prove this identity for $i = 0$. Consider the simplex Δ having $k+1$ vertices. Directly we have $\mathrm{Semi}(\Delta, n) = \check{\Phi}(n, k+1, 0)$. On the other hand we can build this complex in two steps, first by adding a facet F of cardinality k and next by adding the simplex Δ. The first step yields $\check{\Phi}(n, k, 0)$ and the second step $\check{\Phi}(n, k+1, 1)$, proving the identity. The case $i > 0$ follows by applying the derivation G i times. $\square$

$k\backslash i$	0	1	2
0	$\mathbf{c}^2 - 2 \cdot \mathbf{d}$		
1	$\mathbf{c}^2 - \mathbf{d}$	$\mathbf{d}$	
2	$\mathbf{c}^2$	$\mathbf{d}$	0

$k\backslash i$	0	1	2	3
0	$\mathbf{c}^3 - 2 \cdot \mathbf{dc}$			
1	$\mathbf{c}^3 - \mathbf{dc} - \mathbf{cd}$	$\mathbf{dc} - \mathbf{cd}$		
2	$\mathbf{c}^3 - \mathbf{cd}$	$\mathbf{dc}$	$\mathbf{cd}$	
3	$\mathbf{c}^3 + \mathbf{dc}$	$\mathbf{dc} + \mathbf{cd}$	$\mathbf{cd}$	0

Table 1: The **cd**-polynomials $\check{\Phi}(2, k, i)$ and $\check{\Phi}(3, k, i)$.

Recalling that the polynomials $\check{\Phi}(n, k, 0)$ are given in Theorem 14.2, we have the following expression for the **cd**-polynomial $\check{\Phi}(n, k, i)$.

Corollary 14.5. *We have the identity*

$$\check{\Phi}(n, k, i) = \sum_{j=0}^{i} (-1)^j \cdot \binom{i}{j} \cdot \check{\Phi}(n, k - j, 0).$$

15 Concluding remarks

Theorem 6.10 was developed using an ambient smooth manifold M. This result can be developed without the ambient manifold using abstract stratifications. We decided against this approach since it is unnecessarily complex and does not give us any advantages.

As was mentioned in the introduction, finding the linear inequalities that hold among the entries of the **cd**-index of a Whitney stratified manifold expands the program of determining linear inequalities for flag vectors of polytopes. Since the coefficients may be negative, one must ask what should the new minimization inequalities be. Observe that Kalai's convolution [31] still holds. More precisely, let M and N be two linear functionals defined on the **cd**-coefficients of any m-dimensional, respectively, n-dimensional manifold. If both M and N are non-negative then their convolution is non-negative on any $(m + n + 1)$-dimensional manifold.

Other inequality questions are: Can Ehrenborg's lifting technique [15] be extended to stratified manifolds? Is there an associated Stanley–Reisner ring for the barycentric subdivision of a stratified space, and if so, what is the right version of the Cohen–Macaulay property [47]? Finally, what non-linear inequalities hold among the **cd**-coefficients?

One interpretation of the coefficients of the **cd**-index is due to Karu [32] who, for each **cd**-monomial, gave a sequence of operators on sheaves of vector spaces to show the non-negativity of the coefficients of the **cd**-index for Gorenstein* posets [32]. Is there a signed analogue of Karu's construction to explain the negative coefficients occurring in the **cd**-index of quasi-graded posets?

One of the original motivations for this paper was to develop a local **cd**-index in the spirit of Stanley's local h-vector [45] which was used to understand the effect of subdivisions on the h-

vector. The main result of Section 12 implies that such a local **cd**-index would always give the zero polynomial.

Observe that the shelling components $\check{\Phi}(n,k,i)$ can have negative coefficients. Refer to Table 1. For which values of n, k and i do we know that they are non-negative? Is there a combinatorial interpretation of the **cd**-polynomials $\check{\Phi}(n,k,i)$? In the case $n = k$ such interpretations are known in terms of André permutations and Simsun permutations [19, 24, 25].

Given a Whitney stratified space, its face poset with rank function given by dimension and weighted zeta function involving the Euler characteristic (see Definition 6.9 and Theorem 6.10) yield an Eulerian quasi-graded poset. Conversely, given an Eulerian quasi-graded poset $(P, \rho, \bar{\zeta})$ can one construct an associated Whitney stratified space? It is clear that for $x \prec y$ with $\rho(x)+1 = \rho(y)$ one must require $\bar{\zeta}(x, y)$ to be a positive integer since $\mathrm{link}_y\, x$ is a 0-dimensional space consisting of a collection of one or more points. What other conditions on an Eulerian quasi-graded poset are necessary so that it is the face poset of a Whitney stratified space?

As always when the **ab**-index is defined one also has the companion quasisymmetric function. This quasisymmetric function can be defined by the (almost) isomorphism γ in [21, Section 3]. More directly, for a chain $c = \{\widehat{0} = x_0 < x_1 < \cdots < x_k = \widehat{1}\}$, define the composition

$$\rho(c) = (\rho(x_0, x_1), \rho(x_1, x_2), \ldots, \rho(x_{k-1}, x_k)).$$

Then the quasisymmetric function of a quasi-graded poset $(P, \rho, \bar{\zeta})$ is given by

$$F(P, \rho, \bar{\zeta}) = \sum_{c} \bar{\zeta}(c) \cdot M_{\rho(c)},$$

where M is the monomial quasisymmetric function. It is straightforward to observe that F can be viewed as a Hopf algebra morphism as follows.

Theorem 15.1. *The following two identities hold for the quasisymmetric function of a quasi-graded poset:*

$$\Delta(F(P, \rho, \bar{\zeta})) = \sum_{\widehat{0} \leq x \leq \widehat{1}} F([\widehat{0}, x], \rho, \bar{\zeta}) \otimes F([x, \widehat{1}], \rho, \bar{\zeta}),$$

$$F(P \times Q, \rho_{P\times Q}, \bar{\zeta}_{P\times Q}) = F(P, \rho_P, \bar{\zeta}_P) \cdot F(Q, \rho_Q, \bar{\zeta}_Q).$$

See [1] for results on generalized Dehn–Sommerville relations in the setting of combinatorial Hopf algebras.

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